ON THE ASYMPTOTICS OF THE REAL SOLUTIONS TO THE GENERAL SIXTH PAINLEVÉ EQUATION

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Received 9 May 2006; Revised 3 September 2006; Accepted 1 October 2006

We study the general sixth Painlevé equation, develop, and justify the existence of several groups of asymptotics of its real solutions. Our methods also justify the differentiability of the asymptotics. Particular attention is paid to the solutions between 0 and 1. We find the asymptotics of all real solutions between 0 and 1 of the sixth Painlevé equation as $x \to +\infty$.

1. Introduction

The mathematical and physical significance of the six Painlevé transcendents has been well established. In the last 20 to 30 years, many mathematicians have spent dramatic effort on studying the properties of these transcendents. Although it is the most complicated one among the six Painlevé equations, there have been many results about the sixth Painlevé transcendent. In fact, the asymptotics problem of the sixth Painlevé transcendent has been studied in many papers such as [1, 2, 4–7, 9–12], and the connection problem is also studied in the papers [1, 4–7, 10, 11]. In this paper, we study the general sixth Painlevé equation

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{2}\left(\frac{dy}{dx}\right)^2\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}\\ &\quad+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\frac{\beta x}{y^2}+\frac{\gamma(x-1)}{(y-1)^2}+\frac{\delta x(x-1)}{(y-x)^2}\right),\end{aligned} \tag{PVI}$$

where α, β, γ, and δ are parameters. Heuristically, if y is a "small" solution of (PVI), the following equation truncated from (PVI) would be its "major" part as $x \to +\infty$:

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2\left(\frac{1}{y}+\frac{1}{y-1}\right)-\frac{1}{x}\frac{dy}{dx}-\frac{\beta(y-1)}{x^2y}-\frac{\gamma y}{x^2(y-1)}. \tag{1.1}$$

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 69562, Pages 1–10
DOI 10.1155/IJMMS/2006/69562

Letting $t = \ln x$ and applying elementary techniques to (1.1), one may solve it to get three different solutions:

$$y = A + B\sin(a\ln x + b), \tag{1.2}$$

where $A = 1/2 + (\beta+\gamma)/a^2$ and $B^2 = (1/2 + (\beta+\gamma)/a^2)^2 - 2\beta/a^2$;

$$y = \frac{b}{2}x^a + \frac{B^2}{2b}x^{-a} + A, \tag{1.3}$$

where $A = 1/2 - (\beta+\gamma)/a^2$ and $B^2 = 2\beta/a^2 + (1/2 - (\beta+\gamma)/a^2)^2$; and

$$y = \frac{\beta+\gamma}{2}(\ln x + b)^2 + \frac{\beta}{\beta+\gamma}. \tag{1.4}$$

It is reasonable to expect that some solutions of (PVI) take (1.2), (1.3), or (1.4) as their asymptotics. Indeed, many authors [4–7, 9–12] have obtained the corresponding asymptotics as $x \to 0$ that can be used to obtain (1.2), (1.3), and (1.4) by applying the well-known symmetry transformations. In this paper, we will prove the following theorems. The differences of our results are pointed out following each theorem.

Theorem 1.1. *Let $\beta > 0$ and $\gamma < 0$. If $x_0 > 1$, $0 < y_0 < 1$, and y is a solution of (PVI) with $y(x_0) = y_0$, then $0 < y < 1$ for all $x > x_0$ and it satisfies, as $x \to +\infty$,*

$$\begin{aligned} y &= A + B\cos(a\ln x + b) + O(x^{-1}), \\ y' &= -aBx^{-1}\sin(a\ln x + b) + O(x^{-2}), \end{aligned} \tag{1.5}$$

where $A = 1/2 + (\beta+\gamma)/a^2$ and $B^2 = (1/2 + (\beta+\gamma)/a^2)^2 - 2\beta/a^2$.

It is clear that the parameters need to satisfy the condition $(1/2 + (\beta+\gamma)/a^2)^2 \geq 2\beta/a^2$. The complex form of the asymptotics as $x \to 0$ corresponding to this asymptotics has been obtained in many papers [7, 10], but our result provides the conditions on the coefficients of the equation for the real solutions to exist, together with the bound $0 < y < 1$. Our proof of this theorem is also elementary and simple.

Theorem 1.2. *Equation (PVI) has a group of solutions with the following asymptotics:*

$$\begin{aligned} y &= \frac{b}{2}x^a + A + O(x^{(3/2)a-1/2}), \\ y' &= \frac{ab}{2}x^{a-1} + O(x^{(3/2)a-3/2}), \quad \text{as } x \longrightarrow +\infty, \end{aligned} \tag{1.6}$$

where $0 < a < 1/4$ or $3/4 < a < 1$, and $A = 1/2 - (\beta+\gamma)/a^2$.

We have noticed that it makes more sense for this result to be true when $0 < a < 1$. But, it seems to be impossible to prove it using our method. This asymptotics is actually a well-known result [4–6, 9–12], but our result here also estimates the second term of the leading behavior of the solution as well as the differentiability of the asymptotics.

Theorem 1.3. *If $\gamma+\beta \neq 0$, then (PVI) has a group of solutions with the following asymptotics:*

$$\begin{gathered} y \sim \frac{\beta+\gamma}{2}(\ln x+b)^2+\frac{\beta}{\beta+\gamma}, \\ y' \sim (\beta+\gamma)x^{-1}(\ln x+b), \quad \text{as } x \longrightarrow +\infty. \end{gathered} \tag{1.7}$$

Various forms of this result occur in the literature. For example, in [7] Guzzetti has the following result for $x \to 0$:

$$y(x) \sim \begin{cases} x\left\{\dfrac{\theta_x^2-\theta_0^2}{4}\left[\ln x+\dfrac{4r+2\theta_0}{\theta_0^2-\theta_x^2}\right]^2+\dfrac{\theta_0^2}{\theta_0^2-\theta_x^2}\right\}, & \theta_0 \neq \pm\theta_x, \\ x(r \pm \theta_0 \ln x), & \theta_0 = \pm\theta_x. \end{cases} \tag{1.8}$$

The coefficients θ_0 and θ_x come from the isomonodromy deformation theory and r is a free complex parameter. Applying the symmetry transformations $y(x) = xz(t)$ and $x = t^{-1}$ to this result, our result in Theorem 1.3 can be obtained.

It is well known that the transformation $y = x/z$ transforms (PVI) to itself with $(\alpha,\beta,\gamma,\delta)$ changed to $(-\beta,-\alpha,1/2-\delta,1/2-\gamma)$. Hence, based on the previous theorems, one may easily obtain the following one.

Theorem 1.4. *Equation (PVI) has solutions with the following asymptotics:*

$$y \sim \frac{x}{A+B\sin(a\ln x+b)}, \quad \text{as } x \longrightarrow +\infty, \text{ provided that } \alpha < 0,\ \delta > \frac{1}{2}, \tag{1.9}$$

where $A = 1/2+(1-2\alpha-2\delta)/2a^2$ and $B^2 = (1/2+(1-2\alpha-2\delta)/2a^2)^2+2\alpha/a^2$, and

$$y \sim \frac{2x}{(1/2-\alpha-\delta)(\ln x+b)^2}, \quad \text{as } x \longrightarrow +\infty, \text{ provided that } \frac{1}{2} \neq \alpha+\delta. \tag{1.10}$$

2. Proof of Theorems 1.2 and 1.3

In this section, we use the classical successive approximation method to prove Theorem 1.2. The proof of Theorem 1.3 is similar. In fact, Shimomura [12] studied a more general nonlinear ordinary differential equation, applied the successive approximation method to it, and obtained the result as an application. We first denote the functions (1.2), (1.3), and (1.4) as y_0 and substitute $t = \ln x$ and $y = y_0 + y_1$ into (PVI). The new equation is

$$\begin{aligned}
&\frac{d^2 y_1}{dt^2} - \left(\frac{1}{y_0} + \frac{1}{y_0 - 1}\right)\frac{dy_0}{dt}\frac{dy_1}{dt} + \left(\frac{\beta}{y_0^2} - \frac{\gamma}{(y_0-1)^2}\right)y_1 + \frac{1}{2}\left(\frac{dy_0}{dt}\right)^2\left(\frac{1}{y_0^2} + \frac{1}{(y_0-1)^2}\right)y_1 \\
&\quad = G(y_0,y_1)y_1^2 + H(y_0,y_1)y_1\frac{dy_1}{dt} + K(y_0,y_1)\left(\frac{dy_1}{dt}\right)^2 + e^{-t}I(y_0,y_1)\frac{dy_1}{dt} + e^{-t}F(y_0,y_1,t),
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
G(y_0,y_1) &= \left(\frac{1}{2y_0^2(y_0+y_1)} + \frac{1}{2(y_0-1)^2(y_0+y_1-1)}\right)\left(\frac{dy_0}{dt}\right)^2 \\
&\quad + \frac{\beta}{y_0^2(y_0+y_1)} - \frac{\gamma}{(y_0-1)^2(y_0+y_1-1)}, \\
H(y_0,y_1) &= -\left(\frac{1}{y_0(y_0+y_1)} + \frac{1}{(y_0-1)(y_0+y_1-1)}\right)\frac{dy_0}{dt}, \\
K(y_0,y_1) &= \frac{1}{2}\left(\frac{1}{y_0+y_1} + \frac{1}{y_0+y_1-1}\right) + \frac{1}{2}\frac{e^{-t}}{((y_0+y_1)e^{-t}-1)^2}, \\
I(y_0,y_1) &= \frac{1}{((y_0+y_1)e^{-t}-1)^2}\frac{dy_0}{dt} - \frac{1}{((y_0+y_1)e^{-t}-1)(1-e^{-t})}, \\
F(y_0,y_1,t) &= \frac{1}{2}\frac{1}{((y_0+y_1)e^{-t}-1)^2}\left(\frac{dy_0}{dt}\right)^2 - \frac{1}{((y_0+y_1)e^{-t}-1)(1-e^{-t})}\frac{dy_0}{dt} \\
&\quad + \frac{\alpha(y_0+y_1)(y_0+y_1-1)((y_0+y_1)e^{-t}-1)}{(1-e^{-t})^2} + \frac{\beta(y_0+y_1-1)(y_0+y_1-2+e^{-t})}{(y_0+y_1)(1-e^{-t})^2} \\
&\quad + \frac{\gamma(y_0+y_1)}{1-e^{-t}} \pm \frac{\delta(y_0+y_1)(y_0+y_1-1)}{(1-e^{-t})((y_0+y_1)e^{-t}-1)}.
\end{aligned} \tag{2.2}$$

As a routine, we introduce a new function z by using the standard transformation

$$y_1 = \sqrt{y_0(y_0-1)}\,z. \tag{2.3}$$

Now, (2.1) is changed to

$$\begin{aligned}
&\frac{d^2 z}{dt^2} + \left(\frac{\beta(3/2-y_0)}{y_0^2} - \frac{\gamma(y_0+1/2)}{(y_0-1)^2}\right)z \\
&\quad = M(y_0,y_1)z^2 + N(y_0,y_1)z\frac{dz}{dt} + P(y_0,y_1)\left(\frac{dz}{dt}\right)^2 \\
&\qquad + e^{-t}Q(y_0,y_1)z + e^{-t}I(y_0,y_1)\frac{dz}{dt} + \frac{e^{-t}}{\sqrt{y_0(y_0-1)}}F(y_0,y_1,t),
\end{aligned} \tag{2.4}$$

where

$$
\begin{aligned}
M(y_0,y_1) &= \sqrt{y_0(y_0-1)}G(y_0,y_1) + \frac{2y_0-1}{2\sqrt{y_0(y_0-1)}}H(y_0,y_1) + \frac{(2y_0-1)^2}{4y_0^{3/2}(y_0-1)^{3/2}}K(y_0,y_1),\\
N(y_0,y_1) &= \sqrt{y_0(y_0-1)}H(y_0,y_1) + \frac{2y_0-1}{\sqrt{y_0(y_0-1)}}K(y_0,y_1),\\
P(y_0,y_1) &= \sqrt{y_0(y_0-1)}K(y_0,y_1),\\
Q(y_0,y_1) &= \frac{2y_0-1}{2y_0(y_0-1)}\frac{dy_0}{dt}I(y_0,y_1).
\end{aligned}
\tag{2.5}
$$

To prove Theorem 1.2, we assume that y_0 takes the function in (1.3). Then, there exist constants L and t_0 such that, for $|a| < 1$, $t > t_0$, and $|y_1| \ll |y_0|$, the following estimates are true:

$$
\begin{aligned}
&|M(y_0,y_1)| < L, && |N(y_0,y_1)| < L,\\
&|P(y_0,y_1)| < L, && \left|\frac{F(y_0,y_1,t)}{\sqrt{y_0(y_0-1)}}\right| < e^{at}L,\\
&|Q(y_0,y_1)| < e^{at}L, && |I(y_0,y_1)| < e^{at}L.
\end{aligned}
\tag{2.6}
$$

We convert (2.4) into the following integral equation:

$$
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \int_t^{\infty} \begin{pmatrix} t-\tau \\ 1 \end{pmatrix} R(\tau,z_1(\tau),z_2(\tau))\,d\tau, \tag{2.7}
$$

where $z_1 = z$, $z_2 = z'$, and

$$
\begin{aligned}
R(\tau,z,z') = {}& \left(\frac{\gamma(y_0+1/2)}{(y_0-1)^2} - \frac{\beta(3/2-y_0)}{y_0^2}\right)z + M(y_0,y_1)z^2 + N(y_0,y_1)z\frac{dz}{dt} + P(y_0,y_1)\left(\frac{dz}{dt}\right)^2\\
&+ e^{-t}Q(y_0,y_1)z + e^{-t}I(y_0,y_1)\frac{dz}{dt} + \frac{e^{-t}}{\sqrt{y_0(y_0-1)}}F(y_0,y_1,t).
\end{aligned}
\tag{2.8}
$$

In order to apply the successive approximation method to (2.7), we rewrite it as

$$
Z(t) = L(t,Z(t)), \tag{2.9}
$$

where $Z(t) = \binom{z_1}{z_2}$ and $L(t,\tau,z)$ is the right-hand side of (2.7). Now, we can define the sequence

$$\begin{gathered} Z_{-1}(t) = 0, \\ Z_n(t) = L\big(t, Z_{n-1}(t)\big), \quad n = 0,1,2,\dots. \end{gathered} \tag{2.10}$$

We first take care of the case when $|a| < 1/4$. Let t_0 be large enough such that, when $t \geq t_0$,

$$\begin{gathered} \sqrt{y_0(y_0-1)}\,e^{-(1/2-a/2)t} < \frac{1}{2}, \qquad \frac{5L}{(1-a)^2}e^{-(1/2-a/2)t} < \frac{1}{6}, \\ 4e^{-at} < \frac{1}{6}, \qquad 8Le^{-at} < \frac{1}{6}, \qquad \frac{4L}{3(a-1)^2}e^{(a-1)t} < \frac{1}{6}, \\ \frac{2}{(1-a)^2}e^{-at} < \frac{1}{6}, \qquad \frac{8L}{9(a-1)^2}e^{(a-1)t} \leq \frac{1}{6}. \end{gathered} \tag{2.11}$$

Then,

$$\begin{aligned} \big|Z_1(t)\big| &\leq \left| \int_t^\infty \begin{pmatrix} \tau - t \\ 1 \end{pmatrix} \frac{e^{-\tau}}{\sqrt{y_0(y_0-1)}} F(y_0,0,\tau)\,d\tau \right| \\ &\leq \frac{L}{(1-a)^2}e^{-(1-a)t} \leq \frac{1}{2}e^{(a/2-1/2)t}, \quad \text{for } t \geq t_0, \\ \big|Z_1(t) - Z_0(t)\big| &\leq \frac{1}{2}e^{(a/2-1/2)t}, \quad \text{for } t \geq t_0. \end{aligned} \tag{2.12}$$

Assume that

$$\begin{gathered} \big|Z_n(t)\big| \leq \frac{1}{2}e^{(a/2-1/2)t}, \\ \big|Z_n(t) - Z_{n-1}(t)\big| \leq \left(\frac{1}{2}\right)^{n-1} e^{(a/2-1/2)t}, \quad \text{for } t \geq t_0. \end{gathered} \tag{2.13}$$

Then, for $t \geq t_0$,

$$\begin{aligned} \big|Z_{n+1}(t)\big| &\leq \int_t^\infty \begin{pmatrix} \tau - t \\ 1 \end{pmatrix} \left(5Le^{(a-1)\tau} + \frac{1}{2}e^{-((a+1)/2)\tau} + 2Le^{(3/2)(a-1)\tau}\right) d\tau \\ &\leq \frac{1}{2}e^{(a/2-1/2)t}. \end{aligned} \tag{2.14}$$

Using the mean value theorem, we can also get

$$\begin{aligned}|Z_{n+1}(t)-Z_n(t)| &\le \int_t^\infty \binom{\tau-t}{1}\left(e^{-a\tau}+2Le^{-a\tau}+3Le^{(a-1)\tau}\right)|Z_n(\tau)-Z_{n-1}(\tau)|\,d\tau \\ &\le \left(\frac{1}{2}\right)^{n-1}\int_t^\infty \binom{\tau-t}{1}\left(e^{-((a+1)/2)\tau}+2Le^{-((a+1)/2)\tau}+3Le^{(3/2)(a-1)\tau}\right)d\tau \\ &\le \left(\frac{1}{2}\right)^{n} e^{(a/2-1/2)t}.\end{aligned} \tag{2.15}$$

Therefore, the sequence $\{Z_n(t)\}$ converges uniformly to $Z(t)$ and

$$|Z(t)| \le \frac{1}{2}e^{(a/2-1/2)t} \quad \forall t \ge t_0. \tag{2.16}$$

Hence, we have proved that (PVI) has a solution satisfying

$$\begin{aligned} y &= \frac{b}{2}e^{at}+A+O\left(e^{((3/2)a-1/2)t}\right) = \frac{b}{2}x^a+A+O\left(x^{(3/2)a-1/2}\right), \\ y' &= \frac{ab}{2}x^{a-1}+O\left(x^{(3/2)a-3/2}\right), \quad \text{as } x \longrightarrow \infty,\ 0<a<\frac{1}{4}. \end{aligned} \tag{2.17}$$

Applying the transformation $y = x/z$ to (PVI) and using the result we have obtained, we can get the asymptotics for $3/4 < a < 1$ and finish the proof of Theorem 1.2.

3. Proof of Theorem 1.1

We can easily prove Theorems 1.2 and 1.3 using the successive approximation method since the corresponding homogeneous equation is easy to solve. When y_0 takes the expression in (1.2), the corresponding homogeneous equation to (2.4) becomes one of the famous Hill equations [3] whose solutions are very hard to analyze. Thus, we have difficulties to apply the successive approximation method to this case. Fortunately, we can manage to manipulate (PVI) a little bit and apply a method used by Hastings and McLeod [8] to it.

We first prove the first part of the theorem. Suppose that $y(x_1) = 0$ for some $x_1 > x_0$. Since $y(x)$ is analytic near x_1, we have the expansion

$$y(x) = c(x-x_1)^n + O\left((x-x_1)^{n+1}\right), \tag{3.1}$$

where $c \ne 0$ and $n > 0$. Substituting (3.1) into (PVI), we get the equation

$$\begin{aligned} &cn(n-1)(x-x_1)^{n-2}+O\left((x-x_1)^{n-1}\right) \\ &\quad = \frac{c}{2}n^2(x-x_1)^{n-2}+O\left((x-x_1)^{n-1}\right) \\ &\qquad + \frac{\beta}{c(x_1-1)^2}(x-x_1)^{-n}+O\left((x-x_1)^{-n+1}\right). \end{aligned} \tag{3.2}$$

Thus, we have $n = 1$ and $c/2 + \beta/c(x_1 - 1)^2 = 0$. This is impossible when $\beta > 0$ and therefore, $y(x) > 0$ for all $x > x_0$. Similarly, we can prove that $y(x) < 1$ for all $x > x_0$ when $\gamma < 0$. This result enables us to assume that y is a solution between 0 and 1 in this section. We first apply the transformation

$$t = \ln x \tag{3.3}$$

to (PVI) and obtain

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 + \frac{\beta(1-y)}{y} + \frac{\gamma y}{1-y}\\ &\quad + e^{-t}\left\{\frac{1}{2(ye^{-t}-1)}\left(\frac{dy}{dt}\right)^2 - \frac{y-1}{(1-e^{-t})(ye^{-t}-1)}\frac{dy}{dt} + \frac{\alpha y(y-1)(ye^{-t}-1)}{(1-e^{-t})^2}\right.\\ &\qquad \left. + \frac{\beta(y-1)(y-2+e^{-t})}{y(1-e^{-t})^2} + \frac{\gamma y}{1-e^{-t}} + \frac{\delta y(y-1)}{(1-e^{-t})(ye^{-t}-1)}\right\}.\end{aligned} \tag{3.4}$$

Since $0 < y < 1$, we can rewrite (3.4) into

$$\begin{aligned}&\frac{d}{dt}\left\{[y(1-y)]^{-1/2}\frac{dy}{dt}\right\} \cdot 2y^{-1/2}(1-y)^{-1/2}\frac{dy}{dt}\\ &\quad = \frac{2}{y(1-y)}\left[\frac{\beta(1-y)}{y} + \frac{\gamma y}{1-y}\right]\frac{dy}{dt} + \frac{2e^{-t}}{y(1-y)}Q(y,t)\frac{dy}{dt}.\end{aligned} \tag{3.5}$$

Integrating both sides of (3.5), we get

$$\frac{1}{y(1-y)}\left(\frac{dy}{dt}\right)^2 + \frac{2\beta}{y} - \frac{2\gamma}{1-y} = C + \frac{1}{y(1-y)}O(e^{-t}). \tag{3.6}$$

Since $\beta > 0$, $\gamma < 0$, and $2\beta/y - 2\gamma/(1-y)$ dominates $(1/y(1-y))O(e^{-t})$ when t is large, $C = a^2 > 0$, $1/y$, $1/(1-y)$, and dy/dt are all bounded as t goes to infinity. Multiplying both sides of (3.6) by $y(1-y)$ and letting $y = z + r$ where r is a constant to be determined later, we get

$$\left(\frac{dz}{dt}\right)^2 + a^2z^2 + (2ra^2 - 2\beta - 2\gamma - a^2)z + 2\beta - (2\beta + 2\gamma + a^2)r + a^2r^2 = O(e^{-t}). \tag{3.7}$$

We select $r = 1/2 + (\beta + \gamma)/a^2 = A$, then

$$\left(\frac{dz}{dt}\right)^2 + a^2z^2 = a^2B^2 + O(e^{-t}). \tag{3.8}$$

To solve (3.8), we let

$$\begin{aligned}z(t) &= \rho(t)\cos\phi(t),\\ z'(t) &= a\rho(t)\sin\phi(t).\end{aligned} \tag{3.9}$$

We are using two functions $\rho(t)$ and $\phi(t)$ in $z(t)$ and $z'(t)$ to describe the relationship of the function $z(t)$ and its derivative. Because of this relationship, one of $\rho(t)$ and $\phi(t)$ should be depending on another as the product rule of derivatives prescribes. Substituting (3.9) into (3.8), one first gets

$$\rho^2(t) = B^2 + O(e^{-t}). \tag{3.10}$$

Following (3.8), we may also get

$$\frac{d\phi}{dt} = -a + \frac{a(z''z + a^2z^2)}{\rho^2(t)} = -a + O(e^{-t}). \tag{3.11}$$

Integrating (3.11) and combining the result with (3.9) and (3.10), one gets

$$\begin{aligned} z(t) &= (B + O(e^{-t}))\cos(at + b + O(e^{-t})), \\ z'(t) &= a(B + O(e^{-t}))\sin(at + b + O(e^{-t})), \end{aligned} \tag{3.12}$$

and finishes the proof of Theorem 1.1.

Acknowledgment

The authors are deeply grateful to the referee for his/her valuable comments and suggestions.

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