EXISTENCE AND BIFURCATION FOR SOME ELLIPTIC PROBLEMS ON EXTERIOR STRIP DOMAINS

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Received 21 December 2005; Revised 18 July 2006; Accepted 19 July 2006

We consider the semilinear elliptic problem $-\Delta u + u = \lambda K(x)u^p + f(x)$ in Ω, $u > 0$ in Ω, $u \in H_0^1(\Omega)$, where $\lambda \geq 0$, $N \geq 3$, $1 < p < (N+2)/(N-2)$, and Ω is an exterior strip domain in $\mathbb{R}^N$. Under some suitable conditions on $K(x)$ and $f(x)$, we show that there exists a positive constant λ^* such that the above semilinear elliptic problem has at least two solutions if $\lambda \in (0,\lambda^*)$, a unique positive solution if $\lambda = \lambda^*$, and no solution if $\lambda > \lambda^*$. We also obtain some bifurcation results of the solutions at $\lambda = \lambda^*$.

1. Introduction

In this paper, we consider the semilinear elliptic problem

$$-\Delta u + u = \lambda K(x)u^p + f(x) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \qquad (1.1)_\lambda$$

where $\lambda \geq 0$, $N = m + n \geq 3$, $m \geq 2$, $n \geq 1$, $1 < p < (N+2)/(N-2)$, $\omega \subseteq \mathbb{R}^m$ is a bounded smooth domain, $\mathbb{S} = \omega \times \mathbb{R}^n$ is a strip domain, D is a bounded smooth domain in $\mathbb{R}^N$ such that $D \subset\subset \mathbb{S}$, $\Omega = \mathbb{S} \setminus \overline{D}$ is an exterior strip domain, $0 \not\equiv f(x) \geq 0$ in Ω, $f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 3$, and $K(x)$ is a positive, bounded, and continuous function on $\overline{\Omega}$. Moreover $K(x)$ satisfies the following conditions:

(k1) $\lim_{|z|\to\infty} K(y,z) = K_\infty > 0$ uniformly for $y \in \omega$;

(k2) there exist some constants $K_\infty > 0$, $\gamma > (n-1)/2$, and $\vartheta > 0$ such that .2

$$K(y,z) \geq K_\infty - \vartheta \exp\left(-\frac{p+1}{p}\sqrt{1+\mu_1}|z|\right)|z|^{-\gamma} \quad \text{as } |z| \longrightarrow \infty, \text{ uniformly for } y \in \omega, \qquad (1.2)$$

where μ_1 is the first eigenvalue of the Dirichlet problem $-\Delta$ in ω.

If Ω is bounded ($n = 0$ in our case), then $(1.1)_\lambda$ has been studied by many authors: see for instance Bahri and Lions [4] and the references therein. We only consider that Ω

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 73278, Pages 1–26
DOI 10.1155/IJMMS/2006/73278

is unbounded ($n \geq 1$ in our case). If Ω is an exterior domain ($m = 0$ in our case), Zhu and Zhou [18] and Zhu [17] established the existence of multiple positive solutions of equations with structure unlike that here. If Ω is an exterior strip domain, Hsu and Wang [12] have investigated the following problem:

$$-\Delta u + u = u^p + f(x) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \tag{1.3}$$

where $1 < p < (N+2)/(N-2)$ and $N \geq 4$. Hsu and Wang [12] have proved that (1.3) has at least two positive solutions if f is nonzero positive L^2 function with the L^2 norm small enough and the decay fast enough.

Throughout this paper, let $x = (y,z)$ be the generic point of $\mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $N = m+n \geq 3$, $m \geq 2$, $n \geq 1$, $1 < p < (N+2)/(N-2)$, $\mathbb{S}$ is a smooth strip domain in $\mathbb{R}^N$, Ω is a smooth exterior strip domain in $\mathbb{R}^N$, u_0 is the unique positive solution of $(1.1)_0$, and we denote by c and c_i ($i = 1,2,\ldots$) the universal constants, unless otherwise specified. We set

$$\begin{gathered}
\|u\| = \left(\int_\Omega (|\nabla u|^2 + |u|^2)\,dx\right)^{1/2}, \\
\|u\|_{L^q(\Omega)} = \left(\int_\Omega |u|^q dx\right)^{1/q}, \quad 2 \leq q < \infty, \\
\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)|, \\
M = \inf\left\{\int_{\mathbb{S}} (|\nabla u|^2 + |u|^2)\,dx : \int_{\mathbb{S}} |u|^{p+1} dx = 1\right\}.
\end{gathered} \tag{1.4}$$

Now, we state our main results in the following.

Theorem 1.1. *Suppose $f(x) \geq 0$, $f(x) \not\equiv 0$ in Ω, $f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 3$, $K(x)$ is a positive, bounded, and continuous function on $\overline{\Omega}$ and $K(x)$ satisfies conditions (k1) and (k2). Then there is a $\lambda^* > 0$, λ^* depending on K and f, such that*

(i) *equation $(1.1)_\lambda$ has at least two solutions u_λ, U_λ and $u_\lambda < U_\lambda$ if $\lambda \in (0,\lambda^*)$;*
(ii) *equation $(1.1)_{\lambda^*}$ has a unique solution u_{λ^*};*
(iii) *equation $(1.1)_\lambda$ has no positive solutions if $\lambda > \lambda^*$.*

Furthermore,

$$\begin{aligned}
\lambda_1 &\equiv \frac{(p+1)(p-1)^{p-1}M^{(p+1)/2}}{(2p)^p\|K\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}^{p-1}} \\
&\leq \lambda^* \leq \inf_{w\in H_0^1(\Omega)\setminus\{0\}}\left(\frac{\|w\|^2}{p\int_\Omega K u_0^{p-1} w^2 dx}\right) \equiv \lambda_2 \\
&\leq \frac{p\|f\|_{L^2(\Omega)}^2}{(p-1)^2\int_\Omega K u_0^{p+1} dx} \equiv \lambda_3,
\end{aligned} \tag{1.5}$$

where u_λ is the minimal solution of (1.1)$_\lambda$ and U_λ is the second solution of (1.1)$_\lambda$ constructed in Section 5.

Theorem 1.2. *Under the assumptions of Theorem 1.1,*

(i) *u_λ is strictly increasing with respect to λ, u_λ is uniformly bounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$ for all $\lambda\in[0,\lambda^*]$, and*

$$u_\lambda \longrightarrow u_0 \quad \text{in } L^\infty(\Omega)\cap H_0^1(\Omega) \text{ as } \lambda\longrightarrow 0^+; \tag{1.6}$$

(ii) *U_λ is unbounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$, that is,*

$$\lim_{\lambda\to 0^+}||U_\lambda|| = \lim_{\lambda\to 0^+}||U_\lambda||_{L^\infty(\Omega)} = \infty; \tag{1.7}$$

(iii) *moreover, assume that $K(x)$ and $f(x)$ are in $C^\alpha(\Omega)\cap L^2(\Omega)$, then all solutions of (1.1)$_\lambda$ are in $C^{2,\alpha}(\Omega)\cap H^2(\Omega)$, and $(\lambda^*,u_{\lambda^*})$ is a bifurcation point for (1.1)$_\lambda$, and*

$$u_\lambda \longrightarrow u_0 \quad \text{in } C^{2,\alpha}(\Omega)\cap H^2(\Omega) \text{ as } \lambda\longrightarrow 0^+. \tag{1.8}$$

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we assert that there exists a positive constant λ^*, depending on K and f, such that (1.1)$_\lambda$ has a minimal solution for $\lambda\in[0,\lambda^*]$. In Section 4, we establish several lemmas for the regularity and asymptotic behaviors of the solution of (1.1)$_\lambda$. In Section 5, we establish the existence of a second solution U_λ for $\lambda\in(0,\lambda^*)$. In Section 6, we analyize the set of solutions.

2. Preliminaries

In this section, we give some notations and some known results. In order to get the existence of positive solutions of (1.1)$_\lambda$, we consider the energy functional $I_\lambda : H_0^1(\Omega)\to\mathbb{R}$ defined by

$$I_\lambda(u) = \int_\Omega\left[\frac{1}{2}(|\nabla u|^2+|u|^2) - \frac{\lambda}{p+1}K(x)(u^+)^{p+1} - f(x)u\right]dx, \tag{2.1}$$

where $u^\pm(x)=\max\{\pm u(x),0\}$.

Then the critical points of I_λ are the positive solutions of (1.1)$_\lambda$.

Consider the equation

$$-\Delta u + u = \lambda K_\infty u^p \quad \text{in } \mathbb{S}, \quad u>0 \quad \text{in } \mathbb{S}, \quad u\in H_0^1(\mathbb{S}), \tag*{(2.1)$_\lambda$}$$

and its associated energy functional I_λ^∞ defined by

$$I_\lambda^\infty(u) = \int_{\mathbb{S}}\left[\frac{1}{2}(|\nabla u|^2+|u|^2) - \frac{\lambda}{p+1}K_\infty(u^+)^{p+1}\right]dx, \quad u\in H_0^1(\mathbb{S}). \tag{2.2}$$

By Esteban [8] and Lien et al. [14], $(2.1)_\lambda$ has a ground state solution $\overline{u}_\lambda$ such that

$$M_\lambda^\infty = I_\lambda^\infty(\overline{u}_\lambda) = \sup_{t\geq 0} I_\lambda^\infty(t\overline{u}_\lambda), \tag{2.3}$$

and we also have

$$M = \inf\left\{\int_{\mathbb{S}}(|\nabla u|^2+|u|^2) : \int_{\mathbb{S}}|u|^{p+1}=1\right\} = \inf\left\{\int_{\Omega}(|\nabla u|^2+|u|^2) : \int_{\Omega}|u|^{p+1}=1\right\}. \tag{2.4}$$

Now, we quote here a precise asymptotic behavior result of Hsu [10] for positive solutions of $(2.1)_\lambda$ at infinity.

Proposition 2.1. *Let u be a positive solution of $(2.1)_\lambda$ in an unbounded strip $\mathbb{S}=\omega\times\mathbb{R}^n \subseteq \mathbb{R}^{m+n}$, $m\geq 2$, $n\geq 1$, and let ψ be the first positive eigenfunction of the Dirichlet problem $-\Delta\psi=\mu_1\psi$ in ω, then for any $\varepsilon>0$, there exist constants c_ε, $\widetilde{c}_\varepsilon>0$ such that*

$$\begin{aligned} &u(x)\leq c_\varepsilon\psi(y)\exp\left(-\sqrt{1+\mu_1}|z|\right)|z|^{-(n-1)/2+\varepsilon} \\ &\widetilde{c}_\varepsilon\psi(y)\exp\left(-\sqrt{1+\mu_1}|z|\right)|z|^{-(n-1)/2-\varepsilon}\leq u(x) \end{aligned} \quad \text{as } |z|\longrightarrow\infty,\ y\in\varpi. \tag{2.5}$$

3. Existence of minimal solution

In this section, by the standard barrier method, we will establish the existence of minimal positive solution u_λ for all λ in some finite interval $[0,\lambda^*]$ (i.e, for any positive solution u of $(1.1)_\lambda$, then $u\geq u_\lambda$).

Lemma 3.1. *Let condition* (k1) *hold. Then $(1.1)_\lambda$ has a solution u_λ if $0\leq\lambda<\lambda_1$, where λ_1 is given by (1.5).*

Proof. For $\lambda=0$, the existence question is equivalent to the existence of $u_0\in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_0\cdot\nabla\phi+u_0\phi=\int_\Omega f\phi \tag{3.1}$$

for all $\phi\in H_0^1(\Omega)$ since

$$\left|\int_\Omega f\phi\right|\leq\|f\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}\leq\|f\|_{L^2(\Omega)}\|\phi\| \tag{3.2}$$

for all $\phi\in H_0^1(\Omega)$. According to the Lax-Milgram theorem, there exists a unique $u_0\in H_0^1(\Omega)$ that satisfies (3.1). Since $0\not\equiv f\geq 0$ in Ω, by strong maximum principle (see Gilbarg and Trudinger [9]), we conclude that $u_0>0$ in Ω.

We consider next the case $\lambda>0$. We show first that for sufficiently small λ, say $\lambda=\lambda_0$, there exists $t=t_0(\lambda_0)>0$ such that $I_{\lambda_0}(u)>0$ for $\|u\|=t_0$. From the definitions of I_λ and M we have for any $u\in H_0^1(\Omega)$,

$$I_\lambda(u)\geq\frac{1}{2}\|u\|^2-\frac{\lambda}{p+1}\|K\|_{L^\infty(\Omega)}M^{-(p+1)/2}\|u\|^{p+1}-\|f\|_{L^2(\Omega)}\|u\|. \tag{3.3}$$

Set

$$h(t)=\frac{1}{2}t-\lambda c_1 t^p-c_2, \tag{3.4}$$

where $c_1=(1/(p+1))\|K\|_{L^\infty(\Omega)}M^{-(p+1)/2}$ and $c_2=\|f\|_{L^2(\Omega)}$.

It then follows that $h(t)$ achieves a maximum at $t_\lambda=(2p\lambda c_1)^{-(p-1)^{-1}}$. Set $B_{t_\lambda}=\{u\in H_0^1(\Omega):\|u\|<t_\lambda\}$. Then for all $u\in\partial B_{t_\lambda}=\{u\in H_0^1(\Omega):\|u\|=t_\lambda\}$,

$$I_\lambda(u)\geq t_\lambda h(t_\lambda)\geq t_\lambda\left[\frac{t_\lambda(p-1)}{2p-c_2}\right]>0 \tag{3.5}$$

provided that $c_2<t_\lambda(p-1)/2p$, which is satisfied for $\lambda\in(0,\lambda_1)$. Fix a $\lambda_0\in(0,\lambda_1)$, and set $t_0=t_{\lambda_0}$. Let $0\not\equiv\phi\geq 0$, $\phi\in C_0^\infty(\Omega)$, such that $\int_\Omega f\phi dx>0$. Then

$$I_{\lambda_0}(t\phi)=\frac{t^2}{2}\|\phi\|^2-\frac{\lambda_0}{p+1}t^{p+1}\int_\Omega K\phi^{p+1}-t\int_\Omega f\phi<0 \tag{3.6}$$

for sufficiently small $t>0$, and it is easy to see that I_{λ_0} is bounded below on B_{t_0}. Set $\alpha_0=\inf\{I_{\lambda_0}(u)\mid u\in B_{t_0}\}$. Then $\alpha_0<0$, and since $I_{\lambda_0}(u)>0$ on ∂B_{t_0}, the continuity of I_{λ_0} on $H_0^1(\Omega)$ implies that there exists $0<t_1<t_0$ such that $I_{\lambda_0}(u)>\alpha_0$ for all $u\in H_0^1(\Omega)$ and $t_1\leq\|u\|\leq t_0$. By the Ekeland variational principle [7], there exists a sequence $\{u_k\}_{k=1}^\infty\subset B_{t_1}$ such that $I_{\lambda_0}(u_k)=\alpha_0+o(1)$ and $I'_{\lambda_0}(u_k)=o(1)$ strongly in $H^{-1}(\Omega)$, as $k\to\infty$. It is easy to see that $\{u_k\}$ is bounded in $H_0^1(\Omega)$. Hence, there exist a subsequence $\{u_k\}$ and $\overline{u}$ in $\overline{B}_{t_1}$ such that

$$\begin{gathered}u_k\rightharpoonup\overline{u}\quad\text{weakly in } H_0^1(\Omega),\\ u_k\longrightarrow\overline{u}\quad\text{strongly in } L^q_{\text{loc}}(\Omega)\text{ for }2\leq q<\frac{2N}{N-2}\quad\text{as }k\longrightarrow\infty\\ u_k\longrightarrow\overline{u}\quad\text{a.e. in }\Omega.\end{gathered} \tag{3.7}$$

For $\phi\in C_0^\infty(\Omega)$, we get

$$\int_\Omega\nabla u_k\cdot\nabla\phi\longrightarrow\int_\Omega\nabla\overline{u}\cdot\nabla\phi,\qquad\int_\Omega u_k\phi\longrightarrow\int_\Omega\overline{u}\phi,\qquad\int_\Omega K(u_k^+)^p\phi\longrightarrow\int_\Omega K(\overline{u}^+)^p\phi \tag{3.8}$$

as $k\to\infty$. Since $\langle I'_{\lambda_0}(u_k),\phi\rangle=o(1)$ as $k\to\infty$, $I'_{\lambda_0}(\overline{u})=0$ in $H^{-1}(\Omega)$. Therefore $\overline{u}$ is a weak positive solution of $(1.1)_{\lambda_0}$. □

Denote

$$\lambda^*=\sup\{\lambda\geq 0:(1.1)_\lambda\text{ has a positive solution}\}. \tag{3.9}$$

Now, by the standard barrier method, we get the following lemma.

Lemma 3.2. *Let condition* (k1) *hold, then there exists* $\lambda^* > 0$ *such that for each* $\lambda \in [0,\lambda^*)$, *problem* $(1.1)_\lambda$ *has a minimal positive solution* u_λ *and* u_λ *is strictly increasing in* λ.

Proof. By Lemma 3.1 and the definition of λ^*, we deduce that $\lambda^* \geq \lambda_1 > 0$. Now, consider $\lambda \in [0,\lambda^*)$. By the definition of λ^*, we know that there exists $\lambda' > \lambda$ such that $\lambda' < \lambda^*$ and $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'} > 0$, that is,

$$\begin{aligned} -\Delta u_{\lambda'} + u_{\lambda'} &= \lambda' K(x) u_{\lambda'}^p + f(x) \\ &> \lambda K(x) u_{\lambda'}^p + f(x). \end{aligned} \tag{3.10}$$

Then $u_{\lambda'}$ is a supersolution of $(1.1)_\lambda$. From $f(x) \geq 0$ and $f(x) \not\equiv 0$, it is easily verified that 0 is a subsolution of $(1.1)_\lambda$. By the standard barrier method, there exists a solution u_λ of $(1.1)_\lambda$ such that $0 \leq u_\lambda \leq u_{\lambda'}$. Since 0 is not a solution of $(1.1)_\lambda$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_\lambda < u_{\lambda'}$. Again using a result of Amann [2], we can choose a minimum positive solution u_λ of $(1.1)_\lambda$. This completes the proof of Lemma 3.2. □

Now, we consider a solution u of $(1.1)_\lambda$. Let $\sigma_\lambda(u)$ be defined by

$$\sigma_\lambda(u) = \inf\left\{ \int_\Omega \left(|\nabla w|^2 + |w|^2\right) dx : w \in H_0^1(\Omega),\ \int_\Omega pKu^{p-1} w^2 dx = 1 \right\}. \tag{3.11}$$

By the standard direct minimization procedure, we can show that $\sigma_\lambda(u)$ is attained by a function $\varphi_\lambda > 0$, $\varphi_\lambda \in H_0^1(\Omega)$, satisfying

$$-\Delta \varphi_\lambda + \varphi_\lambda = \sigma_\lambda(u) pKu^{p-1} \varphi_\lambda \quad \text{in } \Omega. \tag{3.12}$$

Lemma 3.3. *Assume condition (k1) holds. For* $\lambda \in [0,\lambda^*)$, *let* u_λ *be the minimal solution of* $(1.1)_\lambda$ *and let* $\sigma_\lambda(u_\lambda)$ *be the corresponding number given by (3.11). Then*

(i) $\sigma_\lambda(u_\lambda) > \lambda$ *and is strictly decreasing in* λ, $\lambda \in [0,\lambda^*)$;

(ii) $\lambda^* < \infty$, *and* $(1.1)_{\lambda^*}$ *has a minimal solution* u_{λ^*}.

Proof. Consider $u_{\lambda'}$, u_λ, where $\lambda^* > \lambda' > \lambda \geq 0$. Let φ_λ be a minimizer of $\sigma_\lambda(u_\lambda)$, then by Lemma 3.2, we obtain

$$\int_\Omega pKu_{\lambda'}^{p-1} \varphi_\lambda^2 dx > \int_\Omega pKu_\lambda^{p-1} \varphi_\lambda^2 dx = 1, \tag{3.13}$$

and there is t, $0 < t < 1$, such that

$$\int_\Omega pKu_{\lambda'}^{p-1} (t\varphi_\lambda)^2 = 1. \tag{3.14}$$

Therefore,

$$\sigma_{\lambda'}(u_{\lambda'}) \leq t^2 \|\varphi_\lambda\|^2 < \|\varphi_\lambda\|^2 = \sigma_\lambda(u_\lambda), \tag{3.15}$$

showing the monotonicity of $\sigma_\lambda(u_\lambda)$, $\lambda \in [0,\lambda^*)$.

Consider now $\lambda \in (0,\lambda^*)$. Let $\lambda < \lambda' < \lambda^*$. From (3.12) and the monotonicity of u_λ, we get

$$\begin{aligned}\sigma_\lambda(u_\lambda)p\int_\Omega (u_{\lambda'}-u_\lambda)Ku_\lambda^{p-1}\varphi_\lambda dx &= \int_\Omega \nabla(u_{\lambda'}-u_\lambda)\cdot\nabla\varphi_\lambda dx + \int_\Omega (u_{\lambda'}-u_\lambda)\varphi_\lambda dx \\ &= (\lambda'-\lambda)\int_\Omega Ku_{\lambda'}^p\varphi_\lambda dx + \lambda\int_\Omega K(u_{\lambda'}^p - u_\lambda^p)\varphi_\lambda dx \\ &> \lambda p\int_\Omega K\varphi_\lambda\int_{u_\lambda}^{u_{\lambda'}} t^{p-1}dt\,dx \\ &\geq \lambda p\int_\Omega Ku_\lambda^{p-1}(u_{\lambda'}-u_\lambda)\varphi_\lambda dx,\end{aligned} \tag{3.16}$$

which implies that $\sigma_\lambda(u_\lambda) > \lambda$, $\lambda \in (0,\lambda^*)$. This completes the proof of (i).

We show next that $\lambda^* < \infty$. Let $\lambda_0 \in (0,\lambda^*)$ be fixed. For any $\lambda \geq \lambda_0$, (3.15) and (3.16) imply

$$\sigma_{\lambda_0}(u_{\lambda_0}) \geq \sigma_\lambda(u_\lambda) > \lambda \tag{3.17}$$

for all $\lambda \in [\lambda_0,\lambda^*)$. Thus, $\lambda^* < \infty$.

By (3.11) and $\sigma_\lambda(u_\lambda) > \lambda$, we have

$$\begin{aligned}\int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right)dx - \lambda p\int_\Omega Ku_\lambda^{p+1}dx > 0,\\ \int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right)dx - \int_\Omega \lambda Ku_\lambda^{p+1}dx - \int_\Omega fu_\lambda = 0.\end{aligned} \tag{3.18}$$

Thus

$$\begin{aligned}\int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right)dx &= \int_\Omega \lambda Ku_\lambda^{p+1}dx + \int_\Omega fu_\lambda dx \\ &< \frac{1}{p}\int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right)dx + \|f\|_{L^2(\Omega)}\|u_\lambda\| \\ &< \left(\frac{1}{p}+\frac{\delta}{2}\right)\|u_\lambda\|^2 + \frac{1}{2\delta}\|f\|_{L^2(\Omega)}^2\end{aligned} \tag{3.19}$$

for any $\delta > 0$. Since $p > 1$, we can obtain $\|u_\lambda\| \leq c < +\infty$ for all $\lambda \in (0,\lambda^*)$ by taking δ small enough. By Lemma 3.2, the solution u_λ is strictly increasing with respect to λ; we may suppose that

$$\begin{aligned}&u_\lambda \rightharpoonup u_{\lambda^*} \quad \text{weakly in } H_0^1(\Omega),\\ &u_\lambda \longrightarrow u_{\lambda^*} \quad \text{strongly in } L_{\text{loc}}^q(\Omega) \text{ for } 2 \leq q < \frac{2N}{N-2} \quad \text{as } \lambda \longrightarrow \lambda^*,\\ &u_\lambda \longrightarrow u_{\lambda^*} \quad \text{a.e. in } \Omega,\end{aligned} \tag{3.20}$$

For $\phi \in C_0^\infty(\Omega)$, we get

$$\int_\Omega \nabla u_\lambda \nabla \phi \longrightarrow \int_\Omega \nabla u_{\lambda^*} \nabla \phi, \qquad \int_\Omega u_\lambda \phi \longrightarrow \int_\Omega u_{\lambda^*} \phi, \qquad \lambda \int_\Omega K u_\lambda^p \phi \longrightarrow \lambda^* \int_\Omega K u_{\lambda^*}^p \phi, \tag{3.21}$$

as $\lambda \to \lambda^*$. From $\langle I'_\lambda(u_\lambda), \phi \rangle = 0$ and letting $\lambda \to \lambda^*$, we deduce $I'_{\lambda^*}(u_{\lambda^*}) = 0$ in $H^{-1}(\Omega)$. Hence u_{λ^*} is a positive solution of $(1.1)_{\lambda^*}$.

Let u be any positive solution of $(1.1)_{\lambda^*}$. By adopting the argument as in Lemma 3.1, we have $u \geq u_\lambda$ in Ω for $\lambda \in (0, \lambda^*)$, where u_λ is the minimal solution of $(1.1)_\lambda$. Therefore $u \geq u_{\lambda^*}$ in Ω. This implies that u_{λ^*} is a minimal solution of $(1.1)_{\lambda^*}$. □

In the following lemma, we give an estimate of λ^*.

Lemma 3.4. *If condition (k1) holds, then $\lambda_1 \leq \lambda^* \leq \lambda_2 \leq \lambda_3$, where λ_1, λ_2, and λ_3 are given by (1.5).*

Proof. By Lemma 3.1 and the definition of λ^*, we conclude that $\lambda^* \geq \lambda_1$.

As in Lemma 3.3, we have $\sigma_\lambda(u_\lambda) > \lambda$ for all $\lambda \in (0, \lambda^*)$, so for any $w \in H_0^1(\Omega) \setminus \{0\}$, we have

$$\int_\Omega (|\nabla w + |w|^2) dx > \lambda p \int_\Omega K u_\lambda^{p-1} w^2 dx. \tag{3.22}$$

Let u_0 be the unique solution of $(1.1)\ _0$, then by (3.22) and $u_\lambda > u_0$ for all $\lambda \in (0, \lambda^*]$, we obtain that

$$\int_\Omega (|\nabla w + |w|^2) dx > \lambda p \int_\Omega K u_0^{p-1} w^2 dx, \tag{3.23}$$

that is,

$$\lambda \leq \inf_{w \in H_0^1(\Omega) \setminus \{0\}} \left(\frac{\|w\|^2}{p \int_\Omega K u_0^{p-1} w^2 dx} \right) = \lambda_2. \tag{3.24}$$

This implies that $\lambda^* \leq \lambda_2$.

For all $\lambda \in [0, \lambda^*]$, let u_λ be a minimal solution of $(1.1)_\lambda$ and take $w = u_\lambda$ in (3.22), then we have

$$\begin{aligned} \|u_\lambda\|^2 &= \lambda \int_\Omega K u_\lambda^{p+1} dx + \int_\Omega f u_\lambda dx \\ &< \frac{1}{p} \|u_\lambda\|^2 + \|f\|_{L^2(\Omega)} \|u_\lambda\|. \end{aligned} \tag{3.25}$$

This implies that

$$\|u_\lambda\| \leq \frac{p}{p-1} \|f\|_{L^2(\Omega)}. \tag{3.26}$$

Take $w = u_\lambda$ in (3.24), and by (3.26) and the monotonicity of u_λ, we get

$$\begin{aligned}\lambda_2 &\le \frac{\|u_\lambda\|^2}{p\int_\Omega K u_0^{p-1} u_\lambda^2 dx} \\ &\le \frac{p\|f\|_{L^2(\Omega)}^2}{(p-1)^2\int_\Omega K u_0^{p+1} dx} = \lambda_3.\end{aligned} \tag{3.27}$$

□

4. Asymptotic behaviors of solutions

In this section, we will prove that a solution of $(1.1)_\lambda$ belongs to $C_b(\overline{\Omega})$ and derive several precise estimates on its behavior at infinity. Now, let $\mathbb{N}$ be all natural numbers, let $\mathbb{X}$ be a smooth domain in $\mathbb{R}^N$, and hence we have the extension lemma, embedding lemma, interpolation lemma (see Adams [1] for the proof), and for regularity Lemmas 4.1–4.7.

Lemma 4.1 (extension). *There is a positive constant $c = c(\ell, q)$ such that for any $u \in W^{\ell,q}(\mathbb{X})$, $\ell \in \mathbb{N}$, $1 < q < \infty$, there exists some $\overline{u} \in W^{\ell,q}(\mathbb{R}^N)$ such that $\overline{u} = u$ a.e. in $\mathbb{X}$ and $\|\overline{u}\|_{W^{\ell,q}(\mathbb{R}^N)} \le \|u\|_{W^{\ell,q}(\mathbb{X})}$.*

Lemma 4.2 (embedding). *There exists the following continuous embedding:*

$$W^{\jmath+\ell,q}(\mathbb{X}) \longrightarrow C^{\jmath,\alpha}(\overline{\mathbb{X}}), \quad 0 \le \alpha \le \ell - \frac{N}{q}, \tag{4.1}$$

provided $(\ell - 1)q < N < \ell q$ and $\jmath \in \mathbb{N} \cup \{0\}$.

Lemma 4.3 (interpolation). *Given $\ell \in \mathbb{N}, 1 < q < \infty$, there exists a positive constant $c = c(\ell, q, N)$ such that for any $0 < \varepsilon < 1$, $0 \le \jmath \le \ell - 1$, and any $u \in W^{\ell,q}(\mathbb{X})$,*

$$\|u\|_{W^{\jmath,q}(\mathbb{X})} \le c\varepsilon \|u\|_{W^{\ell,q}(\mathbb{X})} + \frac{c}{\varepsilon^{\jmath/(\ell-\jmath)}} \|u\|_{W^{0,q}(\mathbb{X})}. \tag{4.2}$$

Lemma 4.4 (regularity Lemma 1). *Let $g : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \mathbb{X}$, there holds*

$$|g(x,u)| \le c(|u| + |u|^p) \quad \textit{uniformly in } x \in \mathbb{X}, \tag{4.3}$$

where c, p are some positive constants, $N \ge 3$, and $1 < p < (N+2)/(N-2)$. Also, let $u \in H_0^1(\mathbb{X})$ be a weak solution of equation $-\Delta u = g(x,u) + f(x)$ in $\mathbb{X}$, where $f \in L^{N/2}(\mathbb{X}) \cap L^2(\mathbb{X})$. Then $u \in L^q(\mathbb{X})$ for $q \in [2, \infty)$.

Proof. See Hsu [11]. □

Now, we quote regularity Lemmas 4.5–4.7 (see Gilbarg and Trudinger [9, Theorems 8.8, 9.11, 9.16] for the proof).

Lemma 4.5 (regularity Lemma 2). *Let $\mathbb{X} \subset \mathbb{R}^N$ be a domain, let $g \in L^2(\mathbb{X})$, and let $u \in H^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then for any subdomain $\mathbb{X}' \subset\subset \mathbb{X}$ with $d' = \operatorname{dist}(\mathbb{X}', \partial\mathbb{X}) > 0$, $u \in H^2(\mathbb{X}')$ and*

$$\|u\|_{H^2(\mathbb{X}')} \le c\big(\|u\|_{H^1(\mathbb{X})} + \|g\|_{L^2(\mathbb{X})}\big) \tag{4.4}$$

for some $c = c(N,d')$. Furthermore u satisfies the equation $-\Delta u + u = g$ a.e. in $\mathbb{X}$.

Lemma 4.6 (regularity Lemma 3). *Let $g \in L^2(\mathbb{X})$ and let $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$. Then $u \in H_0^2(\mathbb{X})$ satisfies*

$$\|u\|_{H^2(\mathbb{X})} \le c\|g\|_{L^2(\mathbb{X})}, \tag{4.5}$$

where $c = c(N, \partial\mathbb{X})$.

Lemma 4.7 (regularity Lemma 4). *Let $g \in L^2(\mathbb{X}) \cap L^q(\mathbb{X})$ for some $q \in [2,\infty)$ and let $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then $u \in W^{2,q}(\mathbb{X})$ and u satisfies*

$$\|u\|_{W^{2,q}(\mathbb{X})} \le c\big(\|u\|_{L^q(\mathbb{X})} + \|g\|_{L^q(\mathbb{X})}\big), \tag{4.6}$$

where $c = c(N,q,\partial\mathbb{X})$.

By Lemma 4.7, we obtain the first asymptotic behavior of solution of $(1.1)_\lambda$.

Lemma 4.8 (asymptotic Lemma 1). *Let condition (k1) hold. If u is a weak solution of $(1.1)_\lambda$, then $u(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$.*

Proof. Let u satisfy

$$-\Delta u + u = \lambda K(x)u^p + f(x) \quad \text{in } H^{-1}(\Omega), \tag{4.7}$$

since K is bounded in Ω and $f \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$. Hence $f \in L^{N/2}(\Omega)$ and by Lemma 4.4, we have $u \in L^q(\Omega)$ for $q \in [2,\infty)$. Hence $\lambda K(x)u^p + f(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$. Then by Lemma 4.7, we have $u \in W^{2,q_0}(\Omega)$ for some $q_0 > N/2$. By Lemma 4.2, $u \in C_b(\overline{\Omega})$ and there exists a constant $c > 0$, such that for any $r > 1$,

$$\|u\|_{L^\infty(\overline{B}_r^c)} \le c\|u\|_{W^{2,q_0}(\overline{B}_r^c)}, \tag{4.8}$$

where $\overline{B}_r^c = \{x = (y,z) \in \Omega : |z| > r\}$. Hence $\lim_{|z|\to\infty} u(y,z) = 0$ uniformly for $y \in \omega$. □

Lemma 4.9 (asymptotic Lemma 2). *Let u be a positive solution of $(1.1)_\lambda$ for $\lambda \in [0,\lambda^*]$ and let ψ be the first positive eigenfunction of the Dirichlet problem $-\Delta\psi = \mu_1\psi$ in ω, then there exists a positive constant c such that*

$$u(x) \ge c\psi(y)\exp\Big(-\sqrt{1+\mu_1}|z|\Big)|z|^{-(n-1)/2} \quad \text{as } |z| \longrightarrow \infty,\ y \in \varpi. \tag{4.9}$$

Proof. Let $\Phi(x) = (1 + 1/\sqrt{|z|})\psi(y)\exp(-\sqrt{1+\mu_1}|z|)|z|^{-(n-1)/2}$ for $x = (y,z) \in \overline{\Omega}$ and $|z| > 0$. It is very easy to show that there is a $R_0 > 0$ such that

$$-\Delta\Phi + \Phi \le 0, \quad \forall |z| \le R_0. \tag{4.10}$$

Let u_λ be the minimal solution of $(1.1)_\lambda$, let $q = (q_y, q_z)$, $q_y \in \partial\omega$, $|q_z| = R_0$, and B a small ball in Ω such that $q \in \partial B$. Since $\psi(y) > 0$ for $x = (y,z) \in B$, $\psi(q_y) = 0$, $u(x) > 0$ for $x \in B$, $u_\lambda(q) = 0$, by the strongly maximum principle $(\partial\psi/\partial y)(q_y) < 0$, $(\partial u_\lambda/\partial x)(q) < 0$. Thus

$$\lim_{\substack{x \to q \\ |z| = R_0}} \frac{u_\lambda(x)}{\psi(y)} = \frac{(\partial u_\lambda/\partial x)(q)}{(\partial\psi/\partial y)(q_y)} > 0. \tag{4.11}$$

Note that $u_\lambda(x)\psi^{-1}(y) > 0$ for $x = (y,z)$, $y \in \omega$, $|z| = R_0$. Thus $u_\lambda(x)\psi^{-1}(y) > 0$ for $x = (y,z)$, $y \in \varpi$, $|z| = R_0$.

Since $\Phi(x)$ and $u_\lambda(x)$ are $C^1(\overline{\omega \times \partial B_{R_0}(0)})$, if we set

$$\alpha = \inf_{y \in \varpi,\ |z| = R_0} \left(u_\lambda(x)\Phi^{-1}(x)\right), \tag{4.12}$$

then $\alpha > 0$ and

$$\alpha\Phi(x) \le u_\lambda(x) \quad \text{for } y \in \varpi, \qquad |z| = R_0. \tag{4.13}$$

For $|z| \ge R_0$, we have

$$-\Delta(u_\lambda - \alpha\Phi)(x) + (u_\lambda - \alpha\Phi)(x) = \lambda K(x)u_\lambda^p(x) + f(x) + \alpha(\Delta\Phi + \Phi)(x) \ge 0. \tag{4.14}$$

By the maximum principle, we obtain

$$u_\lambda(x) \ge \alpha\Phi(x) \quad \text{for } y \in \varpi, \qquad |z| \ge R_0. \tag{4.15}$$

Let $c = \alpha > 0$, we get

$$u_\lambda(x) \ge c\psi(y)\exp\left(-\sqrt{1+\mu_1}|z|\right)|z|^{-(n-1)/2} \quad \text{for } y \in \varpi,\ |z| \ge R_0. \tag{4.16}$$

This implies that (4.9) holds for u_λ and hence for arbitrary positive solution u. □

5. Existence of second solution

The existence of a second solution of $(1.1)_\lambda$, $\lambda \in (0,\lambda^*)$, will be established via the mountain pass theorem. When $0 < \lambda < \lambda^*$, we have known that $(1.1)_\lambda$ has a minimal positive solution u_λ by Lemma 3.2, then we need only to prove that $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + v_\lambda$, where v_λ is a solution of the following problem:

$$-\Delta v + v = \lambda K\left[(v+u_\lambda)^p - u_\lambda^p\right] \quad \text{in } \Omega, \quad v \in H_0^1(\Omega), \quad v > 0 \quad \text{in } \Omega. \tag{5.1$_\lambda$}$$

The corresponding variational functional of $(5.1)_\lambda$ is

$$J_\lambda(v) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + v^2\right) - \lambda\int_\Omega\int_0^{v^+} K\left[(s+u_\lambda)^p - u_\lambda^p\right]ds\,dx, \quad v \in H_0^1(\Omega). \tag{5.1}$$

To verify the conditions of the mountain pass theorem, we need the following lemmas.

Lemma 5.1. *For any $\epsilon > 0$, there is a positive constant c_ϵ such that*

$$(\xi + s)^p - \xi^p - p\xi^{p-1}s \le \epsilon \xi^{p-1}s + c_\epsilon s^p, \quad \forall s \ge 0, \xi > 0. \tag{5.2}$$

Proof. From the fact

$$\lim_{t\to 0^+} \frac{(1+t)^p - 1 - pt}{t} = 0, \qquad \lim_{s\to\infty} \frac{(1+t)^p - 1 - pt}{t^p} = 1, \tag{5.3}$$

we obtain that for any $\epsilon > 0$, there is a positive constant c_ϵ such that

$$(1+t)^p - 1 - pt \le \epsilon t + c_\epsilon t^p, \quad \forall t \ge 0. \tag{5.4}$$

Let $\xi > 0$, $s \ge 0$, and take $t = s/\xi$ in (5.4), we can deduce that

$$(\xi + s)^p - \xi^p - p\xi^{p-1}s \le \epsilon \xi^{p-1}s + c_\epsilon s^p. \tag{5.5}$$

□

Lemma 5.2. *There exist positive constants ρ and α, such that*

$$J_\lambda(v) \ge \alpha > 0, \quad v \in H_0^1(\Omega), \quad \|v\| = \rho. \tag{5.6}$$

Proof. For any $\epsilon > 0$ there is by Lemma 5.1 (with $\xi = u_\lambda$) a positive constant c_ϵ such that

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\,dx - \frac{1}{2}\lambda p \int_\Omega K u_\lambda^{p-1}(v^+)^2 dx \\ &\quad - \lambda \int_\Omega \int_0^{v^+} K\Big[(u_\lambda + s)^p - u_\lambda^p - p u_\lambda^{p-1} s\Big]\,ds\,dx \\ &\ge \frac{1}{2}\Big[\int_\Omega (|\nabla v|^2 + v^2)\,dx - \lambda p \int_\Omega K u_\lambda^{p-1}(v^+)^2 dx\Big] \\ &\quad - \lambda \int_\Omega K\left[\frac{\epsilon}{2} u_\lambda^{p-1}(v^+)^2 + c_\epsilon \frac{(v^+)^{p+1}}{p+1}\right] dx. \end{aligned} \tag{5.7}$$

Furthermore, from the definition $\sigma_\lambda(u_\lambda)$ in (3.11), we have

$$\int_\Omega (|\nabla v|^2 + v^2)\,dx \ge \sigma_\lambda(u_\lambda)\, p \int_\Omega K u_\lambda^{p-1}(v^+)^2 dx, \tag{5.8}$$

and, therefore, by (5.7) we obtain

$$J_\lambda(v) \ge \frac{1}{2}\sigma_\lambda(u_\lambda)^{-1}\Big(\sigma_\lambda(u_\lambda) - \lambda - \frac{\epsilon}{2}\lambda\Big)\|v\|^2 - \lambda c_\epsilon (p+1)^{-1} \int_\Omega K(v^+)^{p+1} dx. \tag{5.9}$$

Since $\sigma_\lambda(u_\lambda) > \lambda$, by property (ii) in Lemma 3.3, the boundedness of K and the Sobolev inequality imply that for small $\epsilon > 0$,

$$J_\lambda(v) \geq \frac{1}{4}\sigma_\lambda(u_\lambda)^{-1}(\sigma_\lambda(u_\lambda) - \lambda)\|v\|^2 - \lambda c\|v\|^{p+1}, \tag{5.10}$$

and the conclusion in Lemma 5.2 follows. □

Now, we give the following decomposition lemma for later use.

LEMMA 5.3. *Assume condition (k1) holds. Let $\{v_k\}$ be a $(PS)_c$ sequence of J_λ in $H_0^1(\Omega)$:*

$$\begin{aligned} J_\lambda(v_k) &= c + o(1) \quad \text{as } k \longrightarrow \infty, \\ J'_\lambda(v_k) &= o(1) \quad \text{strongly in } H^{-1}(\Omega). \end{aligned} \tag{5.11}$$

Then there exists a subsequence (still denoted by) $\{v_k\}$ for which the following holds: there exist an integer $l \geq 0$, sequence $\{x_k^i\} \subset \mathbb{R}^N$ of the form $(0, z_k^i) \in \mathbb{S}$ for $1 \leq i \leq l$, a solution v_λ of (5.1)$_\lambda$, and solutions $\overline{v}_\lambda^i$ of (2.1)$_\lambda$ for $1 \leq i \leq l$, such that as $k \rightharpoonup \infty$

$$\begin{aligned} v_k &\rightharpoonup v_\lambda \quad \text{weakly in } H_0^1(\Omega); \\ v_k - \left[v_\lambda + \sum_{i=1}^{l} \overline{v}_\lambda^i(\,\cdot - x_k^i)\right] &\longrightarrow 0 \quad \text{strongly in } H_0^1(\Omega); \\ J_\lambda(v_k) &= J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_\lambda^i) + o(1), \end{aligned} \tag{5.12}$$

where its agreed upon that in the case $l = 0$, the above holds without $\overline{v}_\lambda^i$, $\{x_n^i\}$.

Proof. The proof can be obtained by using the arguments in Bahri and Lions [5] (also see [15, 16]). We omit it. □

Now, let δ be small enough, D^δ a δ-tubular neighborhood of D such that $D^\delta \subset\subset \mathbb{S}$. Let $\eta(x) : \mathbb{S} \to [0,1]$ be a C^∞ cutoff function such that $0 \leq \eta \leq 1$ and

$$\eta(x) = \begin{cases} 0 & \text{if } x \in D; \\ 1 & \text{if } x \in \Omega \setminus D^\delta. \end{cases} \tag{5.13}$$

Let $e_N = (0,0,\dots,0,1) \in \mathbb{R}^N$, denote

$$\begin{aligned} \tau_0 &= 2\sup_{x \in D^\delta} |x| + 1, \\ \widetilde{u}_\tau(x) &= \eta(x)\overline{u}_\lambda(x - \tau e_N), \quad \tau \in [0, \infty), \end{aligned} \tag{5.14}$$

where $\overline{u}_\lambda$ is a ground state solution of (2.1)$_\lambda$.

LEMMA 5.4. *Assume condition* (k1) *holds, then there exist some constants $t_0 > 0$, $\tau_* \geq \tau_0$ such that $J_\lambda(t\widetilde{u}_\tau) < 0$ for all $\tau \geq \tau_*$, $t \geq t_0$.*

Proof. By the inequality $(a+b)^p \geq a^p + b^p$ for all $a \geq 0$, $b \geq 0$, $p > 1$ and $\overline{u}_\lambda$ is a ground state solution of $(2.1)_\lambda$, denote $\eta_\tau(x) = \eta(x+\tau e_N)$, then we have

$$\begin{aligned} J_\lambda(t\widetilde{u}_\tau) &= \frac{1}{2}t^2\int_\Omega \left(|\nabla \widetilde{u}_\tau|^2 + |\widetilde{u}_\tau|^2\right)dx - \frac{1}{p+1}t^{p+1}\int_\Omega \lambda K(x)\widetilde{u}_\tau^{p+1}dx \\ &\quad - \int_\Omega\int_0^{t\widetilde{u}_\tau} \lambda K(x)\left[(s+u_\lambda)^p - u_\lambda^p - s^p\right]ds\,dx \\ &\leq \frac{1}{2}t^2\int_{\mathbb{S}}(-\Delta\overline{u}_\lambda + \overline{u}_\lambda)(\eta_\tau^2\overline{u}_\lambda)dx + \frac{1}{2}t^2\int_{\mathbb{S}}|\nabla\eta_\tau|^2|\overline{u}_\lambda|^2dx \\ &\quad - \frac{1}{p+1}t^{p+1}\int_{\mathbb{S}}\lambda K(x)\eta^{p+1}(x)\overline{u}_\lambda^{p+1}(x-\tau e_N)dx \\ &\leq \frac{1}{2}t^2\int_{\mathbb{S}}\lambda K_\infty\overline{u}_\lambda^{p+1}dx + \frac{1}{2}t^2\left(\max_{x\in\mathbb{S}}|\nabla\eta|^2\right)\int_{\mathbb{S}}|\overline{u}_\lambda|^2dx \\ &\quad - \frac{t^{p+1}}{p+1}\int_{\mathbb{S}}\lambda K(x)\eta^{p+1}(x)\overline{u}_\lambda^{p+1}(x-\tau e_N)dx. \end{aligned} \tag{5.15}$$

Set $B_1(\tau e_N) = \{x=(y,z)\in\mathbb{S} : y\in\omega, |z-\tau e_N| < 1\}$. By condition (k1), there exists $\tau_* \geq \tau_0$ such that $K(x) \geq K_\infty/2$ for $x \in B_1(\tau e_N)$ for all $\tau \geq \tau_*$ and note that $\eta(x) \equiv 1$ on $B_1(\tau e_N)$ for $\tau \geq \tau_*$, then we obtain that

$$\begin{aligned} &\int_{\mathbb{S}}\lambda K(x)\eta^{p+1}(x)\overline{u}_\lambda^{p+1}(x-\tau e_N)dx \\ &\quad \geq \int_{B_1(\tau e_N)}\frac{\lambda}{2}K_\infty\overline{u}_\lambda^{p+1}(x-\tau e_N)dx \\ &\quad = \int_{\{x=(y,z)\in\mathbb{S}:y\in\omega,|z|\leq 1\}}\frac{\lambda}{2}K_\infty\overline{u}_\lambda^{p+1}(x)dx = c > 0, \end{aligned} \tag{5.16}$$

where c is independent of τ. Combining (5.15) and (5.16), there exist some positive constants c_1, c_2, independent of τ, such that

$$J_\lambda(t\widetilde{u}_\tau) \leq c_1t^2 - c_2t^{p+1} \quad \forall\tau \geq \tau_*. \tag{5.17}$$

From (5.17), we conclude the result. □

Lemma 5.5. *Assume conditions (k1) and (k2) hold, then there exists a constant $\tau^* > 0$, such that the following inequality holds for $\tau \geq \tau^*$:*

$$0 < \sup_{t\geq 0}J_\lambda(t\widetilde{u}_\tau) < I_\lambda^\infty(\overline{u}_\lambda) = M_\lambda^\infty. \tag{5.18}$$

Proof. From (5.6), we easily see that the left-hand of (5.18) holds and we need only to show that the right-hand side of (5.18) holds. By Lemma 5.4, we have that there exists a constant $t_2 > 0$ such that

$$\sup_{t\geq 0} J_\lambda(t\widetilde{u}_\tau) = \sup_{0\leq t\leq t_2} J_\lambda(t\widetilde{u}_\tau) \quad \text{for any } \tau \geq \tau_*. \tag{5.19}$$

Since J is continuous in $H_0^1(\Omega)$ and $J(0) = 0$, there exists a constant $t_1 > 0$ such that

$$J_\lambda(t\widetilde{u}_\tau) < M_\lambda^\infty \quad \text{for any } \tau \in (0,\infty),\ 0 \leq t < t_1. \tag{5.20}$$

Then, to prove (5.18) we now need only to prove the following inequality:

$$\sup_{t_1\leq t\leq t_2} J_\lambda(t\widetilde{u}_\tau) < M_\lambda^\infty \quad \text{for } \tau \text{ large enough.} \tag{5.21}$$

By the definition of J_λ, we get

$$\begin{aligned} J_\lambda(t\widetilde{u}_\tau) = {} & \frac{t^2}{2}\int_\Omega \left(|\nabla \widetilde{u}_\tau|^2 + \widetilde{u}_\tau^2\right)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{S}} \lambda K_\infty \widetilde{u}_\tau^{p+1}dx \\ & + \frac{t^{p+1}}{p+1}\int_\Omega \lambda(K_\infty - K(x))\widetilde{u}_\tau^{p+1}dx \\ & - \int_\Omega\int_0^{t\widetilde{u}_\tau} \lambda K(x)\left[(s+u_\lambda)^p - u_\lambda^p - s^p\right]ds\,dx. \end{aligned} \tag{5.22}$$

Since $\overline{u}_\lambda$ is a ground state solution of $(2.1)_\lambda$, denote $\eta_\tau(x) = \eta(x+\tau e_N)$, then we have

$$\begin{aligned} J_\lambda(t\widetilde{u}_\tau) \leq {} & \frac{t^2}{2}\int_{\mathbb{S}} \left(-\Delta\overline{u}_\lambda + \overline{u}_\lambda\right)\left(\eta_\tau^2\overline{u}_\lambda\right)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{S}} \lambda K_\infty \overline{u}_\lambda^{p+1}dx \\ & + \frac{t_2^2}{2}\int_{\mathbb{S}} |\nabla\eta_\tau|^2|\overline{u}_\lambda|^2dx \\ & + \frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}} \lambda K_\infty\left(\overline{u}_\lambda^{p+1} - \widetilde{u}_\tau^{p+1}\right)dx \\ & + \frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}} \lambda(K_\infty - K(x))^+\widetilde{u}_\tau^{p+1}dx \\ & - \int_\Omega\int_0^{t\widetilde{u}_\tau} \lambda K(x)\left[(s+u_\lambda)^p - u_\lambda^p - s^p\right]ds\,dx. \end{aligned} \tag{5.23}$$

It follows from (2.5) that for any $\epsilon > 0$, there exists a constant $c_1 > 0$, independent of τ, such that, for all $\tau \geq \tau_*$,

$$\begin{gathered}\frac{t_2^2}{2}\int_{\mathbb{S}}|\nabla\eta_\tau|^2|\overline{u}_\lambda|^2dx \leq c_1\exp\left(-2\sqrt{1+\mu_1}\tau\right)\tau^{-n+1+2\epsilon},\\ \frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}}\lambda K_\infty\left(\overline{u}_\lambda^{p+1}-\widetilde{u}_\tau^{p+1}\right)dx \leq c_1\exp\left(-2\sqrt{1+\mu_1}\tau\right)\tau^{-n+1+2\epsilon}.\end{gathered} \tag{5.24}$$

From condition (k2) and (2.5), there exists a constant $\tau_1 > 0$ such that, for all $\tau \geq \tau_1$,

$$\begin{aligned}&\frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}}\lambda(K_\infty-K(x))^+\widetilde{u}_\tau^{p+1}dx\\ &\quad\leq c\left(\int_{\mathbb{S}\cap\{|z|\geq\tau/(p+1)\}}+\int_{\mathbb{S}\cap\{|z|\leq\tau/(p+1)\}}\right)(K_\infty-K(x))^+\overline{u}_\lambda^{p+1}(x+\tau e_N)dx\\ &\quad\leq c\exp\left(-\sqrt{1+\mu_1}\tau\right)\left(\frac{p}{p+1}\right)\tau^{-\gamma}+c_1\exp\left(-\sqrt{1+\mu_1}\tau\right)\left(\frac{1}{p+1}\tau\right)^{(p+1)(-(n-1)/2+\epsilon)}\\ &\quad\leq c_2\exp\left(-\sqrt{1+\mu_1}\tau\right)\tau^{-\gamma_0},\end{aligned} \tag{5.25}$$

where $c_2 > 0$ is a constant independent of τ and $\gamma_0 = \min\{\gamma,(p+1)(-(n-1)/2+\epsilon)\}$.

Let $\omega_0 \subset\subset \omega$ be a smooth bounded domain in $\mathbb{R}^m$. Set $D_1(\tau e_N) = \{x = (y,z) \in \mathbb{S} : y \in \omega_0, |z-\tau e_N| < 1\}$. Noting that $(a+b)^p \geq a^p + b^p$ for all $a \geq 0$, $b \geq 0$, $p > 1$, then for $\tau \geq \tau_0$, we have $\eta(x) = 1$ on $D_1(\tau e_N)$ and

$$\begin{aligned}&\int_\Omega\int_0^{t\widetilde{u}_\tau}\lambda K(x)[(s+u_\lambda)^p-u_\lambda^p-s^p]ds\,dx\\ &\quad\geq\int_{D_1(\tau e_N)}\int_0^{t\widetilde{u}_\tau}\lambda K(x)[(s+u_\lambda)^p-u_\lambda^p-s^p]ds\,dx\\ &\quad=\int_{D_1(\tau e_N)}\int_0^{t\widetilde{u}_\tau}\lambda K(x)\Big([(s+u_\lambda)^{p-1}-s^{p-1}]s+[(s+u_\lambda)^{p-1}-u_\lambda^{p-1}]u_\lambda\Big)ds\,dx\\ &\quad\geq\int_{D_1(\tau e_N)}\int_0^{t\widetilde{u}_\tau}\lambda K(x)\Big[(s+u_\lambda)^{p-1}-u_\lambda^{p-1}\Big]u_\lambda ds\,dx\\ &\quad=\int_{D_1(\tau e_N)}\lambda K(x)\left[\frac{(t\widetilde{u}_\tau+u_\lambda)^p-u_\lambda^p}{p\widetilde{u}_\tau}-tu_\lambda^{p-1}\right]\widetilde{u}_\tau u_\lambda dx.\end{aligned} \tag{5.26}$$

By Lemma 4.8, there exist some constants $\tau_2 \geq \tau_0 + \tau_1$ and $\alpha > 0$, such that

$$\frac{(t\widetilde{u}_\tau+u_\lambda)^p-u_\lambda^p}{p\widetilde{u}_\tau}-tu_\lambda^{p-1}\geq\alpha\quad\text{for }\tau\geq\tau_2,\ x\in D_1(\tau e_N),\ t\in[t_1,t_2], \tag{5.27}$$

then by (k1), (4.9), and (5.26), there exists a constant $\tau_3 \geq \tau_2$ such that $K(x) \geq K_\infty/2$ for $x \in D_1(\tau e_N)$ and

$$\begin{aligned}
&\int_\Omega \int_0^{t\widetilde{u}_\tau} \lambda K(x)\left[(s+u_\lambda)^p - u_\lambda^p - s^p\right] ds\, dx \\
&\quad \geq \frac{1}{2}\lambda\alpha K_\infty \int_{D_1(\tau e_N)} \overline{u}_\lambda(x-\tau e_N) u_\lambda(x) dx \\
&\quad \geq c \int_{D_1(\tau e_N)} \overline{u}_\lambda(x-\tau e_N) \exp\left[-(\tau+1)\sqrt{1+\mu_1}\right](\tau+1)^{-(n-1)/2} \\
&\quad \geq c_3 \exp\left(-\sqrt{1+\mu_1}\tau\right)\tau^{-(n-1)/2},
\end{aligned} \tag{5.28}$$

where $c_3 > 0$ is a constant independent of τ for all $\tau \geq \tau_3$ and $t \in [t_1, t_2]$.

From (5.23)–(5.28), we get, for $\tau \geq \tau_3 + \tau_*$ and $t \in [t_1, t_2]$,

$$\begin{aligned}
J_\lambda(t\widetilde{u}_\tau) \leq{}& M_\lambda^\infty + 2c_1 \exp\left(-2\sqrt{1+\mu_1}\tau\right)\tau^{-n+1+2\epsilon} \\
&+ c_2 \exp\left(-\sqrt{1+\mu_1}\tau\right)\tau^{-\gamma_0} - c_3 \exp\left(-\sqrt{1+\mu_1}\tau\right)\tau^{-(n-1)/2},
\end{aligned} \tag{5.29}$$

where c_i, $1 \leq i \leq 3$, are independent of τ.

Let $\epsilon = p(n-1)/4(p+1)$ and by $\gamma > (n-1)/2$, we have $\gamma_0 > (n-1)/2$. Hence, we can find some constant $\tau^* > \tau_3 + \tau_*$ large enough such that

$$2c_1 \exp\left(-2\sqrt{1+\mu_1}\tau\right)\tau^{-n+1+2\epsilon} + c_2 \exp\left(-\sqrt{1+\mu_1}\tau\right)\tau^{-\gamma_0} - c_3 \exp\left(-\sqrt{1+\mu_1}\tau\right)\tau^{-(n-1)/2} < 0 \tag{5.30}$$

and (5.18) is proved. □

Proposition 5.6. *Let conditions (k1) and (k2) hold, then (5.1)$_\lambda$ has at least one solution for $\lambda \in (0, \lambda^*)$.*

Proof. For the constant τ^* in Lemma 5.5, by Lemma 5.4, we know that there is a constant $t_0 > 0$ such that $J_\lambda(t_0\widetilde{u}_{\tau^*}) < 0$. We set

$$\Gamma = \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) : \gamma(0) = 0,\ \gamma(1) = t_0\widetilde{u}_{\tau^*}\right\}, \tag{5.31}$$

then, from (5.6) and (5.18) we get

$$0 < c = \inf_{\gamma\in\Gamma} \max_{0\leq s\leq 1} J_\lambda(\gamma(s)) < M_\lambda^\infty. \tag{5.32}$$

Applying the mountain pass lemma of Ambrosetti and Rabinowitz [3], there exists a $(PS)_c$-sequence $\{v_k\}$ such that

$$J_\lambda(v_k) \longrightarrow c, \qquad J_\lambda'(v_k) \longrightarrow 0 \quad \text{in } H^{-1}(\Omega). \tag{5.33}$$

By Lemma 5.3, there exist a subsequence, still denoted by $\{v_k\}$, an integer $l \geq 0$, a solution v_λ of $(5.1)_\lambda$, and solutions $\overline{v}_\lambda^i$ of $(2.1)_\lambda$, for $1 \leq i \leq l$, such that

$$c = J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_\lambda^i). \tag{5.34}$$

By the strongly maximum principle, to complete the proof, we only need to prove $v_\lambda \not\equiv 0$ in Ω. We proceed by contradiction. Assume that $v_\lambda \equiv 0$ in Ω. From (5.32) and (5.34), we have $l \geq 1$ and

$$0 < M_\lambda^\infty \leq l M_\lambda^\infty \leq \sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_\lambda^i) = c < M_\lambda^\infty. \tag{5.35}$$

This implies $v_\lambda \not\equiv 0$ in Ω. □

6. Properties and bifurcation of solutions

Denote $A = \{(\lambda, u) : u \text{ satisfies } (1.1)_\lambda, \lambda \in [0, \lambda^*]\}$. By Lemma 4.8, we have $A \subset \mathbb{R} \times L^\infty(\Omega) \cap H_0^1(\Omega)$. Moreover, we assume that $f(x), K(x) \in C^\alpha(\Omega) \cap L^2(\Omega)$. By elliptic regular theory [9], we can deduce that $A \subset \mathbb{R} \times C^{2,\alpha}(\Omega) \cap H^2(\Omega)$.

For each $(\lambda, u) \in A$, let $\sigma_\lambda(u)$ denote the number defined by (3.11), which is the first eigenvalue of the problem (3.12).

Lemma 6.1. *Let u be a solution and let u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. Then*

(i) $\sigma_\lambda(u) > \lambda$ *if and only if* $u = u_\lambda$;
(ii) $\sigma_\lambda(U_\lambda) < \lambda$, *where U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 5.*

Proof. Now, let $\phi \geq 0$ and $\phi \in H_0^1(\Omega)$. Since u and u_λ are the solution of $(1.1)_\lambda$, then

$$\begin{aligned} &\int_\Omega \nabla\phi \cdot \nabla(u_\lambda - u)\,dx + \int_\Omega \phi(u_\lambda - u)\,dx \\ &\quad = \lambda \int_\Omega K(u_\lambda^p - u^p)\phi\,dx = \lambda \int_\Omega \left(\int_u^{u_\lambda} t^{p-1}dt\right) pK\phi\,dx \geq \lambda \int_\Omega pKu^{p-1}(u_\lambda - u)\phi\,dx. \end{aligned} \tag{6.1}$$

Let $\phi = (u - u_\lambda)^+ \geq 0$ and $\phi \in H_0^1(\Omega)$. If $\phi \not\equiv 0$, then (6.1) implies

$$-\int_\Omega (|\nabla\phi|^2 + \phi^2)\,dx \geq -\lambda \int_\Omega pKu^{p-1}\phi^2\,dx \tag{6.2}$$

and, therefore, the definition of $\sigma_\lambda(u)$ implies

$$\begin{aligned} &\int_\Omega (|\nabla\phi|^2 + \phi^2)\,dx \\ &\quad \leq \lambda \int_\Omega pKu^{p-1}\phi^2\,dx < \sigma_\lambda(u) \int_\Omega pKu^{p-1}\phi^2\,dx \leq \int_\Omega (|\nabla\phi|^2 + \phi^2)\,dx, \end{aligned} \tag{6.3}$$

which is impossible. Hence $\phi \equiv 0$, and $u = u_\lambda$ in Ω. On the other hand, by Lemma 3.3, we also have that $\sigma_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

By (i), we get that $\sigma_\lambda(U_\lambda) \le \lambda$ for $\lambda \in (0,\lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ cannot occur. We proceed by contradiction. Set $w_\lambda = U_\lambda - u_\lambda$; we have

$$-\Delta w_\lambda + w_\lambda = \lambda K[U_\lambda^p - (U_\lambda - w_\lambda)^p], \quad w_\lambda > 0 \text{ in } \Omega. \tag{6.4}$$

By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem

$$-\Delta\varphi + \varphi = \lambda pKU_\lambda^{p-1}\varphi, \quad \varphi \in H_0^1(\Omega), \tag{6.5}$$

possesses a positive solution φ_λ.

Multiplying (6.4) by φ_λ and (6.5) by w_λ, integrating, and subtracting we deduce that

$$\begin{aligned} 0 &= \int_\Omega \lambda K[U_\lambda^p - (U_\lambda - w_\lambda)^p - pU_\lambda^{p-1}w_\lambda]\varphi_\lambda dx \\ &= -\frac{1}{2}p(p-1)\int_\Omega \lambda K\xi_\lambda^{p-2}w_\lambda^2\varphi_\lambda dx, \end{aligned} \tag{6.6}$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w_\lambda \equiv 0$, that is, $U_\lambda = u_\lambda$ for $\lambda \in (0,\lambda^*)$. This is a contradiction. Hence, we have $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0,\lambda^*)$. □

LEMMA 6.2. *Let u_λ be the minimal solution of (1.1)$_\lambda$ for $\lambda \in [0,\lambda^*]$ and $\sigma_\lambda(u_\lambda) > \lambda$. Then for any $g(x) \in H^{-1}(\Omega)$, problem*

$$-\Delta w + w = \lambda pKu_\lambda^{p-1}w + g(x), \quad w \in H_0^1(\Omega), \tag{6.4$_\lambda$}$$

has a solution.

Proof. Consider the functional

$$\Phi(w) = \frac{1}{2}\int_\Omega (|\nabla w|^2 + w^2)dx - \frac{1}{2}\lambda p\int_\Omega Ku_\lambda^{p-1}w^2dx - \int_\Omega g(x)w\,dx, \tag{6.7}$$

where $w \in H_0^1(\Omega)$. From Hölder inequality and Young's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned} \Phi(w) &\ge \frac{1}{2}\left(1 - \lambda\sigma_\lambda(u_\lambda)^{-1}\right)\|w\|^2 - \frac{1}{2}\epsilon\|w\|^2 - \frac{C_\epsilon}{2}\|g\|_{H^{-1}(\Omega)}^2 \\ &\ge -C\|g\|_{H^{-1}(\Omega)}^2 \end{aligned} \tag{6.8}$$

if we choose ϵ small.

Now, let $\{w_k\} \subset H_0^1(\Omega)$ be the minimizing sequence of variational problem

$$d = \inf\{\Phi(w) \mid w \in H_0^1(\Omega)\}. \tag{6.9}$$

From (6.8) and $\sigma_\lambda(u_\lambda) > \lambda$, we can also deduce that $\{w_k\}$ is bounded in $H_0^1(\Omega)$ if we choose ϵ small. So we may suppose that

$$\begin{aligned} &w_k \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \\ &w_k \longrightarrow w \quad \text{strongly in } L_{\text{loc}}^q(\Omega) \text{ for } 2 \le q < \frac{2N}{N-2} \quad \text{as } k \longrightarrow \infty. \\ &w_k \longrightarrow w \quad \text{a.e. in } \Omega, \end{aligned} \tag{6.10}$$

By Fatou's lemma,

$$\|w\|^2 \leq \liminf_{k\to\infty} \|w_k\|^2, \tag{6.11}$$

and by the weak convergence we have

$$\int_\Omega g w_k dx \longrightarrow \int_\Omega g w\, dx \quad \text{as } k \longrightarrow \infty. \tag{6.12}$$

By Lemma 4.8, we have $u_\lambda(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$. It follows that there exists a constant $c_1 > 0$ such that

$$|u_\lambda(x)| \leq c_1 \quad \forall x \in \Omega. \tag{6.13}$$

Furthermore, for any $\varepsilon > 0$, there exists $R > 0$ such that $|u_\lambda^{p-1}(x)| < \varepsilon$ for all $x = (y,z) \in \Omega$ and $|y| \geq R$. Let $\Omega_R = \{x = (y,z) \in \Omega : |z| < R\}$, then we have

$$\begin{aligned} &\left| \int_\Omega K u_\lambda^{p-1} (w_k^2 - w^2) dx \right| \\ &\quad \leq \|K\|_{L^\infty(\Omega)} \left(\int_{\Omega_R} u_\lambda^{p-1} |w_k - w|^2 dx + \int_{\Omega\setminus\Omega_R} u_\lambda^{p-1} |w_k - w|^2 dx \right) \\ &\quad \leq c_2 \int_{\Omega_R} |w_k - w|^2 dx + \varepsilon \int_{\Omega\setminus\Omega_R} |w_k - w|^2 dx. \end{aligned} \tag{6.14}$$

From $w_k \to w$ strongly in $L^q_{\text{loc}}(\Omega)$ for $2 \leq q < 2N/(N-2)$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} \int_{\Omega_R} |w_k - w|^2 dx = 0. \tag{6.15}$$

Since $\{w_k\}$ is bounded in $H_0^1(\Omega)$, this implies that there exists a constant $c_3 > 0$ such that

$$\int_{\Omega\setminus\Omega_R} |w_k - w|^2 dx \leq c_3. \tag{6.16}$$

Therefore, we conclude that

$$\lim_{k\to\infty} \left| \int_\Omega K u_\lambda^{p-1} (w_k^2 - w^2) dx \right| \leq c_3 \varepsilon. \tag{6.17}$$

Take $\varepsilon \to 0$, we obtain

$$\int_\Omega K u_\lambda^{p-1} w_k^2 dx \longrightarrow \int_\Omega K u_\lambda^{p-1} w^2 dx \quad \text{as } k \longrightarrow \infty. \tag{6.18}$$

Therefore

$$\Phi(w) \leq \lim_{n\to\infty} \Phi(w_k) = d \tag{6.19}$$

and $\Phi(w) = d$ which gives that w is a solution of $(6.4)_\lambda$. □

Remark 6.3. From Lemma 6.2, we know that $(6.4)_\lambda$ has a solution $w \in H_0^1(\Omega)$. Now, we also assume that $K(x)$, $f(x)$, and $g(x)$ are in $C^\alpha(\Omega) \cap L^2(\Omega)$, then by Lemmas 4.4 and 4.6, we have that $w \in H_0^2(\Omega)$. The standard elliptic regular theory yields $w \in C^{2,\alpha}(\Omega)$.

Lemma 6.4. *Suppose u_{λ^*} is a solution of $(1.1)_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution u_{λ^*} is unique.*

Proof. Define $F : \mathbb{R} \times H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$F(\lambda,u) = \Delta u - u + \lambda K(u^+)^p + f(x). \tag{6.20}$$

Since $\sigma_\lambda(u_\lambda) \geq \lambda$ for $\lambda \in (0,\lambda^*)$, so $\sigma_{\lambda^*}(u_{\lambda^*}) \geq \lambda^*$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $F_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. From Lemma 6.2, F_u maps $\mathbb{R} \times H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Applying the implicit function theorem to F, we can find a neighborhood $(\lambda^* - \delta, \lambda^* + \delta)$ of λ^* such that $(1.1)_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of λ^*. Hence, we obtain $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$-\Delta w + w = \lambda^* K\left[(w + u_{\lambda^*})^p - u_{\lambda^*}^p\right], \quad w > 0 \text{ in } \Omega. \tag{6.21}$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta \phi + \phi = \lambda^* p K u_{\lambda^*}^{p-1} \phi, \quad \phi \in H_0^1(\Omega) \tag{6.22}$$

possesses a positive solution ϕ_1.

Multiplying (6.21) by ϕ_1 and (6.22) by w, integrating, and subtracting we deduce that

$$\begin{aligned} 0 &= \int_\Omega \lambda^* K\left[(w + u_{\lambda^*})^p - u_{\lambda^*}^p - p u_{\lambda^*}^{p-1} w\right]\phi_1 dx \\ &= \frac{1}{2}p(p-1)\int_\Omega \lambda^* K \xi_{\lambda^*}^{p-2} w^2 \phi_1 dx, \end{aligned} \tag{6.23}$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. Thus $w \equiv 0$. □

Proposition 6.5. *Let u_λ be the minimal solution of $(1.1)_\lambda$. Then u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in [0,\lambda^*]$ and*

$$u_\lambda \longrightarrow u_0 \quad \text{in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+, \tag{6.24}$$

where u_0 is the unique positive solution of $(1.1)_0$.

Proof. By Lemmas 4.8, 3.3, and 6.4, we can deduce $\|u_\lambda\|_{L^\infty(\Omega)} \leq \|u_{\lambda^*}\|_{L^\infty(\Omega)} \leq c$ for $\lambda \in [0,\lambda^*]$. By (3.26), we have $\|u_\lambda\| \leq (p/(p+1))\|f\|_{H^{-1}}$. Hence, u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in [0,\lambda^*]$.

Now, let $w_\lambda = u_\lambda - u_0$, then w_λ satisfies the following equation:

$$-\Delta w_\lambda + w_\lambda = \lambda K u_\lambda^p \quad \text{in } \Omega, \tag{6.8$_\lambda$}$$

and by u_λ being uniformly bounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$, we have

$$\begin{aligned} ||w_\lambda||^2 &= \int_\Omega \lambda K u_\lambda^p w_\lambda dx \\ &\le \lambda ||K||_{L^\infty(\Omega)} ||u_\lambda||_{L^\infty(\Omega)}^{p-1} ||u_\lambda||_{L^2(\Omega)} ||w_\lambda||_{L^2(\Omega)} \le c\lambda, \end{aligned} \tag{6.25}$$

where c is independent of λ. Hence, we obtain $u_\lambda \to u_0$ in $H_0^1(\Omega)$ as $\lambda \to 0^+$.

By Lemma 4.4, $u_\lambda \in L^q(\Omega)$ for all $q \in [2,\infty)$ and u_λ is uniformly bounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$, then for any $q \in [2,\infty)$, there exists a positive constant c_q, independent of u_λ, $\lambda \in [0,\lambda^*]$, such that

$$||K u_\lambda^p||_{L^q(\Omega)} \le c_q. \tag{6.26}$$

Now, let $q_0 = N/2+1 > N/2$ and by Lemma 4.4, we have $\lambda K u_\lambda^p \in L^{q_0}(\Omega)$. Apply Lemmas 4.2, 4.4, 4.6, to $(6.8)_\lambda$ and by (6.25) and (6.26), we obtain

$$\begin{aligned} ||w_\lambda||_{L^\infty(\Omega)} &\le c_1 ||w_\lambda||_{W^{2,q_0}(\Omega)} \\ &\le c_2\Big(||\lambda K u_\lambda^p||_{L^{q_0}(\Omega)} + ||w_\lambda||_{L^{q_0}(\Omega)}\Big) \\ &\le c_3\lambda + c_2 ||w_\lambda||_{L^\infty(\Omega)}^{1-2/q_0} ||w_\lambda||_{L^2(\Omega)}^{2/q_0} \le c\left(\lambda+\lambda^{1/q_0}\right), \end{aligned} \tag{6.27}$$

where c is independent of λ. Hence, we obtain $u_\lambda \to u_0$ in $L^\infty(\Omega)$ as $\lambda \to 0^+$. □

Proposition 6.6. *For $\lambda \in (0,\lambda^*)$, let U_λ be the positive solution of $(1.1)_\lambda$ with $U_\lambda > u_\lambda$, then U_λ is unbounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$, that is,*

$$\lim_{\lambda\to 0^+} ||U_\lambda|| = \lim_{\lambda\to 0^+} ||U_\lambda||_{L^\infty(\Omega)} = \infty. \tag{6.28}$$

Proof. Let φ_λ be a minimizer of $\sigma_\lambda(U_\lambda)$ for $\lambda \in (0,\lambda^*)$, that is,

$$\int_\Omega pKU_\lambda^{p-1}\varphi_\lambda^2 dx = 1, \quad ||\varphi_\lambda||^2 = \sigma_\lambda(U_\lambda). \tag{6.29}$$

(i) First, we show that $\{U_\lambda : \lambda \in (0,\lambda_0)\}$ is unbounded in $L^\infty(\Omega)$ for any $\lambda_0 \in (0,\lambda^*)$. We proceed by contradiction. Assume to the contrary that there exists $c_0 > 0$ such that

$$||U_\lambda||_\infty \le c_0 < \infty \quad \forall \lambda \in (0,\lambda_0), \tag{6.30}$$

by (6.29) and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0,\lambda_0)$, we obtain

$$1 = \int_\Omega pKU_\lambda^{p-1}\varphi_\lambda^2 dx \le c||\varphi_\lambda||^2 = c\sigma_\lambda(U_\lambda) < c\lambda, \tag{6.31}$$

where $c = p\|K\|_{L^\infty(\Omega)} c_0^{p-1}$. This is a contradiction for all $\lambda < 1/c$. Hence, for any $\lambda_0 \in (0,\lambda^*)$, $\{U_\lambda : \lambda \in (0,\lambda^*)\}$ is unbounded in $L^\infty(\Omega)$. From this result, it is easy to see that $\lim_{\lambda\to 0^+} \|U_\lambda\|_\infty = \infty$.

(ii) Now, we show that $\{U_\lambda : \lambda \in (0,\lambda_0)\}$ is unbounded in $H_0^1(\Omega)$ for any $\lambda_0 \in (0,\lambda^*)$. If not, then there exists a constant $c_0 > 0$ independent of λ, such that

$$||U_\lambda|| \le c_0 \quad \forall \lambda \in (0,\lambda_0). \tag{6.32}$$

By (6.29), (6.32), Hölder inequality, Sobolev embedding theorem, and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0,\lambda^*)$, we have

$$\begin{aligned} 1 &= \int_\Omega pKU_\lambda^{p-1}\varphi_\lambda^2 dx \le p\|K\|_{L^\infty(\Omega)}||U_\lambda||_{L^{p+1}(\Omega)}^{p-1}||\varphi_\lambda||_{L^{p+1}(\Omega)}^2 \\ &\le c_1||U_\lambda||^{p-1}||\varphi_\lambda||^2 \le c_1c_0^{p-1}||\varphi_\lambda||^2 = c_1c_0^{p-1}\sigma_\lambda(U_\lambda) < c_1c_0^{p-1}\lambda, \end{aligned} \tag{6.33}$$

where c_1 is a constant independent of λ. Now, let $\lambda \to 0^+$, then we obtain a contradiction. Hence, $\{U_\lambda : \lambda \in (0,\lambda^*)\}$ is unbounded in $H_0^1(\Omega)$ and $\lim_{\lambda\to 0^+} \|U_\lambda\| = +\infty$. □

In order to get bifurcation results we need the following bifurcation theorem which can be found in Crandall and Rabinowitz [6].

Theorem 6.7. *Let X, Y be Banach spaces. Let $(\bar{\lambda},\bar{x}) \in \mathbb{R}\times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda},\bar{x})$ into Y. Let the null space $N(F_x(\bar{\lambda}, \bar{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and codim $R(F_x(\bar{\lambda},\bar{x})) = 1$. Let $F_\lambda(\bar{\lambda},\bar{x}) \notin R(F_x(\bar{\lambda},\bar{x}))$. If Z is the complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda,x) = F_x(\bar{\lambda},\bar{x})$ near $(\bar{\lambda},\bar{x})$ form a curve $(\lambda(s),x(s)) = (\bar{\lambda}+\tau(s),\bar{x}+sx_0+z(s))$, where $s \to (\tau(s),z(s)) \in \mathbb{R}\times Z$ is continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

Proof of Theorems 1.1 and 1.2. Theorem 1.1 now follows from Lemmas 3.2, 3.3, 6.1, 6.4, and Proposition 5.6. The conclusions (i) and (ii) of Theorem 1.2 follow immediately from Lemma 3.3, Remark 6.2 and Propositions 6.5, 6.6. Now we are going to prove that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point in $C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ by using an idea in [13]. We also assume that $K(x)$ and $f(x)$ are in $C^\alpha(\Omega)\cap L^2(\Omega)$ and define

$$F : \mathbb{R}^1 \times C^{2,\alpha}(\Omega)\cap H^2(\Omega) \longrightarrow C^\alpha(\Omega)\cap L^2(\Omega) \tag{6.34}$$

by

$$F(\lambda,u) = \Delta u - u + \lambda K(u^+)^p + f(x), \tag{6.35}$$

where $C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ and $C^\alpha(\Omega)\cap L^2(\Omega)$ are endowed with the natural norm; then they become Banach spaces. It can be verified easily that $F(\lambda,u)$ is differentiable. From Lemma 6.2 and Remark 6.3, we know that

$$F_u(\lambda,u)w = \Delta w - w + \lambda pKu_\lambda^{p-1}w \tag{6.36}$$

is an isomorphism of $\mathbb{R}^1 \times C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ onto $C^\alpha(\Omega)\cap L^2(\Omega)$. It follows from implicit function theorem that the solutions of $F(\lambda,u) = 0$ near (λ,u_λ) are given by a continuous curve.

Now we are going to prove that $(\lambda^*,u_{\lambda^*})$ is a bifurcation point of F. We show first that at the critical point $(\lambda^*,u_{\lambda^*})$, Theorem 6.7 applies. Indeed, from Lemma 6.4, problem

(6.22) has a solution $\phi_1 > 0$ in Ω. By the standard elliptic regular theory, we have $\phi_1 \in C^{2,\alpha}(\Omega) \cap H^2(\Omega)$ if $f \in C^{\alpha}(\Omega) \cap L^2(\Omega)$. Thus $F_u(\lambda^*, u_{\lambda^*})\phi = 0$, $\phi \in C^{2,\alpha}(\Omega) \cap H^2(\Omega)$ has a solution $\phi_1 > 0$. This implies that $N(F_u(\lambda^*, u_{\lambda^*})) = \text{span}\{\phi_1\} = 1$ is one dimensional and codim $R(F_u(\lambda^*, u_{\lambda^*})) = 1$ by the Fredholm alternative. It remains to check that $F_\lambda(\lambda^*, u_{\lambda^*}) \notin R(F_u(\lambda^*, u_{\lambda^*}))$.

Assuming the contrary would imply the existence of $v \not\equiv 0$ such that

$$\Delta v - v + \lambda^* p K u_{\lambda^*}^{p-1} v = K u_{\lambda^*}^p, \quad v \in H_0^1(\Omega). \tag{6.37}$$

From $F_u(\lambda^*, u_{\lambda^*})\phi_1 = 0$, we conclude that $\int_\Omega K u_{\lambda^*}^p \phi_1 dx = 0$. This is impossible because $K(x) \geq 0$, $K(x) \not\equiv 0$, $u_{\lambda^*}(x) > 0$, and $\phi_1(x) > 0$ in Ω.

Applying Theorem 6.7, we conclude that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point near which the solution of $(1.1)_\lambda$ forms a curve $(\lambda^* + \tau(s), u_{\lambda^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. We claim that $\tau''(0) < 0$ which implies that the bifurcation curve turns strictly to the left in (λ, u) plane. In order to obtain that $\tau''(0) < 0$, we need the following lemma. □

Lemma 6.8. *Suppose condition* (k1) *holds, then*

$$\int_\Omega K u_{\lambda^*}^{p-2} \phi_1^3 dx < +\infty. \tag{6.38}$$

Proof. Since $u_{\lambda^*}(x) \to 0$ as $|x| \to \infty$, there is $R_1 > 0$ such that

$$0 = \Delta\phi_1 - \phi_1 + \lambda^* p K u_{\lambda^*}^{p-1} \phi_1 \leq \Delta\phi_1 - \frac{1}{4}\phi_1, \quad \text{for } y \in \omega, \qquad |z| \geq R_1. \tag{6.39}$$

It is well-known that the Dirichlet equation $\Delta w - (1/4)w = -w^p$ in $\mathbb{S}$ has a positive ground-state solution, denoted by $\overline{w}$ (see [14] and the references there). We can modify the proof in Hsu [10] and obtain that for any $\varepsilon > 0$ with $0 < \varepsilon < 1/4 + \mu_1$, there exist constants $c_\varepsilon > 0$ and $R_2 > 0$ such that

$$\overline{w}(y,z) \leq c_\varepsilon \psi(y) \exp\left(-\sqrt{\frac{1}{4} + \mu_1 - \varepsilon}\,|z|\right) \quad \text{for } y \in \omega,\ |z| \geq R_2, \tag{6.40}$$

where ψ is the first positive eigenfunction of the Dirichlet problem $-\Delta\psi = \mu_1\psi$ in ω. Now, let $\varepsilon = (1/2)\mu_1$. Since $\Delta\overline{w} - (1/4)\overline{w} = -\overline{w}^p \leq 0$ in Ω, hence by the maximum principle we obtain that there exist constants $c_1 > 0$ and $R_3 > 0$ such that

$$\phi_1(y,z) \leq c_1 \psi(y) \exp\left(-\frac{1}{2}\sqrt{1 + 2\mu_1}\,|z|\right) \quad \text{for } y \in \omega,\ |z| \geq R_3. \tag{6.41}$$

Let $q \in \partial\Omega$, and B a small ball in Ω such that $q \in \partial B$. Since $\phi_1(x) > 0$ for $x \in B$, $\phi_1(q) = 0$, $u_{\lambda^*}(x) > 0$ for $x \in B$, $u(q) = 0$, by the strongly maximum principle $(\partial\phi_1/\partial x)(q) < 0$, $(\partial u_{\lambda^*}/\partial x)(q) < 0$. Thus

$$\lim_{x \to q} \frac{u_{\lambda^*}(x)}{\phi_1(x)} = \frac{(\partial u_{\lambda^*}/\partial x)(q)}{(\partial\phi_1/\partial x)(q)} > 0, \tag{6.42}$$

and we have $u_{\lambda^*}^{-1}\phi_1 \in C^1(\overline{\Omega})$ and $u_{\lambda^*}^{-1}\phi_1 > 0$ on $\overline{\Omega}$. Therefore, there exists $c_2 > 0$ such that

$$u_{\lambda^*}^{-1}(x)\phi_1(x) \le c_2 \quad \text{for } x \in \Omega_{R_0}, \tag{6.43}$$

where $\Omega_{R_0} = \{x = (y,z) \in \Omega : |z| < R_0\}$.

Now, by (4.9), (6.41), and (6.43), there exists $c_3 > 0$ such that

$$u_{\lambda^*}^{-1}(x)\phi_1^2(x) \le c_3 \quad \text{for } x \in \overline{\Omega}. \tag{6.44}$$

From (6.41), (6.44) and Hölder's inequality, we derive

$$\begin{aligned}
&\int_\Omega K u_{\lambda^*}^{p-2}\phi_1^3 dx \\
&\quad \le c_3 \int_\Omega K u_{\lambda^*}^{p-1}\phi_1 dx \\
&\quad \le c\left(\int_\Omega u_{\lambda^*}^{p+1} dx\right)^{(p-1)/(p+1)} \left(\int_\omega [\psi(y)]^{p+1/2} dy \cdot \int_{\mathbb{R}^n} e^{-(p+1)/4\sqrt{1+2\mu_1}|z|} dz\right)^{2/(p+1)} < \infty.
\end{aligned} \tag{6.45}$$

□

Since $\lambda = \lambda^* + \tau(s)$, $u = u_{\lambda^*} + s\phi_1 + z(s)$ in

$$-\Delta u + u - \lambda K u^p - f = 0, \quad u > 0,\ u \in C^{2,\alpha}(\Omega) \cap H^2(\Omega). \tag{6.46}$$

Differentiating (6.46) in s twice, we have

$$-\Delta u_{ss} + u_{ss} - \lambda p K u^{p-1} u_{ss} - 2\lambda_s p K u^{p-1} u_s - \lambda p(p-1) K u^{p-2} u_s^2 - \lambda_{ss} K u^p = 0. \tag{6.47}$$

Setting here $s = 0$ and using the facts that $\tau'(0) = 0$, $u_s = \phi_1(x)$, and $u = u_{\lambda^*}$ as $s = 0$, we obtain

$$-\Delta u_{ss} + u_{ss} - \lambda^* p K u_{\lambda^*}^{p-1} u_{ss} - \lambda^* p(p-1) K u_{\lambda^*}^{p-2}\phi_1^2 - \tau''(0) K u_{\lambda^*}^p = 0. \tag{6.48}$$

Multiplying $F_u(\lambda^*, u_{\lambda^*})\phi_1 = 0$ by u_{ss} and (6.48) by ϕ_1, integrating, and subtracting the result, and by (6.38) we obtain

$$\int_\Omega \lambda^* p(p-1) K u_{\lambda^*}^{p-2}\phi_1^3 dx + \tau''(0)\int_\Omega K u_{\lambda^*}^p \phi_1 dx = 0, \tag{6.49}$$

which immediately gives $\tau''(0) < 0$. Thus

$$\begin{aligned}
u_\lambda &\longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \text{ as } \lambda \longrightarrow \lambda^*, \\
U_\lambda &\longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \text{ as } \lambda \longrightarrow \lambda^*.
\end{aligned} \tag{6.50}$$

Using Lemma 6.2, Remark 6.3, the implicit function theorem, and the uniqueness of the positive ground-state solution of $(1.1)_0$, we can easily prove that

$$u_\lambda \longrightarrow u_0 \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \text{ as } \lambda \longrightarrow 0^+, \tag{6.51}$$

which proves Theorem 1.2.

Remark 6.9. If $\Omega = \mathbb{S}$, $\Omega = \mathbb{R}^N$, or $\Omega = \mathbb{R}^N \setminus D$, the proof still holds after simple modification.

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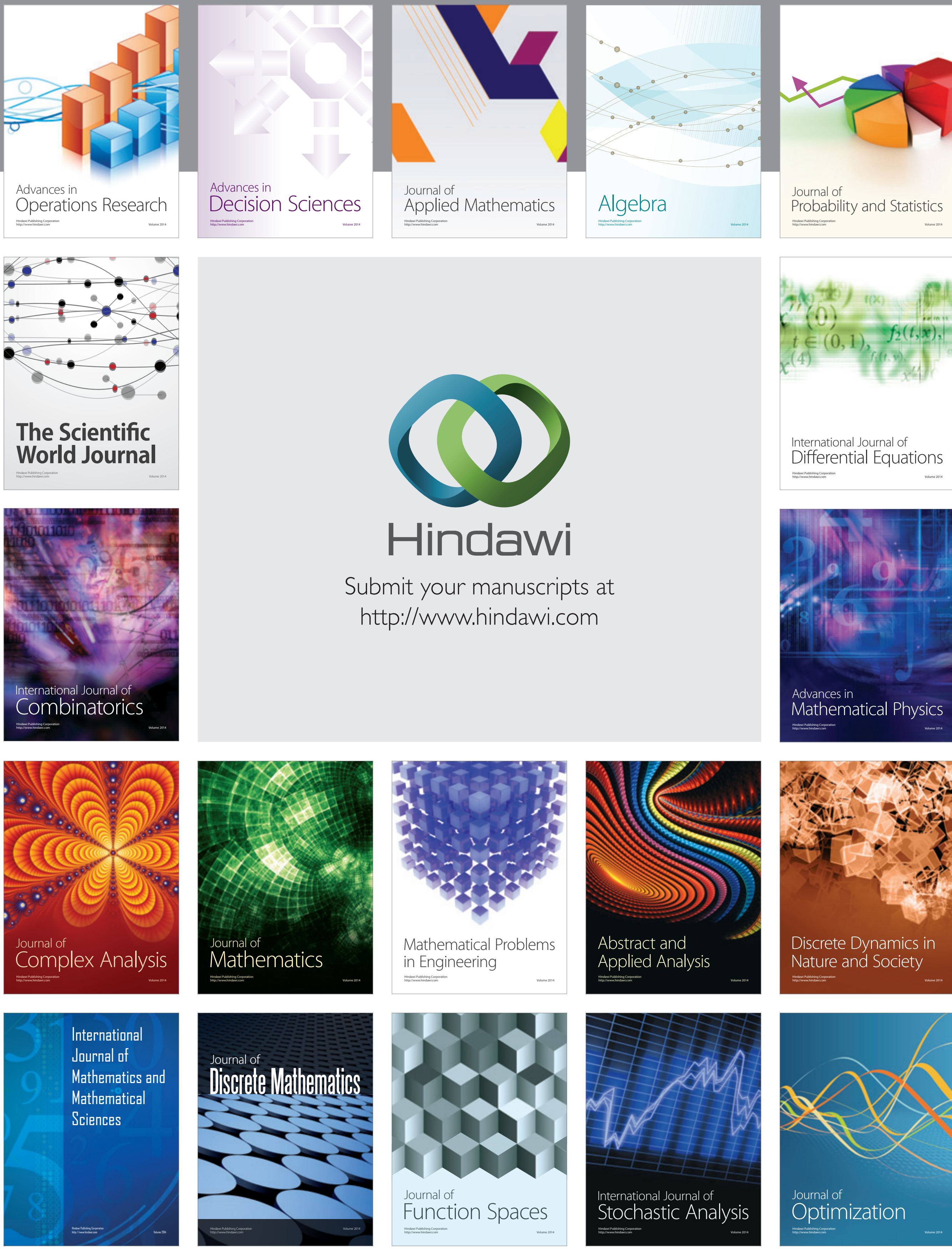
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