MINIMAL BRATTELI DIAGRAMS AND THE DIMENSION GROUPS OF AF C^*-ALGEBRAS

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A method is described which identifies a wide variety of AF algebra dimension groups with groups of continuous functions. The results here generalize the well-known fact that commutative AF algebras have dimension groups which can be identified with groups of integer-valued continuous functions.

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1. Introduction

An important and widely studied class of operator algebras is the approximately finite-dimensional (AF) C^*-algebras. Their name stems from the fact that each element of the algebra can be approximated, to arbitrary precision, by an element from a finite-dimensional subalgebra. As such elements are direct sums of matrices, the AF algebras can be thought of as builtup from relatively simple matrix algebras.

Despite their reasonably simple structure, AF algebras have come to occupy a prominent place in the study of C^*-algebras. It was Glimm [12] who studied the important class of AF algebras which have come to be known as UHF algebras. However, the first completely general treatment of AF algebras was initiated by Bratteli [2] in his seminal paper. It was here that the infinite graphs which bear Bratteli's name were introduced as a means for representing AF algebras. These Bratteli diagrams, as they came to be called, provide a graphical representation of the AF algebra's structure, and will play a prominent role in the present paper's discussion.

Subsequent to [2], Elliott [10] proved that dimension groups are complete isomorphism invariants for AF algebras. These ordered groups with order units can be realized as the K_0 groups (see, e.g., [1]) of the corresponding algebras, and it is this vantage point which has led to them being considered by a number of authors [9, 11, 13, 18].

It is the structure of an AF algebra's dimension group that will be the focus of this paper. As motivation, we offer for consideration the situation when the AF algebra is commutative. To be specific, let X be a compact metric space with a basis consisting of sets which are simultaneously open and closed (clopen), such as the Cantor middle-thirds

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set. Then $C(X)$, the continuous complex-valued functions on X, is a commutative AF algebra, and the associated dimension group can be shown to be order isomorphic to the scaled ordered group $(C(X,\mathbb{Z}), C(X,\mathbb{Z}^+), \chi_X)$ where $C(X,\mathbb{Z})$ are the continuous functions $f : X \to \mathbb{Z}$, $C(X,\mathbb{Z}^+)$ are those functions in $C(X,\mathbb{Z})$ which only take nonnegative integer values, and χ_X is the function identically equal to 1. For commutative AF algebras such as this, the spectrum X can be identified with the set of all infinite paths in the associated Bratteli diagram. Therefore, in such a case, the dimension group can be conveniently described as the set of integer-valued continuous functions whose domain is, at least in some sense, equal to the Bratteli diagram.

It was a generalization of this situation which was considered in [19]. There, Bratteli diagrams with a certain uniqueness condition were studied, and for the associated AF algebras, an analogue to the result mentioned above for commutative AF algebras was established. The present paper will generalize these results further by expanding on the results of [19]. In particular, we will show that a large class of AF algebras have dimension groups which can be viewed as groups of continuous functions. This will involve a consideration of special Bratteli diagrams on which an order structure has been defined. These *ordered Bratteli diagrams* were considered by [15] in the context of partial dynamical systems. Although partial dynamical systems do not play a role in the present paper, as in [15], the existence of a minimal subset of the Bratteli diagram will be important.

Although the results presented here are explicitly for AF algebras and their dimension groups, some of the results involve graph theoretic notions, and may therefore be of interest to those working in such areas.

2. Preliminaries

Consider a graph with vertex sets $V(n)$, $n \geq 0$, and edge sets $E(n)$, $n \geq 1$, which connect the vertices in $V(n-1)$ with the vertices in $V(n)$. We will assume that $V(0)$ is a singleton set. For $e \in E(n)$, let $s(e) \in V(n-1)$ be the vertex at level $n-1$ to which e is connected and let $r(e) \in V(n)$ be the vertex at level n to which e is connected. Assuming that s and r are surjective for each $n \geq 1$, the resulting infinite graph will be called a *Bratteli diagram* (see Example 2.1 below).

For a given Bratteli diagram, we will let X represent the set of all infinite paths in the diagram. That is,

$$X = \{(e_1, e_2, \dots) : e_i \in E(i),\ r(e_i) = s(e_{i+1})\ \forall i \geq 1\}. \tag{2.1}$$

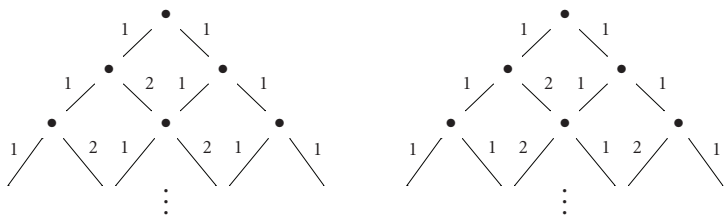
We can then topologize X by letting, for each path of edges $p = (e_1, e_2, \dots, e_n)$ from $V(0)$ to $V(n)$,

$$C(p) = \{(f_1, f_2, \dots) \in X : f_i = e_i\ \forall 1 \leq i \leq n\}. \tag{2.2}$$

By giving X the smallest topology for which each set $C(p)$ is open, one can verify that X is compact and that each set $C(p)$ is also closed, and therefore clopen. Consequently, X is a 0-dimensional compact Hausdorff space.

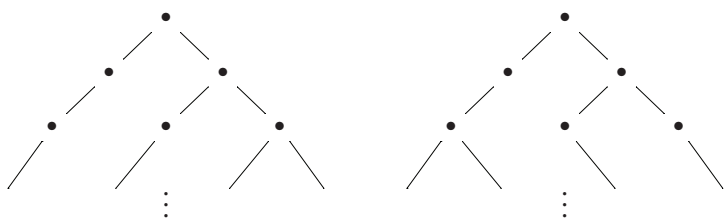
In this paper, the object of interest will not just be a Bratteli diagram, but rather an *ordered Bratteli diagram*. These are Bratteli diagrams where each set $E(n)$ is given a partial order satisfying the property that two elements $e, e' \in E(n)$ are comparable if and only if $r(e) = r(e')$. It is through a consideration of this ordering that the set $X_{\min} \subset X$ is defined as $X_{\min} = \{(e_1, e_2, \ldots) \in X : e_i \text{ is minimal in } E(i) \text{ for all } i\}$. We note that the set $X_{\min}$ is a closed subset of X, and, to illustrate that $X_{\min}$ depends on the ordering, consider the following example.

Example 2.1. To see that nonhomeomorphic copies of $X_{\min}$ are possible when different orders are placed on a given Bratteli diagram, consider the diagram with the two different orders as shown below:



(2.3)

The resulting possibilities for the set $X_{\min}$ will correspond to the following graphs:



(2.4)

The sets of all infinite paths in these diagrams are homeomorphic to the closures of $\{1/n : n \in \mathbb{Z}^+\}$ and $\{\pm(1 - 1/n) : n \in \mathbb{Z}^+\}$, respectively. Since the former has one nonisolated point and the latter two such points, they are clearly not homeomorphic.

3. Minimal Bratteli diagrams

For a given Bratteli diagram, there exists a sequence of matrices with nonnegative integer entries $\overline{A}_{n,n+1} \in M_{m_{n+1},m_n}$, $n \geq 0$, which completely describes the diagram. In Example 2.1, the given diagram has, for example,

$$\overline{A}_{0,1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \overline{A}_{1,2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{3.1}$$

In particular, the rth row of $\overline{A}_{n,n+1}$ describes the edges connected to the rth vertex in $V(n+1)$. The first entry of this row gives the number of edges connecting the first vertex in $V(n)$ to the rth vertex in $V(n+1)$, the second entry the number of edges connecting the second vertex in $V(n)$ to the rth vertex in $V(n+1)$, and so forth. These *multiplicity*

matrices, as they completely describe the Bratteli diagram, also give, up to unitary equivalence, a complete description of the associated AF algebra. Because deleting or inserting levels of edges and vertices in Bratteli diagrams (telescoping and microscoping in the terminology of [8]) does not affect the associated AF algebra (see [7, 17]), for the purposes of this paper we will assume, without loss of generality, that the sequence $\{m_n\}$ is either (i) constant or (ii) strictly increasing.

At this point, as it will be necessary for our main result, we make the assumption that each of the multiplicity matrices has full rank. That is, for all $n \geq 0$, we assume $\operatorname{rank}(\overline{A}_{n,n+1}) = m_n$. Since any AF algebra has many different sequences of subalgebras which can be used to describe it, this means the results presented here will be valid for any AF algebra for which there exists a sequence of subalgebras whose multiplicity matrices have full rank.

With this in mind, we see that in case (i), where the sequence $\{m_n\}$ is constant, we are assuming that each of the multiplicity matrices is invertible. Such a situation, which is treated extensively by [3–6], will not be the focus of this paper. It is the situation in case (ii), when the sequence $\{m_n\}$ is strictly increasing, to which we will devote our primary efforts. As such, this paper can be seen as an attempt to generalize some of the ideas in [3–6]. In this case it is possible to further assume, without loss of generality, that $m_n = n+1$, a fact whose verification we leave to the reader. It will be under these assumptions that we will proceed.

As illustrated in Example 2.1, for a given Bratteli diagram, different possibilities may exist for the set $X_{\min}$. For the proof of this paper's main result, it will be necessary for $X_{\min}$ to be chosen so that it is, in some sense, large. Our assumption that each multiplicity matrix has full rank is sufficient to guarantee that there always exists an ordering such that the resulting ordered Bratteli diagram gives an appropriate $X_{\min}$. The remainder of this section will be devoted to a consideration of precisely what we mean by the word "large."

To begin that discussion, we consider the situation where $m_n = n+1$ and $\operatorname{rank}(\overline{A}_{n,n+1}) = n+1$, for all $n \geq 0$. For the moment, we adopt a general perspective. Let $n \geq 1$ be given. We will consider graphs with the following properties.

Properties 3.1. (I) There are $2n+1$ vertices which are arranged so that n vertices appear in one horizontal row (which we will refer to as level 1) and the remaining $n+1$ vertices appear in a horizontal row below the first (which we will refer to as level 2).

(II) The only edges are those connecting vertices at different levels. In other words, no vertices at the same level are connected by an edge, and consequently such graphs are bipartite.

(III) Every vertex is connected to another by at least one edge.

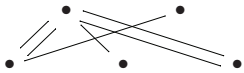
We label the set of all such graphs G_n.

Remark 3.2. The AF algebras with $m_n = n+1$, for all $n \geq 0$, are built from graphs of this form.

Definition 3.3. Given a graph $\Gamma \in G_n$, we will call the graph γ a *reduction of* Γ if $\gamma \in G_n$ is a subgraph of Γ obtained by deleting only edges. A graph in G_n will be called *minimal* if it has $n+1$ edges. Such subgraphs can therefore be thought of as *minimal* spanning trees.

Remark 3.4. Any graph in G_n must have at least $n+1$ edges by Properties 3.1(III). Therefore, no nontrivial reductions of minimal graphs exist.

Example 3.5. It is easy to see that minimal reductions are not unique. The graph



(3.2)

is an element of G_2, with both

(3.3)

being minimal reductions.

It is easy to see that not all Bratteli diagrams will have minimal reductions at each level. However, for our purposes, we will be interested in those Bratteli diagrams which, at each level, do have minimal reductions. In particular, if a Bratteli diagram does have a minimal reduction at each level, then this means that there exists at least one ordering on this diagram's edges so that for the resulting ordered diagram, the set $X_{\min}$ will itself be a Bratteli diagram. One might call such a graph corresponding to $X_{\min}$ a *minimal Bratteli diagram* since deleting any more edges will create a subgraph which is no longer a Bratteli diagram. This will be an important object for our main result. We are therefore interested in answering the following somewhat more general question: for any $n \geq 1$, how can we decide which elements of G_n have a minimal reduction (in G_n)?

At this point we will begin to make use of the assumption we have made about the multiplicity matrices $\overline{A}_{n,n+1}$. Namely, that each has full rank. Under such circumstances we can guarantee that a graph $\Gamma \in G_n$ has a minimal reduction for all $n \geq 1$.

Theorem 3.6. *Let $\Gamma \in G_n$ for any $n \geq 1$ and suppose M_Γ is the multiplicity matrix which describes Γ. If* $\operatorname{rank}(M_\Gamma) = n$, *then Γ has a minimal reduction.*

Proof. We will proceed via induction on n. The induction basis is provided by the case $n = 1$, where it is easy to see that all graphs $\Gamma \in G_1$ have minimal reductions.

Next, suppose the result holds for n and let $\Gamma \in G_{n+1}$. We will assume $\operatorname{rank}(M_\Gamma) = n+1$ and write $M_\Gamma = [a_{ij}]$, $1 \leq i \leq n+2$, $1 \leq j \leq n+1$. Before proceeding further we remark that by permuting the rows and columns of M_Γ we are merely rearranging the vertices in the graph. For example, a row permutation amounts to rearranging the vertices at level 2 and a column permutation amounts to rearranging the vertices at level 1. Of course, if it is possible to obtain a minimal reduction of this permuted form of the original graph, then, by reversing the permutations, we will also have a minimal reduction of the original graph. We will therefore work with various matrices obtained from M_Γ through row and column permutations in order to obtain our result.

We consider two cases.

(a) There exists $j_0 \in \{1,\dots,n+1\}$ such that the submatrix $[a_{ij}]$, $1 \le i \le n+2$, $1 \le j \le n+1$, $j \ne j_0$, has only nonzero rows.

(b) For every $j_0 \in \{1,\dots,n+1\}$, the submatrix $[a_{ij}]$, $1 \le i \le n+2$, $1 \le j \le n+1$, $j \ne j_0$, has a zero row.

Consider case (a). Permute the columns of M_Γ so that the submatrix that results from omitting the last column of the permuted matrix has only nonzero rows. For notational convenience, we will continue to write M_Γ and $[a_{ij}]$ for these permuted forms of the original multiplicity matrix. Now, permute the rows so that the top $n+1$ rows are linearly independent, which we can do since $\operatorname{rank}(M_\Gamma) = n+1$. Thus, the submatrix $[a_{ij}]$, $1 \le i, j \le n+1$, is nonsingular.

Because $[a_{ij}]$, $1 \le i, j \le n+1$, is nonsingular, the matrix $[a_{ij}]$, $1 \le i \le n+1$, $1 \le j \le n$, has rank n. Thus, there exists i, which, without loss of generality, we may suppose equals $n+1$, such that the submatrix $[a_{ij}]$, $1 \le i, j \le n$, is nonsingular.

At this point we make an assumption that is justified later (Lemma 3.7). Assume $a_{n+1,n+1} \ne 0$. Permute the last two rows of M_Γ to finally arrive at a matrix with the following characteristics:

(i) omitting the $(n+1)$st column leaves all nonzero rows;

(ii) the first n rows of the submatrix $[a_{ij}]$, $1 \le i \le n+1$, $1 \le j \le n$, are linearly independent;

(iii) the entry $a_{n+2,n+1}$ is nonzero.

Since (i) and (ii) hold, the submatrix $[a_{ij}]$, $1 \le i \le n+1$, $1 \le j \le n$, is the multiplicity matrix for a graph in G_n with rank n. Thus, by the induction hypothesis, there exists a minimal reduction of this graph (which is just a subgraph of Γ). Since (iii) holds, there is at least one edge connecting the last vertex at level 1 with the last vertex at level 2. By deleting all other edges which connect these last two vertices to any others, we obtain a minimal reduction of Γ.

Next, consider case (b). The assumption implies that at least $n+1$ rows have exactly one nonzero entry and for different rows, the columns in which these entries appear are different. We may assume, without loss of generality, that these rows are the first $n+1$ rows. So, the first $n+1$ vertices at level 2 are connected to exactly one vertex at level 1, and different level 2 vertices are connected to different level 1 vertices. Finally, since the last row is not zero, the last vertex at level 2 is connected to at least one vertex at level 1. Thus, a minimal reduction in G_{n+1} is possible. □

We now justify an assumption made in the proof of the previous theorem.

Lemma 3.7. *Given an invertible matrix $B = [b_{ij}] \in M_n(\mathbb{C})$, there exists k, $1 \le k \le n$, such that the submatrix $[b_{ij}]$, $1 \le i, j \le n$, $i \ne k$, $j \ne n$, is nonsingular and $b_{k,n} \ne 0$.*

Proof. First, if there exists $1 \le k \le n$ such that $b_{k,1} = \cdots = b_{k,n-1} = 0$, then it must be that $b_{k,n} \ne 0$. Furthermore, since the dimension of the set span $\{[b_{i,1},\dots,b_{i,n-1}] : 1 \le i \le n\}$ is $n-1$, the desired result is achieved. To complete the proof we consider the case where $[b_{i,1},\dots,b_{i,n-1}] \ne 0$, for every $1 \le i \le n$.

Assume, without loss of generality, that the first i_0 rows of the matrix B end in 0 (i.e., $b_{i,n} = 0$, for all $1 \le i \le i_0$) and the remaining rows end in a nonzero number (i.e., $b_{i,n} \ne 0$,

for all $i_0+1 \le i \le n$). Of course, $i_0 < n$ since B is nonsingular. If we assume that there exist scalars $\alpha_1,\dots,\alpha_{i_0}$ such that

$$\alpha_1[b_{1,1},\dots,b_{1,n-1}]+\cdots+\alpha_{i_0}[b_{i_0,1},\dots,b_{i_0,n-1}]=0, \tag{3.4}$$

then $\alpha_1[b_{1,1},\dots,b_{1,n-1},b_{1,n}]+\cdots+\alpha_{i_0}[b_{i_0,1},\dots,b_{i_0,n-1},b_{i_0,n}]=0$ as well since $b_{1,n}=\cdots=b_{i_0,n}=0$. Because these later vectors are linearly independent, it must be that $\alpha_1=\cdots=\alpha_{i_0}=0$, and therefore, the set $\{[b_{i,1},\dots,b_{i,n-1}]:1\le i\le i_0\}$ is linearly independent.

We know the set $\{[b_{i,1},\dots,b_{i,n-1}]:1\le i\le n\}$ is linearly dependent, and so there exist coefficients $\alpha_1,\dots,\alpha_n$, not all zero, such that

$$\sum_{i=1}^{n}\alpha_i[b_{i,1},\dots,b_{i,n-1}]=0. \tag{3.5}$$

Furthermore, since $\{[b_{i,1},\dots,b_{i,n-1}]:1\le i\le i_0\}$ is a linearly independent set, it must be that at least one coefficient α_{i_1}, such that $i_1>i_0$, is nonzero. But then, $\mathbb{C}^{n-1}=\operatorname{span}\{[b_{i,1},\dots,b_{i,n-1}]:1\le i\le n\}$, which in turn is just $\operatorname{span}\{[b_{i,1},\dots,b_{i,n-1}]:1\le i\le n,\ i\ne i_1\}$, implying $\{[b_{i,1},\dots,b_{i,n-1}]:1\le i\le n,\ i\ne i_1\}$ is linearly independent. Hence, the submatrix $[b_{ij}]$, $1\le i\le n$, $i\ne i_1$, $1\le j\le n-1$, is nonsingular and $b_{i_1,n}\ne 0$. □

This proves that for those AF algebras which are of type (ii), a minimal Bratteli diagram corresponding to the set $X_{\min}$ exists under the assumption that each multiplicity matrix has full rank. We note that for algebras of type (i), there is an analogous result.

The results obtained here will be used in the subsequent section to prove a result about the K_0 groups of a large class of AF algebras. Despite that motive for their inclusion, they are interesting in their own right. After all, being able to reduce a Bratteli diagram in the way described here means that an order exists so that the corresponding ordered Bratteli diagram has a sub-Bratteli diagram which is minimal spanning tree. Specifically, deleting any more edges will result in a subgraph which is no longer itself a Bratteli diagram. Of course, all Bratteli diagrams are reducible to sub-Bratteli diagrams (possibly in a trivial way). However, the reductions here to the level of $X_{\min}$ are as far as one can go. Deleting any more edges, without deleting any vertices, will result in a subgraph which is no longer a Bratteli diagram.

4. Dimension groups and minimal Bratteli diagrams

As is well known, if X is a 0-dimensional (basis consisting of clopen sets) compact metric space, then there exists a sequence $\{E_n\}_{n=0}^{\infty}$ of successively finer partitions of X which generate the topology and consist of clopen sets. Therefore, $\{C(E_n)\}_{n=0}^{\infty}$, where $C(E_n)$ consists of those functions constant on elements of the partition E_n, is an increasing sequence of finite-dimensional C^*-algebras (isomorphic to $\mathbb{C}^{|E_n|}$). Since $C(X)=\overline{\bigcup_{n\ge 0}C(E_n)}$, it follows that $C(X)$ is AF.

For any $n\ge 0$, the dimension group $K_0(C(E_n))$ of $C(E_n)$ is easily seen to be isomorphic to $(C(E_n,\mathbb{Z}),C(E_n,\mathbb{Z}^+),\chi_X)$, where $C(E_n,\mathbb{Z})$ are the continuous functions from X to $\mathbb{Z}$

constant on the elements of E_n. As such, one can conclude that

$$K_0(C(X)) = \lim_{\rightarrow} K_0(C(E_n)) \cong (C(X,\mathbb{Z}), C(X,\mathbb{Z}^+), \chi_X). \tag{4.1}$$

One aspect of this well-known example which we want to draw attention to is that $X_{\min} = X$. Therefore, in a natural way, the dimension group of $C(X)$ can be realized as a group of continuous functions on $X_{\min}$. In the context of AF groupoids, [16] presents additional examples for which this also holds true, and [19] uses dimension groups of this form to show that $X_{\min}$ is an AF algebra isomorphism invariant under certain hypotheses. We intend in this section to generalize the results of [16, 19] in order to demonstrate that the dimension groups of those AF algebras in a certain class can be realized in this same way, as groups of continuous functions on $X_{\min}$.

Here, as in the previous section, we begin with a Bratteli diagram such that the sequence $\{m_n\}_{n=0}^{\infty}$ is strictly monotonically increasing and each multiplicity matrix $\overline{A}_{n,n+1}$ has full rank. As mentioned earlier, we may also assume, without loss of generality, that $m_n = n+1$, for all $n \geq 0$. It follows by Theorem 3.6 that there exists an ordering on the diagram such that $X_{\min}$ is itself a Bratteli diagram. We will therefore proceed by assuming that, in fact, ours is an ordered Bratteli diagram with such an order.

For each $n \geq 1$, let $v_n \in V(n)$ be a vertex in the Bratteli diagram, and so, by extension, also a vertex in the graph corresponding to $X_{\min}$. Define $C(v_n) = \{(x_1,x_2,\ldots) \in X_{\min} : r(x_n) = v_n = s(x_{n+1})\}$. Note that these sets are simply the clopen basis elements of X restricted to $X_{\min}$, and therefore act as basis elements for the topology $X_{\min}$ inherits from X. It now follows from Theorem 3.6 that for each $n \geq 1$, there exist unique vertices $r_n, r_n' \in V(n)$, and $a_n \in V(n-1)$ such that $C(r_n) \cup C(r_n') \subset C(a_n)$ and that there exists a bijection

$$\sigma_n : \{s \in V(n) : s \neq r_n, r_n'\} \longrightarrow \{b \in V(n-1) : b \neq a_n\} \tag{4.2}$$

such that $C(s) \subset C(\sigma_n(s))$.

To achieve our result, we will now define, for all $n \geq 1$, a linear map $R_n : \mathbb{C}^{n+1} \to C(X_{\min},\mathbb{C})$ by $R_n(\alpha_1,\ldots,\alpha_{n+1}) = \sum_{l=0}^{n} \alpha_{l+1}\chi_{C(r_l)}$, where $r_0 = 1$. Note that we are now assuming that a specific choice for $X_{\min}$ has been made. Thus, the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{r_n'\}_{n=1}^{\infty}$ are fixed. The following technical result about the maps R_n will be useful to us in what follows.

Lemma 4.1. *For all $n \geq 1$, if $(\alpha_1,\ldots,\alpha_{n+1})^T \in \mathbb{C}^{n+1}$, then there exists $(\beta_1,\ldots,\beta_{n+1})^T \in \mathbb{C}^{n+1}$ such that if $v(1,n), v(2,n),\ldots,v(n+1,n)$ are the vertices in $V(n)$, then*

$$R_n(\beta_1,\ldots,\beta_{n+1}) = \sum_{l=1}^{n+1} \alpha_l \chi_{C(v(l,n))}. \tag{4.3}$$

Furthermore, the map R_n injects.

Proof. For $n = 1$, set $f = \alpha_1\chi_{C(v(1,1))} + \alpha_2\chi_{C(v(2,1))}$. Let $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 - \alpha_1$. Then,

$$R_1(\beta_1,\beta_2) = \beta_1\chi_{C(v(1,0))} + \beta_2\chi_{C(v(2,1))} = \alpha_1\chi_{C(v(1,0))} + (\alpha_2 - \alpha_1)\chi_{C(v(2,1))}. \tag{4.4}$$

We note that $C(v(1,1)) \cup C(v(2,1)) = C(v(1,0))$, and so,

$$R_1(\beta_1,\beta_2) = \alpha_1\chi_{C(v(1,1))} + \alpha_1\chi_{C(v(2,1))} + (\alpha_2 - \alpha_1)\chi_{C(v(2,1))} = f. \tag{4.5}$$

This provides the basis for a proof by induction, and in fact illustrates the strategy that the inductive step must employ. As the details are not difficult, the remainder of the proof is left to the reader. □

Now since, by assumption, $\overline{A}_{n,n+1}$ has full rank, for all $n \geq 0$, it is possible by adding appropriate columns $[a_{1,n+2},\dots,a_{n+2,n+2}]^T$ of nonnegative integers to create a sequence $\{A_{n,n+1}\}_{n=0}^{\infty}$ of nonsingular matrices where

$$A_{n,n+1} = \begin{bmatrix} & a_{1,n+2} \\ \overline{A}_{n,n+1} & \vdots \\ & a_{n+2,n+2} \end{bmatrix} \in M_{n+2}. \tag{4.6}$$

Remark 4.2. At this point, any choice for $[a_{1,n+2},\dots,a_{n+2,n+2}]^T$ which makes $A_{n,n+1}$ nonsingular is appropriate. In fact, there is no a priori reason that the $a_{i,n+2}$ cannot be elements of $\mathbb{C}$. However, as we will see in a moment, in certain instances a somewhat more restrictive choice will be desirable.

To compute the dimension group of an AF algebra $\mathfrak{A}$, $K_0(\mathfrak{A})$, where $\mathfrak{A}$ is the direct limit $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ of the finite-dimensional subalgebras $\mathfrak{A}_n$, we utilize the fact that $K_0(\mathfrak{A}) = \varinjlim K_0(\mathfrak{A}_n)$. However, to be more explicit about the nature of $\varinjlim K_0(\mathfrak{A}_n)$, we first define, for all $n \geq 1$,

$$A_n = [A_{0,1}^{-1} \oplus I_{n-1}] \cdots [A_{n-2,n-1}^{-1} \oplus I_1] A_{n-1,n}^{-1}. \tag{4.7}$$

Then, let $\Phi_n : \mathbb{Z}^{n+1} \to C(X_{\min},\mathbb{Q})$ be given by $\Phi_n = R_n \circ A_n$, for all $n \geq 1$.

Letting $n \geq 1$ be given, it is clear that

$$A_n^{-1} = A_{n-1,n}[A_{n-2,n-1} \oplus I_1] \cdots [A_{1,2} \oplus I_{n-2}][A_{0,1} \oplus I_{n-1}] \tag{4.8}$$

and, by the multiplicativity of the determinant, that

$$|A_n^{-1}| = |A_{n-1,n}| \cdot |A_{n-2,n-1}| \cdot \cdots \cdot |A_{1,2}| \cdot |A_{0,1}|. \tag{4.9}$$

As, for example, in [14, pages 20-21], we can then write

$$A_n = \frac{1}{|A_n^{-1}|} \operatorname{adj}(A_n^{-1}), \tag{4.10}$$

where $\operatorname{adj}(A_n^{-1})$ is the *adjugate* or *classical adjoint* of the matrix A_n^{-1}. In defining each matrix $A_{n,n+1}$, if we choose columns of integers, then each matrix A_n^{-1} will be a product of matrices with integer entries, and therefore itself must have integer entries. But, by the definition of $\operatorname{adj}(A_n^{-1})$, it will then follow that $\operatorname{adj}(A_n^{-1})$ has integer entries as well.

We will now define the set G_n by

$$G_n = \left\{ \frac{a}{|A_n^{-1}|} : a \in \mathbb{Z} \right\}. \tag{4.11}$$

Then, G_n is embedded in G_{n+1} by inclusion, and we write

$$G = \lim_{\rightarrow} G_n = \bigcup_{n \geq 1} G_n \subset \mathbb{Q}. \tag{4.12}$$

It is then clear that for $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{Z}^{n+1}$, $A_n(\alpha_1, \ldots, \alpha_{n+1})^T$ is an element of G_{n+1}. Giving G the discrete topology, we therefore see that Φ_n will be a map from $\mathbb{Z}^{n+1}$ to $C(X_{\min}, G)$, which since $C(X_{\min}, G) \subset C(X_{\min}, \mathbb{Q})$, explains why Φ_n maps to rational-valued functions.

To show that $K_0(\mathfrak{A})$ is (at least) a subgroup of $C(X_{\min}, \mathbb{Q})$, we would like to show that the diagram

$$\begin{tikzcd} K_0(\mathfrak{A}_n) \cong \mathbb{Z}^{n+1} \arrow[r, "\phi_{n*}"] \arrow[dr, "\Phi_n"'] & K_0(\mathfrak{A}_{n+1}) \cong \mathbb{Z}^{n+2} \arrow[d, "\Phi_{n+1}"] \\ & C(X_{\min}, \mathbb{Q}) \end{tikzcd} \tag{4.13}$$

commutes, where ϕ_{n*} is the homomorphism corresponding to the multiplicity matrix $\overline{A}_{n,n+1}$.

Let $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{Z}^{n+1}$. Then, $\Phi_{n+1} \circ \phi_{n*}(\alpha_1, \ldots, \alpha_{n+1})^T$ can be written as

$$\left(R_{n+1} \circ [A_{0,1}^{-1} \oplus I_n] \cdots [A_{n-1,n}^{-1} \oplus I_1] A_{n,n+1}^{-1} \right) \circ A_{n,n+1}(\alpha_1, \ldots, \alpha_{n+1}, 0)^T. \tag{4.14}$$

If we define $(\beta_1^{n-1}, \ldots, \beta_{n+1}^{n-1})^T = A_{n-1,n}^{-1}(\alpha_1, \ldots, \alpha_{n+1})^T$ and, in general, for $2 \leq i < n$, $(\beta_1^{n-i}, \ldots, \beta_{n-i+2}^{n-i})^T = A_{n-i,n-i+1}^{-1}(\beta_1^{n-i+1}, \ldots, \beta_{n-i+2}^{n-i+1})^T$, we see that

$$\begin{aligned} &\Phi_{n+1} \circ \phi_{n*}(\alpha_1, \ldots, \alpha_{n+1})^T \\ &\quad = R_{n+1} \circ [A_{0,1}^{-1} \oplus I_n] \cdots [A_{n-2,n-1}^{-1} \oplus I_2] (\beta_1^{n-1}, \ldots, \beta_{n+1}^{n-1}, 0)^T \\ &\quad = R_{n+1}(\beta_1^0, \beta_2^0, \beta_3^1, \ldots, \beta_n^{n-2}, \beta_{n+1}^{n-1}, 0) = \beta_1^0 \chi_{C(v(1,0))} + \sum_{l=1}^{n} \beta_{l+1}^{l-1} \chi_{C(v(r_l))}. \end{aligned} \tag{4.15}$$

However, the following calculation yields:

$$\Phi_n(\alpha_1, \ldots, \alpha_{n+1})^T = R_n(\beta_1^0, \beta_2^0, \beta_3^1, \ldots, \beta_n^{n-2}, \beta_{n+1}^{n-1}) = \beta_1^0 \chi_{C(v(1,0))} + \sum_{l=1}^{n} \beta_{l+1}^{l-1} \chi_{C(v(r_l))}. \tag{4.16}$$

Thus, $\Phi_n = \Phi_{n+1} \circ \phi_{n*}$, and we conclude that the diagram commutes. By the universal property of the direct limit, it follows that there exists a homomorphism $\Psi : K_0(\mathfrak{A}) \to C(X_{\min}, \mathbb{Q})$. Since each of the maps Φ_n is injective, it follows that, in fact, Ψ is an injection. We summarize all of this in the following theorem.

Theorem 4.3. *Let $(V(n),E(n),r,s)$ be a Bratteli diagram such that $V(n) = n+1$ for all $n \geq 0$. If each of the multiplicity matrices describing this diagram has full rank, then the dimension group of the corresponding AF algebra is a subgroup of $C(X_{\min},\mathbb{Q})$, where $X_{\min}$ is a compact 0-dimensional Hausdorff space corresponding to a subgraph of the Bratteli diagram.*

This result, when applied to a commutative AF algebra $C(X)$, will give $K_0(C(X))$ as $(C(X,\mathbb{Z}),C(X,\mathbb{Z}^+),\chi_X)$ since in this case $X = X_{\min}$ for any choice of ordering on the corresponding Bratteli diagram. We point the reader to [19] for an application of this result which shows that, in fact, $X_{\min}$ is an isomorphism invariant for a certain class of AF algebras.

Remark 4.4. Considering $C(X_{\min},\mathbb{Q})$ as an ordered group with positive cone $C(X_{\min},\mathbb{Q}^+)$, this homomorphism is not necessarily order preserving. That is, in general, $K_0^+(\mathfrak{A}) \ncong K_0(\mathfrak{A}) \cap C(X_{\min},\mathbb{Q}^+)$. We will say more about this in a moment. It is true, however, that the order unit of $K_0(\mathfrak{A})$ in $C(X_{\min},\mathbb{Q})$ is $\chi_{X_{\min}}$.

The following theorem provides conditions under which more information about the structure of $K_0(\mathfrak{A})$ is available. It should also be noted that this theorem will apply to the GICAR algebra whose Bratteli diagram appears in Example 2.1.

Theorem 4.5. *If, in constructing the matrices $A_{n,n+1}$, integer-valued columns can be chosen so that $|A_{n,n+1}| = 1$, for all $n \geq 0$, then $K_0(\mathfrak{A}) \cong C(X_{\min},\mathbb{Z})$ and the order unit is $\chi_{X_{\min}}$. In general, however, $K_0^+(\mathfrak{A}) \ncong C(X_{\min},\mathbb{Z}^+)$.*

Proof. In this case each matrix A_n is invertible and has integer entries. Thus, $K_0(\mathfrak{A})$ is a subgroup of $C(X_{\min},\mathbb{Z})$. If we let $f \in C(X_{\min},\mathbb{Z})$, then by continuity and the compactness of $X_{\min}$, there exists $n \geq 1$ and $(\alpha_1,\dots,\alpha_{n+1})^T \in \mathbb{Z}^{n+1}$ such that $f = \sum_{l=1}^{n+1} \alpha_l \chi_{B(l,n)}$. By Lemma 4.1, there exists $(\beta_1,\dots,\beta_{n+1})^T \in \mathbb{Z}^{n+1}$ such that $f = R_n(\beta_1,\dots,\beta_{n+1})$. Thus, $A_n^{-1}(\beta_1,\dots,\beta_{n+1})^T \in \mathbb{Z}^{n+1} \cong K_0(\mathfrak{A}_n)$, and we can obtain f as $\Phi_n(A_n^{-1}(\beta_1,\dots,\beta_{n+1})^T)$. □

This theorem implies that whenever two AF algebras satisfy the hypotheses of Theorem 4.5 and each has an ordered Bratteli diagram which yields the same $X_{\min}$, then the only aspect of their dimension groups which distinguishes them is the positive cone. This, of course, makes it clear why, in general, $K_0^+(\mathfrak{A})$ is not $C(X_{\min},\mathbb{Z}^+)$. At the beginning of this section we commented on how if X is a 0-dimensional compact metric space, then the AF algebra $C(X)$ has dimension group $(C(X,\mathbb{Z}),C(X,\mathbb{Z}^+),\chi_X)$. Therefore, given $\mathfrak{A}$ satisfying the hypotheses of Theorem 4.5, for any $X_{\min}$ associated with $\mathfrak{A}$,

$$K_0(C(X_{\min})) \cong (C(X_{\min},\mathbb{Z}),C(X_{\min},\mathbb{Z}^+),\chi_{X_{\min}}). \tag{4.17}$$

So, by Elliott's theorem [10] and Theorem 4.5, unless $C(X_{\min}) \cong \mathfrak{A}$, it must be that $K_0^+(\mathfrak{A}) \subset C(X_{\min},\mathbb{Z})$ is distinct from $C(X_{\min},\mathbb{Z}^+)$.

One also sees that the flexibility that may exist in choosing an ordering (and therefore $X_{\min}$) is of no help here. In particular, recall that $X_{\min}$ is specified by the ordering on the diagram. However, as the arguments above apply to any choice for this ordering, much flexibility exists in our choice for $X_{\min}$. Despite this flexibility, any $X_{\min}$ will result in $K_0^+(\mathfrak{A})$ being, in general, different from $C(X_{\min},\mathbb{Z}^+)$. Despite this, [16] shows that useful characterizations of $K_0^+(\mathfrak{A})$ as a subset of $C(X_{\min},\mathbb{Z})$ do exist in specific cases.

Furthermore, circumstances under which the topological structure of $X_{\min}$ is useful in distinguishing between AF algebras are discussed in [19].

Remark 4.6. To this point, we have only been concerned with those AF algebras $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$ such that $m_n = n+1$, for all $n \geq 0$. The case where $m_n = L \in \mathbb{Z}^+$, for all $n \geq 0$ and L fixed, is simpler and more transparent, and in fact culminates in analogous results. Furthermore, it is in part treated by [3–6], where numerical invariants are considered.

It should be noted that the converse of Theorem 4.3 does not hold. In particular, consider the Bratteli diagram where $V(n) = n+1$, for all $n \geq 0$, and which has an edge connecting every vertex at a given level to every vertex at both the previous and subsequent levels. Such a diagram corresponds to a UHF algebra (see, e.g., [7]), and so its dimension group will be isomorphic to a subgroup of $C(X_{\min}, \mathbb{Q})$ which consists of functions of the form $\alpha\chi_{X_{\min}}$. However, the corresponding multiplicity matrices will each have rank 1.

This leads one to ask if there is some way to characterize those AF algebras which have dimension groups of the form $C(X, \mathbb{Q})$. Another issue to address, which would take its lead from [3–6], might be potential relationships between the isomorphism class of an algebra and the characteristics of the full-rank multiplicity matrices, possibly in terms of determinants of submatrices.

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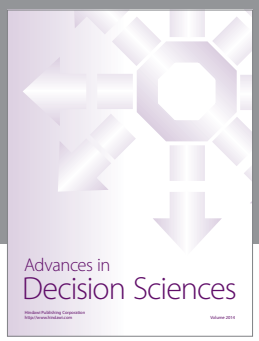

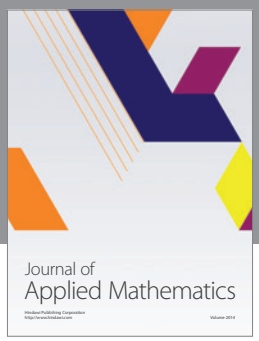

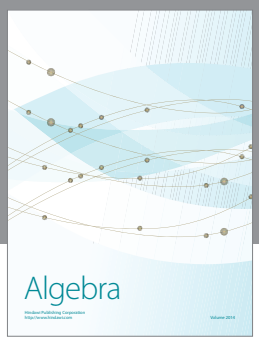

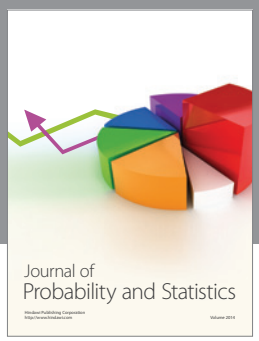

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