INTUITIONISTIC FUZZY H_v-IDEALS

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The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy sets to H_v-rings. We introduce the notion of an intuitionistic fuzzy H_v-ideal of an H_v-ring and then some related properties are investigated. We state some characterizations of intuitionistic fuzzy H_v-ideals. Also we investigate some natural equivalence relations on the set of all intuitionistic fuzzy H_v-ideals of an H_v-ring.

1. Introduction and preliminaries

Hyperstructure theory was born in 1934 when Marty [11] defined hypergroups as a generalization of groups. This theory has been studied in the following decades and nowadays by many mathematicians. A recent book [3] contains a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities. Vougiouklis in the fourth Algebraic Hyperstructures and Applications Congress (1990) [15] introduced the notion of H_v-structures. The H_v-structures are hyperstructures where the equality is replaced by the nonempty intersection. The main tool in the study of H_v-structure is the fundamental structure which is the same as in the classical hyperstructures. In this paper, we deal with H_v-rings. H_v-rings are the largest class of algebraic systems that satisfy ring-like axioms. In [4], Darafsheh and Davvaz defined the H_v-ring of fractions of a commutative hyperring which is a generalization of the concept of ring of fractions. For the notion of an H_v-near-ring module, you can see [7]. In [13], Spartalis studied a wide class of H_v-rings resulting from an arbitrary ring by using the P-hyperoperations. In [18], Vougiouklis introduced the classes of H_v-rings useful in the theory of representations.

A *hyperstructure* is a nonempty set H together with a map $* : H \times H \to \mathcal{P}^*(H)$ called *hyperoperation*, where $\mathcal{P}^*(H)$ denotes the set of all nonempty subsets of H. The image of the pair (x,y) is denoted by $x * y$. If $x \in H$ and $A,B \subseteq H$, then by $A * B$, $A * x$, and $x * B$,

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we mean

$$A * B = \bigcup_{a\in A,\, b\in B} a * b, \qquad A * x = A * \{x\}, \qquad x * B = \{x\} * B. \tag{1.1}$$

A hyperstructure $(H, *)$ is called an H_v-*semigroup* if

$$(x * (y * z)) \cap ((x * y) * z) \neq \varnothing \quad \forall x, y, z \in H. \tag{1.2}$$

Defintion 1.1. An H_v-*ring* is a system $(R, +, \cdot)$ with two hyperoperations satisfying the following ring-like axioms:

(i) $(R, +, \cdot)$ is an H_v-*group*, that is,

$$\begin{gathered} ((x+y)+z) \cap (x+(y+z)) \neq \varnothing \quad \forall\ x, y \in R, \\ a+R = R+a = R \quad \forall\ a \in R; \end{gathered} \tag{1.3}$$

(ii) $(R, \cdot)$ is an H_v-semigroup;

(iii) $(\cdot)$ is *weak distributive* with respect to $(+)$, that is, for all $x, y, z \in R$,

$$\begin{gathered} (x \cdot (y+z)) \cap (x \cdot y + x \cdot z) \neq \varnothing, \\ ((x+y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \varnothing. \end{gathered} \tag{1.4}$$

An H_v-ring $(R, +, \cdot)$ is called *dual* H_v-*ring* if $(R, \cdot, +)$ is an H_v-ring. If both operations $(+)$ and $(\cdot)$ are weak commutative, then R is called a *weak commutative dual* H_v-*ring*.

We see that H_v-rings are a nice generalization of rings. For more definitions, results, and applications on H_v-rings, see [4, 5, 7, 8, 13–15, 17, 18].

Example 1.2 (cf. Vougiouklis [18]). Let $(H, *)$ be an H_v-group, then for every hyperoperation $(\circ)$ such that $\{x, y\} \subseteq x \circ y$ for all $x, y \in H$, the hyperstructure $(H, *, \circ)$ is a dual H_v-ring.

Example 1.3 (cf. Dramalidis [8]). On the set $\mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers, we define three hyperoperations:

$$\begin{gathered} x \uplus y = \{r(x+y) \mid r \in [0,1]\}, \\ x \otimes y = \{x + r(y-x) \mid r \in [0,1]\}, \\ x \Box y = \{x + ry \mid r \in [0,1]\}. \end{gathered} \tag{1.5}$$

Then the hyperstructure $(\mathbb{R}^n, *, \circ)$, where $*, \circ \in \{\uplus, \otimes, \Box\}$, is a weak commutative dual H_v-ring.

Defintion 1.4. Let R be an H_v-ring. A nonempty subset I of R is called a *left* (resp., *right*) H_v-*ideal* if the following axioms hold:

(i) $(I, +)$ is an H_v-subgroup of $(R, +)$,

(ii) $R \cdot I \subseteq I$ (resp., $I \cdot R \subseteq I$).

2. Fuzzy sets and intuitionistic fuzzy sets

The concept of a fuzzy subset of a nonempty set was first introduced by Zadeh [19].

Let X be a nonempty set, a mapping $\mu : X \to [0,1]$ is called a fuzzy subset of X. The complement of μ, denoted by μ^c, is the fuzzy set of X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Note that using fuzzy subsets, we can introduce on any ring the structure of H_v-ring.

Example 2.1 (cf. Davvaz [5]). Let $(R,+,\cdot)$ be an ordinary ring and let μ be a fuzzy subset of R. We define hyperoperations $\uplus$, $\otimes$, $*$ on R as follows:

$$\begin{aligned} x \uplus y &= \{t \mid \mu(t) = \mu(x+y)\}, \\ x \otimes y &= \{t \mid \mu(t) = \mu(x \cdot y)\}, \\ x * y = y * x &= \{t \mid \mu(x) \leq \mu(t) \leq \mu(y)\} \quad (\text{if } \mu(x) \leq \mu(y)). \end{aligned} \tag{2.1}$$

Then $(R,*,*)$, $(R,*,\otimes)$, $(R,*,\uplus)$, $(R,\uplus,*)$, and $(R,\uplus,\otimes)$ are H_v-rings.

Rosenfeld [12] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since then, many papers concerning various fuzzy algebraic structures have appeared in the literature. In [5–7], Davvaz applied the concept of fuzzy set theory in the algebraic hyperstructures, in particular in [5] he defined the concept of fuzzy H_v-ideal of an H_v-ring which is a generalization of the concept of fuzzy ideal.

Defintion 2.2. Let $(R,+,\cdot)$ be an H_v-ring and μ a fuzzy subset of R. Then μ is said to be a *left* (resp., *right*) *fuzzy H_v-ideal* of R if the following axioms hold:

(1) $\min\{\mu(x),\mu(y)\} \leq \inf\{\mu(z) \mid z \in x+y\}$ for all $x,y \in R$,

(2) for all $x,a \in R$, there exists $y \in R$ such that $x \in a+y$ and

$$\min\{\mu(a),\mu(x)\} \leq \mu(y), \tag{2.2}$$

(3) for all $x,a \in R$ there exists $z \in R$ such that $x \in z+a$ and

$$\min\{\mu(a),\mu(x)\} \leq \mu(z), \tag{2.3}$$

(4) $\mu(y) \leq \inf\{\mu(z) \mid z \in x \cdot y\}$ (resp., $\mu(x) \leq \inf\{\mu(z) \mid z \in x \cdot y\}$) for all $x,y \in R$.

Example 2.3 (cf. Davvaz [5]). Let $(R,+,\cdot)$ be an ordinary ring and let μ be a fuzzy ideal of R. We consider the H_v-ring $(R,\uplus,\otimes)$ defined in Example 2.1. Then μ is a left fuzzy H_v-ideal of $(R,\uplus,\otimes)$.

The concept of intuitionistic fuzzy set was introduced by Atanassov [1] as a generalization of the notion of fuzzy set. Some fundamental operations on intuitionistic fuzzy sets are defined by Atanassov in [2]. In [9], Kim et al. introduced the notion of an intuitionistic fuzzy subquasigroup of a quasigroup. Also in [10], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of semirings.

Defintion 2.4. An *intuitionistic fuzzy set* A of a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}, \tag{2.4}$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\lambda_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Defintion 2.5. For every two intuitionistic fuzzy sets A and B, define the following operations:

(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
(2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$,
(3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$,
(4) $A \cup B = \{x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$,
(5) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) \mid x \in X\}$,
(6) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) \mid x \in X\}$.

For the sake of simplicity, we will use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x) \mid x \in X\}$.

Defintion 2.6. Let $(R, +, \cdot)$ be an ordinary ring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in R is called a *left* (resp., *right*) *intuitionistic fuzzy ideal* of R if

(1) $\min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(x - y)$ for all $x, y \in R$,
(2) $\mu_A(y) \leq \mu_A(x \cdot y)$ (resp., $\mu_A(x) \leq \mu_A(x \cdot y)$) for all $x, y \in R$,
(3) $\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
(4) $\lambda_A(x \cdot y) \leq \lambda_A(y)$ (resp., $\lambda_A(x \cdot y) \leq \lambda_A(x)$) for all $x, y \in R$.

3. Intuitionistic fuzzy H_v-ideals

In what follows, let R denote an H_v-ring, and we start by defining the notion of intuitionistic fuzzy H_v-ideals.

Defintion 3.1. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in R is called a *left* (resp., *right*) *intuitionistic fuzzy H_v-ideal* of R if

(1) $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) \mid z \in x + y\}$ for all $x, y \in R$,
(2) for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\min\{\mu_A(a), \mu_A(x)\} \leq \min\{\mu_A(y), \mu_A(z)\}, \tag{3.1}$$

(3) $\mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$ (resp., $\mu_A(x) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$) for all $x, y \in R$,
(4) $\sup\{\lambda_A(z) \mid z \in x + y\} \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
(5) for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda_A(a), \lambda_A(x)\}, \tag{3.2}$$

(6) $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(y)$ (resp., $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(x)$) for all $x, y \in R$.

Example 3.2. Let μ be a left fuzzy H_v-ideal of $(R,\uplus,\otimes)$ defined in Example 2.3. Then, as it is not difficult to see, $A=(\mu_A,\mu_A^c)$ is a left intuitionistic fuzzy H_v-ideal of $(R,\uplus,\otimes)$.

Here we present all the proofs for left H_v-ideals. For right H_v-ideals, similar results hold as well.

Lemma 3.3. *If $A=(\mu_A,\lambda_A)$ is a left intuitionistic fuzzy H_v-ideal of R, then so is $\Box A=(\mu_A,\mu_A^c)$.*

Proof. It is sufficient to show that μ_A^c satisfies the conditions (4), (5), (6) of Definition 3.1. For $x,y\in R$, we have

$$\min\{\mu_A(x),\mu_A(y)\}\le\inf\{\mu_A(z)\mid z\in x+y\},\tag{3.3}$$

and so

$$\min\{1-\mu_A^c(x),1-\mu_A^c(y)\}\le\inf\{1-\mu_A^c(z)\mid z\in x+y\}.\tag{3.4}$$

Hence

$$\min\{1-\mu_A^c(x),1-\mu_A^c(y)\}\le 1-\sup\{\mu_A^c(z)\mid z\in x+y\},\tag{3.5}$$

which implies that

$$\sup\{\mu_A^c(z)\mid z\in x+y\}\le 1-\min\{1-\mu_A^c(x),1-\mu_A^c(y)\}.\tag{3.6}$$

Therefore

$$\sup\{\mu_A^c(z)\mid z\in x+y\}\le\max\{\mu_A^c(x),\mu_A^c(y)\},\tag{3.7}$$

in this way, Definition 3.1(4) is verified.

Now, let $a,x\in R$, then there exist $y,z\in R$ such that $x\in(a+y)\cap(z+a)$ and

$$\min\{\mu_A(a),\mu_A(x)\}\le\min\{\mu_A(y),\mu_A(z)\}.\tag{3.8}$$

So

$$\min\{1-\mu_A^c(a),1-\mu_A^c(x)\}\le\min\{1-\mu_A^c(y),1-\mu_A^c(z)\}.\tag{3.9}$$

Hence

$$\max\{\mu_A^c(y),\mu_A^c(z)\}\le\max\{\mu_A^c(a),\mu_A^c(x)\},\tag{3.10}$$

and Definition 3.1(5) is satisfied.

For the condition (6), let $x,y\in R$, then since μ_A is a left fuzzy H_v-ideal of R, we have

$$\mu_A(y)\le\inf\{\mu_A(z)\mid z\in x\cdot y\},\tag{3.11}$$

and so

$$1-\mu_A^c(y)\le\inf\{1-\mu_A^c(z)\mid z\in x\cdot y\},\tag{3.12}$$

which implies that

$$\sup\{\mu_A^c(z) \mid z \in x \cdot y\} \leq \mu_A^c(y). \tag{3.13}$$

Therefore Definition 3.1(6) is satisfied. □

Lemma 3.4. *If $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v-ideal of R, then so is $\Diamond A = (\lambda_A^c, \lambda_A)$.*

The proof is similar to the proof of Lemma 3.3.

Combining the above two lemmas, it is not difficult to see that the following theorem is valid.

Theorem 3.5. *$A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v-ideal of R if and only if $\Box A$ and $\Diamond A$ are left intuitionistic fuzzy H_v-ideals of R.*

Corollary 3.6. *$A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v-ideal of R if and only if μ_A and λ_A^c are left fuzzy H_v-ideals of R.*

Defintion 3.7. For any $t \in [0,1]$ and fuzzy set μ of R, the set

$$U(\mu;t) = \{x \in R \mid \mu(x) \geq t\} \quad (\text{resp., } L(\mu;t) = \{x \in R \mid \mu(x) \leq t\}) \tag{3.14}$$

is called an *upper* (resp., *lower*) *t-level cut* of μ.

Theorem 3.8. *If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of R, then for every $t \in \operatorname{Im}(\mu_A) \cap \operatorname{Im}(\lambda_A)$, the sets $U(\mu_A;t)$ and $L(\lambda_A;t)$ are H_v-ideals of R.*

Proof. Let $t \in \operatorname{Im}(\mu_A) \cap \operatorname{Im}(\lambda_A) \subseteq [0,1]$ and let $x, y \in U(\mu_A;t)$. Then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$, and so $\min\{\mu_A(x), \mu_A(y)\} \geq t$. It follows from Definition 3.1(1) that $\inf\{\mu_A(z) \mid z \in x + y\} \geq t$. Therefore for all $z \in x + y$, we have $z \in U(\mu_A;t)$, so $x + y \subseteq U(\mu_A;t)$. Hence for all $a \in U(\mu_A;t)$, we have $a + U(\mu_A;t) \subseteq U(\mu_A;t)$ and $U(\mu_A;t) + a \subseteq U(\mu_A;t)$. Now, let $x \in U(\mu_A;t)$, then there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \min\{\mu(y), \mu(z)\}$. Since $x, a \in U(\mu_A;t)$, we have $t \leq \min\{\mu_A(x), \mu_A(a)\}$, and so $t \leq \min\{\mu_A(y), \mu_A(z)\}$, which implies that $y \in U(\mu_A;t)$ and $z \in U(\mu_A;t)$. This proves that $U(\mu_A;t) \subseteq a + U(\mu_A;t)$ and $U(\mu_A;t) \subseteq U(\mu_A;t) + a$.

Now, for every $x \in R$ and $y \in U(\mu_A;t)$, we show that $x \cdot y \subseteq U(\mu_A;t)$. Since A is a left intuitionistic fuzzy H_v-ideal of R, we have

$$t \leq \mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}. \tag{3.15}$$

Therefore, for every $z \in x \cdot y$, we get $\mu_A(z) \geq t$, which implies that $z \in U(\mu_A;t)$, so $x \cdot y \subseteq U(\mu_A;t)$.

If $x, y \in L(\lambda_A;t)$, then $\max\{\lambda_A(x), \lambda_A(y)\} \leq t$. It follows from Definition 3.1(4) that $\sup\{\lambda_A(z) \mid z \in x + y\} \leq t$. Therefore for all $z \in x + y$, we have $z \in L(\lambda_A;t)$, so $x + y \subseteq L(\lambda_A;t)$. Hence for all $a \in L(\lambda_A;t)$, we have $a + L(\lambda_A;t) \subseteq L(\lambda_A;t)$ and $L(\lambda_A;t) + a \subseteq L(\lambda_A;t)$. Now, let $x \in L(\lambda_A;t)$, then there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and $\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda(a), \lambda(x)\}$. Since $x, a \in L(\lambda_A;t)$, we have $\max\{\lambda_A(a), \lambda_A(x)\} \leq t$, and so $\max\{\lambda_A(y), \lambda_A(z)\} \leq t$. Thus $y \in L(\lambda_A;t)$ and $z \in L(\lambda_A;t)$. Hence $L(\lambda_A;t) \subseteq a + L(\lambda_A;t)$ and $L(\lambda_A;t) \subseteq L(\lambda_A;t) + a$.

Now, we show that $x \cdot y \subseteq L(\lambda_A;t)$ for every $x \in R$ and $y \in L(\lambda_A;t)$. Since A is a left intuitionistic fuzzy H_v-ideal of R, we have

$$\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(y) \leq t. \tag{3.16}$$

Therefore, for every $z \in x \cdot y$, we get $\lambda_A(z) \leq t$, which implies that $z \in L(\lambda_A;t)$, so $x \cdot y \subseteq L(\lambda_A;t)$. □

Theorem 3.9. *If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set of R such that all nonempty levels $U(\mu_A;t)$ and $L(\lambda_A;t)$ are H_v-ideals of R, then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v-ideal of R.*

Proof. Assume that all nonempty levels $U(\mu_A;t)$ and $L(\lambda_A;t)$ are H_v-ideals of R. If $t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and $t_1 = \max\{\lambda_A(x), \lambda_A(y)\}$ for $x, y \in R$, then $x, y \in U(\mu_A;t_0)$ and $x, y \in L(\lambda_A;t_1)$. So $x + y \subseteq U(\mu_A;t_0)$ and $x + y \subseteq L(\lambda_A;t_1)$. Therefore for all $z \in x + y$, we have $\mu_A(z) \geq t_0$ and $\lambda_A(z) \leq t_1$, that is,

$$\begin{aligned} \inf\{\mu_A(z) \mid z \in x + y\} &\geq \min\{\mu_A(x), \mu_A(y)\}, \\ \sup\{\lambda_A(z) \mid z \in x + y\} &\leq \max\{\lambda_A(x), \lambda_A(y)\}, \end{aligned} \tag{3.17}$$

which verifies the conditions (1) and (4) of Definition 3.1.

Now, if $t_2 = \min\{\mu_A(a), \mu_A(x)\}$ for $x, a \in R$, then $a, x \in U(\mu_A;t_2)$. So there exist $y_1, z_1 \in U(\mu_A;t_2)$ such that $x \in a + y_1$ and $x \in z_1 + a$. Also we have $t_2 \leq \min\{\mu_A(y_1), \mu_A(z_1)\}$. Therefore, Definition 3.1(2) is verified. If we put $t_3 = \max\{\lambda_A(a), \lambda_A(x)\}$, then $a, x \in L(\lambda_A;t_3)$. So there exist $y_2, z_2 \in L(\lambda_A;t_3)$ such that $x \in a + y_2$ and $x \in z_2 + a$, and we have $\max\{\lambda_A(y_2), \lambda_A(y_2)\} \leq t_3$, and so Definition 3.1(5) is verified.

Now, we verify the conditions (3) and (6). Let $t_4 = \mu_A(y)$ and $t_5 = \lambda_A(y)$ for some $x, y \in R$. Then $y \in U(\mu_A;t_4)$, $y \in L(\lambda_A, t_5)$. Since $U(\mu_a;t_4)$ and $L(\lambda_A, t_5)$ are H_v-ideals of R, then $x \cdot y \subseteq U(\mu_A;t_4)$ and $x \cdot y \in L(\lambda_A, t_5)$. Therefore for every $z \in x \cdot y$, we have $z \in U(\mu_A;t_4)$ and $z \in L(\lambda_A, t_5)$ which imply that $\mu_A(z) \geq t_4$ and $\lambda_A(z) \leq t_5$. Hence

$$\begin{aligned} \inf\{\mu_A(z) \mid z \in x \cdot y\} &\geq t_4 = \mu_A(y), \\ \sup\{\lambda_A(z) \mid z \in x \cdot y\} &\leq t_5 = \lambda_A(y). \end{aligned} \tag{3.18}$$

This completes the proof. □

Corollary 3.10. *Let I be a left H_v-ideal of an H_v-ring R. If fuzzy sets μ and λ are defined on R by*

$$\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1 & \text{if } x \in R \setminus I, \end{cases} \qquad \lambda(x) = \begin{cases} \beta_0 & \text{if } x \in I, \\ \beta_1 & \text{if } x \in R \setminus I, \end{cases} \tag{3.19}$$

where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$, then $A = (\mu, \lambda)$ is an intuitionistic fuzzy H_v-ideal of R and $U(\mu;\alpha_0) = I = L(\lambda;\beta_0)$.

Corollary 3.11. *Let χ_S be the characteristic function of a left H_v-ideal I of R. Then $A = (\chi_I, \chi_I^c)$ is a left intuitionistic fuzzy H_v-ideal of R.*

Theorem 3.12. *If $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v-ideal of R, then for all $x \in R$,*

$$\begin{aligned} \mu_A(x) &= \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}, \\ \lambda_A(x) &= \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\}. \end{aligned} \tag{3.20}$$

Proof. Let $\delta = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A;\alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \le \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \le \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A;\beta)$, and so

$$\beta \in \{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}. \tag{3.21}$$

Hence

$$\mu_A(x) = \beta \le \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\} = \delta. \tag{3.22}$$

Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}. \tag{3.23}$$

Now let $\eta = \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\}$. Then

$$\inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\} < \eta + \varepsilon \tag{3.24}$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in L(\lambda_A;\alpha)$. Since $\lambda_A(x) \le \alpha$ and ε is arbitrary, it follows that $\lambda_A(x) \le \eta$.

To prove that $\lambda_A(x) \ge \eta$, let $\lambda_A(x) = \zeta$. Then $x \in L(\lambda_A;\zeta)$, and thus $\zeta \in \{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\}$. Hence

$$\inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\} \le \zeta, \tag{3.25}$$

that is, $\eta \le \zeta = \lambda_A(x)$. Consequently

$$\lambda_A(x) = \eta = \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\}, \tag{3.26}$$

which completes the proof. □

4. Relations

Let $\alpha \in [0,1]$ be fixed and let IF(R) be the family of all intuitionistic fuzzy left H_v-ideals of an H_v-ring R. For any $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from IF(R), we define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on IF(R) as follows:

$$\begin{aligned} (A,B) \in \mathfrak{U}^\alpha &\Longleftrightarrow U(\mu_A;\alpha) = U(\mu_B;\alpha), \\ (A,B) \in \mathfrak{L}^\alpha &\Longleftrightarrow L(\lambda_A;\alpha) = L(\lambda_B;\alpha). \end{aligned} \tag{4.1}$$

These two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations. Hence IF(R) can be divided into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = (\mu_A, \lambda_A) \in \mathrm{IF}(R)$, respectively. The corresponding quotient sets will be denoted as $\mathrm{IF}(R)/\mathfrak{U}^\alpha$ and $\mathrm{IF}(R)/\mathfrak{L}^\alpha$, respectively.

For the family $LI(R)$ of all left H_v-ideals of R, we define two maps U_α and L_α from $\mathrm{IF}(R)$ to $LI(R) \cup \{\varnothing\}$ putting

$$U_\alpha(A) = U(\mu_A;\alpha), \qquad L_\alpha(A) = L(\lambda_A;\alpha) \tag{4.2}$$

for each $A = (\mu_A, \lambda_A) \in \mathrm{IF}(R)$.

It is not difficult to see that these maps are well defined.

Lemma 4.1. *For any $\alpha \in (0,1)$, the maps U_α and L_α are surjective.*

Proof. Let $\mathbf{0}$ and $\mathbf{1}$ be fuzzy sets on R defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in R$. Then $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathrm{IF}(R)$ and $U_\alpha(\mathbf{0}_\sim) = L_\alpha(\mathbf{0}_\sim) = \varnothing$ for any $\alpha \in (0,1)$. Moreover, for any $K \in LI(R)$, we have $I_\sim = (\chi_K, \chi_K^c) \in \mathrm{IF}(R)$, $U_\alpha(I_\sim) = U(\chi_K;\alpha) = K$, and $L_\alpha(I_\sim) = L(\chi_K^c;\alpha) = K$. Hence U_α and L_α are surjective. □

Theorem 4.2. *For any $\alpha \in (0,1)$, the sets $\mathrm{IF}(R)/\mathfrak{U}^\alpha$ and $\mathrm{IF}(R)/\mathfrak{L}^\alpha$ are equipotent to $LI(R) \cup \{\varnothing\}$.*

Proof. Let $\alpha \in (0,1)$. Putting $U_\alpha^*([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A)$ and $L_\alpha^*([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A)$ for any $A = (\mu_A, \lambda_A) \in \mathrm{IF}(R)$, we obtain two maps:

$$U_\alpha^* : \mathrm{IF}(R)/\mathfrak{U}^\alpha \longrightarrow LI(R) \cup \{\varnothing\}, \qquad L_\alpha^* : \mathrm{IF}(R)/\mathfrak{L}^\alpha \longrightarrow LI(R) \cup \{\varnothing\}. \tag{4.3}$$

If $U(\mu_A;\alpha) = U(\mu_B;\alpha)$ and $L(\lambda_A;\alpha) = L(\lambda_B;\alpha)$ for some $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from $\mathrm{IF}(R)$, then $(A,B) \in \mathfrak{U}^\alpha$ and $(A,B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$, which means that $U*_\alpha$ and L_α^* are injective.

To show that the maps U_α^* and L_α are surjective, let $K \in LI(R)$. Then for $I_\sim = (\chi_K, \chi_K^c) \in \mathrm{IF}(R)$, we have $U_\alpha^*([I_\sim]_{\mathfrak{U}^\alpha}) = U(\chi_K;\alpha) = K$ and $L_\alpha^*([I_\sim]_{\mathfrak{L}^\alpha}) = L(\chi_K^c;\alpha) = K$. Also $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathrm{IF}(R)$. Moreover, $U_\alpha^*([\mathbf{0}_\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0};\alpha) = \varnothing$ and $L_\alpha^*([\mathbf{0}_\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1};\alpha) = \varnothing$. Hence U_α^* and L_α^* are surjective. □

Now for any $\alpha \in [0,1]$, we have the new relation $\mathfrak{R}^\alpha$ on $\mathrm{IF}(R)$ putting

$$(A,B) \in \mathfrak{R}^\alpha \Longleftrightarrow U(\mu_A;\alpha) \cap L(\lambda_A;\alpha) = U(\mu_B;\alpha) \cap L(\lambda_B;\alpha), \tag{4.4}$$

where $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$. Obviously, $\mathfrak{R}^\alpha$ is an equivalence relation.

Lemma 4.3. *The map $I_\alpha : \mathrm{IF}(R) \to LI(R) \cup \{\varnothing\}$ defined by*

$$I_\alpha(A) = U(\mu_A;\alpha) \cap L(\lambda_A;\alpha), \tag{4.5}$$

where $A = (\mu_A, \lambda_A)$, is surjective for any $\alpha \in (0,1)$.

Proof. Indeed, if $\alpha \in (0,1)$ is fixed, then for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathrm{IF}(R)$, we have

$$I_\alpha(\mathbf{0}_\sim) = U(\mathbf{0};\alpha) \cap L(\mathbf{1};\alpha) = \varnothing, \tag{4.6}$$

and for any $K \in LI(R)$, there exists $I_\sim = (\chi_K, \chi_K^c) \in \mathrm{IF}(R)$ such that $I_\alpha(I_\sim) = U(\chi_K;\alpha) \cap L(\chi_K^c;\alpha) = K$. □

Theorem 4.4. *For any $\alpha \in (0,1)$, the quotient set $\mathrm{IF}(R)/\mathfrak{R}^\alpha$ is equipotent to $LI(R) \cup \{\varnothing\}$.*

Proof. Let $I_\alpha^* : \mathrm{IF}(R)/\mathfrak{R}^\alpha \to LI(R) \cup \{\varnothing\}$, where $\alpha \in (0,1)$, be defined by the formula

$$I_\alpha^*([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \quad \text{for each } [A]_{\mathfrak{R}^\alpha} \in \mathrm{IF}(R)/\mathfrak{R}^\alpha. \tag{4.7}$$

If $I_\alpha^*([A]_{\mathfrak{R}^\alpha}) = I_\alpha^*([B]_{\mathfrak{R}^\alpha})$ for some $[A]_{\mathfrak{R}^\alpha}, [B]_{\mathfrak{R}^\alpha} \in \mathrm{IF}(R)/\mathfrak{R}^\alpha$, then

$$U(\mu_A;\alpha) \cap L(\lambda_A;\alpha) = U(\mu_B;\alpha) \cap L(\lambda_B;\alpha), \tag{4.8}$$

which implies that $(A,B) \in \mathfrak{R}^\alpha$ and, in the consequence, $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. Thus I_α^* is injective.

It is also onto because $I_\alpha^*(\mathbf{0}_\sim) = I_\alpha(\mathbf{0}_\sim) = \varnothing$ for $\mathbf{0}_\sim = (\mathbf{0},\mathbf{1}) \in \mathrm{IF}(R)$, and $I_\alpha^*(I_\sim) = I_\alpha(K) = K$ for $K \in LI(R)$ and $I_\sim = (\chi_K, \chi_K^c) \in \mathrm{IF}(R)$. □

The relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* is a ring. γ^* is called the fundamental equivalence relation on R and R/γ^* is called the fundamental ring, see [16].

According to [16], if $\mathcal{U}$ denotes the set of all finite polynomials of elements of R over $\mathbb{N}$, then a relation γ can be defined on R as follows:

$$x\gamma y \quad \text{iff } \{x,y\} \subseteq u \text{ for some } u \in \mathcal{U}. \tag{4.9}$$

According to [16], the transitive closure of γ is the fundamental relation γ^*, that is, $a\gamma^* b$ if and only if there exist $x_1,\dots,x_{m+1} \in R$; $u_1,\dots,u_m \in \mathcal{U}$ with $x_1 = a$, $x_{m+1} = b$ such that

$$\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1,\dots,m. \tag{4.10}$$

Suppose that $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then both the sum $\oplus$ and the product $\odot$ on R/γ^* are defined as follows:

$$\begin{aligned}\gamma^*(a) \oplus \gamma^*(b) &= \gamma^*(c) \quad \forall c \in \gamma^*(a) + \gamma^*(b),\\ \gamma^*(a) \odot \gamma^*(b) &= \gamma^*(d) \quad \forall c \in \gamma^*(a) \cdot \gamma^*(b).\end{aligned} \tag{4.11}$$

We denote the unit of the group $(R/\gamma^*, \oplus)$ by ω_R.

Defintion 4.5. Let R be an H_v-ring and μ a fuzzy subset of R. The fuzzy subset μ_{γ^*} on R/γ^* is defined as follows:

$$\begin{aligned}\mu_{\gamma^*} &: R/\gamma^* \longrightarrow [0,1],\\ \mu_{\gamma^*}(\gamma^*(x)) &= \sup\{\mu(a) \mid a \in \gamma^*(x)\}.\end{aligned} \tag{4.12}$$

Theorem 4.6. *Let R be an H_v-ring and $A = (\mu_A, \lambda_A)$ a left intuitionistic fuzzy H_v-ideal of R. Then $A/\gamma^* = (\mu_{\gamma^*}, \lambda_{\gamma^*})$ is a left intuitionistic fuzzy ideal of the fundamental ring R/γ^*.*

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