CLASSICAL ORTHOGONAL POLYNOMIALS AND LEVERRIER-FADDEEV ALGORITHM FOR THE MATRIX PENCILS $sE - A$

JAVIER HERNÁNDEZ AND FRANCISCO MARCELLÁN

Received 16 May 2005; Revised 27 March 2006; Accepted 25 April 2006

In this contribution we present an extension of the Leverrier-Faddeev algorithm for the simultaneous computation of the determinant and the adjoint matrix $B(s)$ of a pencil $sE - A$ where E is a singular matrix but $\det(sE - A) \not\equiv 0$. Using a previous result by the authors we express $B(s)$ and $\det(sE - A)$ in terms of classical orthogonal polynomials.

1. Introduction

Consider a linear, time-invariant, multivariable singular system described in the state space as follows:

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1.1}$$

where $E \in \mathbb{C}^{n\times n}$ is a singular matrix, x is the n-dimensional state vector, u is the m-dimensional input vector, y is the r-dimensional output vector, and A, B, and C are matrices with complex entries and appropriate dimension.

We can take the Laplace transform of our system (1.1). If $\det(sE - A) \not\equiv 0$, then the following transfer function appears:

$$H(s) = C(sE - A)^{-1}B, \tag{1.2}$$

which, in general, is a strictly proper rational matrix (see [1, 5] and references therein).

The computation of $(sE - A)^{-1}$ can be carried out by using the Cramer rule, which requires the evaluation of n^2 determinants of $(n-1)\times(n-1)$ polynomial matrices. Clearly, this is not a practical procedure for large n. We will describe an extension of the classical Leverrier-Faddeev algorithm using families of classical orthogonal polynomials following our previous contribution [2] when instead of a singular matrix E we used I_n. Here we generalize a recent result [6] based on the Chebyshev polynomials, a very

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 74507, Pages 1–13
DOI 10.1155/IJMMS/2006/74507

particular family of classical orthogonal polynomials. Notice that in [3, 5] an alternative approach using the canonical basis (x^n) in the linear space of polynomials with complex coefficients was given for linear pencils. Along the paper, we will assume that the pencil $sE-A$ is regular, that is, $\det(sE-A)\not\equiv 0$.

The structure of the manuscript is the following. In Section 2 we summarize our algorithm presented in [2] as well as we introduce the basic background about monic classical orthogonal polynomials. In Section 3 we describe the algorithm to find the adjoint matrix $B(s)$ as well as the determinant of a regular pencil $sE-A$, where E is a singular matrix. We also cover a gap in [6] concerning the connection between $\det(sE-A)$ and the adjoint matrix of $(sE-A)$. Finally, in Section 4, some numerical examples in order to test our algorithm will be shown.

2. Leverrier-Faddeev algorithm and classical orthogonal polynomials

For a matrix $A\in\mathbb{C}^{n\times n}$ an algorithm attributed to Leverrier, Faddeev, and others allows the simultaneous determination of the characteristic polynomial of A and the adjoint matrix of sI_n-A. As it is shown in [1], if

$$\begin{aligned} p_A(s) &= \det\left(sI_n-A\right) = s^n+\sum_{k=0}^{n-1}\hat{a}_{n-k}s^k, \\ \widetilde{A}(s) &= \operatorname{Adj}\left(sI_n-A\right) = s^{n-1}I_n+\sum_{k=0}^{n-2}s^k\widehat{B}_{n-k-1}, \end{aligned} \tag{2.1}$$

then the relation between the coefficients $(\hat{a}_k)$ and the matrices $(\widehat{B}_k)$ follows by identification of the coefficients of the monomials in the following two equations:

$$\begin{aligned} (sI_n-A)\widetilde{A}(s) &= p_A(s)I_n, \\ \frac{dp_A(s)}{ds} &= \operatorname{tr}\widetilde{A}(s). \end{aligned} \tag{2.2}$$

From a numerical point of view, the accuracy of this algorithm is not so good. This is the reason why in [2] we have presented an alternative approach using in (2.1) the representation of $p_A(s)$ and $\widetilde{A}(s)$ in terms of a family of monic classical orthogonal polynomials.

The main reason to do it is related to the following fact.

Proposition 2.1 (see [4]). *$(P_n)_{n=0}^{\infty}$ is a family of monic classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) if and only if there exist sequences of real numbers (r_n) and (s_n) such that*

$$P_n(s)=\frac{P'_{n+1}(s)}{n+1}+r_n\frac{P'_n(s)}{n}+s_n\frac{P'_{n-1}(s)}{n-1}\quad \text{for } n\geqslant 2. \tag{2.3}$$

The coefficients that appear in (2.3) are given in Table 2.1.

Notice that the Hermite case appears when $r_n=s_n=0$, $n\geqslant 2$. The Laguerre case appears when $s_n=0$, $n\geqslant 2$. Finally, the Jacobi and the Bessel cases are related to the case $s_n\neq 0$ for every $n\geqslant 2$.

Table 2.1. Coefficients in the relation of Proposition 2.1.

	r_n	s_n
Hermite	0	0
Laguerre	n	0
Jacobi	$\dfrac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$-\dfrac{4n(n-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$
Bessel	$\dfrac{4n}{(2n+\alpha)(2n+\alpha+2)}$	$\dfrac{4n(n-1)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}$

Table 2.2. Coefficients in the three-term recurrence relation (2.4).

	β_n	γ_n
Hermite	0	$\dfrac{n}{2}$
Laguerre	$2n+\alpha+1$	$n(n+\alpha)$
Jacobi	$\dfrac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\dfrac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$
Bessel	$-\dfrac{2\alpha}{(2n+\alpha)(2n+\alpha+2)}$	$-\dfrac{4n(n+\alpha)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}$

The second ingredient for our algorithm is the fact that if $(P_n)_{n=0}^{\infty}$ is a family of monic classical orthogonal polynomials, then the following three-term recurrence relation holds:

$$\begin{gathered} sP_n(s) = P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s), \quad n \geqslant 1 \text{ with } \gamma_n \neq 0, \\ P_0(s) = 1, \qquad P_1(s) = s - \beta_0. \end{gathered} \tag{2.4}$$

The coefficients that appear in (2.4) are given in Table 2.2.

If we expand the characteristic polynomial $p_A(s)$ of A as well as the adjoint matrix $\widetilde{A}(s)$ of $sI_n - A$ in terms of the above basis of monic classical orthogonal polynomials, that is,

$$p_A(s) = P_n(s) + \sum_{k=0}^{n-1} \hat{a}_{n-k} P_k(s), \qquad \widetilde{A}(s) = P_{n-1}(s) I_n + \sum_{k=0}^{n-2} P_k(s) \widehat{B}_{n-k-1}, \tag{2.5}$$

and take into account (2.2) together with (2.3) and (2.4), then we get the following.

Proposition 2.2 (see [2]). (i) *For* $k = 1, \dots, n$,

$$k\hat{a}_k = (\beta_{n-k} - r_{n-k}) \operatorname{tr} \widehat{B}_{k-1} + (\gamma_{n-k+1} - s_{n-k+1}) \operatorname{tr} \widehat{B}_{k-2} - \operatorname{tr}(A\widehat{B}_{k-1}); \tag{2.6}$$

Data: $\{\beta_k\}_{k=0}^{n-1}$, $\{\gamma_k\}_{k=1}^{n}$, $\{r_k\}_{k=0}^{n-1}$, $\{s_k\}_{k=1}^{n}$.
Initial Condition: $\widehat{B}_{-1} = 0$, $\widehat{B}_0 = I_n$.
For $k = 1,2,\dots,n-1$

$$\begin{aligned}\widehat{a}_k &= (1/k)\big[(\beta_{n-k} - r_{n-k})\operatorname{tr}\widehat{B}_{k-1} + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr}\widehat{B}_{k-2} - \operatorname{tr}(A\widehat{B}_{k-1})\big],\\ \widehat{B}_k &= A\widehat{B}_{k-1} + \widehat{a}_k I_n - \gamma_{n-k+1}\widehat{B}_{k-2} - \beta_{n-k}\widehat{B}_{k-1}.\end{aligned} \tag{2.8}$$

End (For)

$$\widehat{a}_n = (1/n)\big[(\beta_0 - r_0)\operatorname{tr}\widehat{B}_{n-1} + (\gamma_1 - s_1)\operatorname{tr}\widehat{B}_{n-2} - \operatorname{tr}(A\widehat{B}_{n-1})\big]. \tag{2.9}$$

Algorithm 2.1

(ii) *for* $k = 1, 2, \dots, n-1$,

$$\widehat{B}_k = A\widehat{B}_{k-1} + \widehat{a}_k I_n - \gamma_{n-k+1}\widehat{B}_{k-2} - \beta_{n-k}\widehat{B}_{k-1}, \tag{2.7}$$

with the convention $\widehat{B}_{-1} = 0$, $r_0 = 0$, $s_1 = 0$.

Indeed the algorithm to find (a_k) and (B_k) is in Algorithm 2.1.

3. Regular pencils

Now, we are interested in the computation of $a(s) = \det(sE - A)$, assuming $sE - A$ is a regular pencil, and $B(s) = \operatorname{Adj}(sE - A)$, where $A, E \in \mathbb{C}^{n\times n}$ and E is a singular matrix. If in the expressions of the previous section we replace A by $A(s) = -sE + A$, then we get

$$\widetilde{a}(\lambda, s) := \det(\lambda I_n - A(s)) = P_n(\lambda) + \sum_{k=0}^{n-1}\widehat{a}_{n-k}(s)P_k(\lambda) \tag{3.1}$$

as well as

$$\widetilde{B}(\lambda, s) := \operatorname{Adj}(\lambda I_n - A(s)) = P_{n-1}(\lambda)I_n + \sum_{k=0}^{n-2}P_k(\lambda)\widehat{B}_{n-k-1}(s). \tag{3.2}$$

Thus, from (2.6) and (2.7) we get

$$\begin{aligned}k\widehat{a}_k(s) &= (\beta_{n-k} - r_{n-k})\operatorname{tr}\widehat{B}_{k-1}(s) - \operatorname{tr}(A(s)\widehat{B}_{k-1}(s))\\ &\quad + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr}\widehat{B}_{k-2}(s), \quad k = 1,\dots,n\end{aligned} \tag{3.3}$$

as well as

$$\widehat{B}_k(s) = \widehat{a}_k(s)I_n - \gamma_{n-k+1}\widehat{B}_{k-2}(s) - \beta_{n-k}\widehat{B}_{k-1}(s) + A(s)\widehat{B}_{k-1}(s) \tag{3.4}$$

for $k = 1,\dots,n-1$. Thus, if $\lambda = 0$ in (3.1) and (3.2), then we get

$$a(s) := \det(sE - A) = \widetilde{a}(0,s) = P_n(0) + \sum_{k=0}^{n-1} \widehat{a}_{n-k}(s)P_k(0), \tag{3.5}$$

$$B(s) := \operatorname{Adj}\,(sE - A) = \widetilde{B}(0,s) = P_{n-1}(0)I_n + \sum_{k=0}^{n-2} P_k(0)\widehat{B}_{n-k-1}(s). \tag{3.6}$$

Taking into account $\deg(P_k(s)) = k$ for all $k \geqslant 0$, (3.3), and (3.4), we can assure that the degrees of the polynomial $\widehat{a}_k(s)$, $k = 1,2,\dots,n$, and the polynomial matrix $\widehat{B}_k(s)$, $k = 1, 2,\dots, n-1$, are at most equal to k. Thus for $\widehat{a}_k(s)$ and $\widehat{B}_k(s)$ we get the expansions

$$\begin{aligned} \widehat{a}_k(s) &= \sum_{j=0}^{k} a_{k,j}P_j(s), \quad a_{k,j} \in \mathbb{C}, \\ \widehat{B}_k(s) &= \sum_{j=0}^{k} P_j(s)B_{k,j}, \quad B_{k,j} \in \mathbb{C}^{n\times n}. \end{aligned} \tag{3.7}$$

Substituting (3.7) in (3.3), we get

$$\begin{aligned} k\sum_{j=0}^{k} a_{k,j}P_j(s) = \operatorname{tr}\Bigg(&(\beta_{n-k} - r_{n-k}) \sum_{j=0}^{k-1} P_j(s)B_{k-1,j} + (\gamma_{n-k+1} - s_{n-k+1}) \sum_{j=0}^{k-2} P_j(s)B_{k-2,j} \\ &+ (sE - A)\sum_{j=0}^{k-1} P_j(s)B_{k-1,j}\Bigg). \end{aligned} \tag{3.8}$$

Applying in the right-hand side the three-term recurrence relation, we get

$$\begin{aligned} k\sum_{j=0}^{k} a_{k,j}P_j(s) = {}& \operatorname{tr}(EB_{k-1,k-1})P_k(s) \\ &+ \big[(\beta_{n-k} - r_{n-k})\operatorname{tr} B_{k-1,k-1} + \beta_{k-1}\operatorname{tr}(EB_{k-1,k-1}) \\ &\quad - \operatorname{tr}(AB_{k-1,k-1}) + \operatorname{tr}(EB_{k-1,k-2})\big]P_{k-1}(s) \\ &+ \sum_{j=1}^{k-2}\big[\gamma_{j+1}\operatorname{tr}(EB_{k-1,j+1}) + \beta_j \operatorname{tr}(EB_{k-1,j}) \\ &\qquad + (\beta_{n-k} - r_{n-k})\operatorname{tr} B_{k-1,j} + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr} B_{k-2,j} \\ &\qquad - \operatorname{tr}(AB_{k-1,j}) + \operatorname{tr}(EB_{k-1,j-1})\big]P_j(s) \\ &+ \big[\gamma_1 \operatorname{tr}(EB_{k-1,1}) + \beta_0 \operatorname{tr}(EB_{k-1,0}) + (\beta_{n-k} - r_{n-k})\operatorname{tr} B_{k-1,0} \\ &\quad + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr} B_{k-2,0} - \operatorname{tr}(AB_{k-1,0})\big]P_0(s). \end{aligned} \tag{3.9}$$

Thus, for $k = 1,2,\dots,n$,

$$\begin{aligned}
ka_{k,0} &= \gamma_1 \operatorname{tr}(EB_{k-1,1}) + \beta_0 \operatorname{tr}(EB_{k-1,0}) + (\beta_{n-k} - r_{n-k}) \operatorname{tr} B_{k-1,0} \\
&\quad - \operatorname{tr}(AB_{k-1,0}) + (\gamma_{n-k+1} - s_{n-k+1}) \operatorname{tr} B_{k-2,0}, \\
&\;\;\vdots \\
ka_{k,j} &= \gamma_{j+1} \operatorname{tr}(EB_{k-1,j+1}) + \beta_j \operatorname{tr}(EB_{k-1,j}) + \operatorname{tr}(EB_{k-1,j-1}) \\
&\quad + (\gamma_{n-k+1} - s_{n-k+1}) \operatorname{tr} B_{k-2,j} + (\beta_{n-k} - r_{n-k}) \operatorname{tr} B_{k-1,j} \\
&\quad - \operatorname{tr}(AB_{k-1,j}), \quad j = 1,\dots,k-2, \\
&\;\;\vdots \\
ka_{k,k-1} &= (\beta_{n-k} - r_{n-k}) \operatorname{tr} B_{k-1,k-1} + \operatorname{tr}(EB_{k-1,k-2}) \\
&\quad + \beta_{k-1} \operatorname{tr}(EB_{k-1,k-1}) - \operatorname{tr}(AB_{k-1,k-1}), \\
ka_{k,k} &= \operatorname{tr}(EB_{k-1,k-1}).
\end{aligned} \tag{3.10}$$

In an analogous way, substituting (3.7) in (3.4),

$$\begin{aligned}
\sum_{j=0}^{k} P_j(s) B_{k,j} &= \sum_{j=0}^{k} a_{k,j} P_j(s) I_n - \gamma_{n-k+1} \sum_{j=0}^{k-2} P_j(s) B_{k-2,j} \\
&\quad - \beta_{n-k} \sum_{j=0}^{k-1} P_j(s) B_{k-1,j} + (-sE + A) \sum_{j=0}^{k-1} P_j(s) B_{k-1,j}.
\end{aligned} \tag{3.11}$$

Using again the three-term recurrence relation, we get

$$\begin{aligned}
\sum_{j=0}^{k} P_j(s) B_{k,j} &= P_k(s)\left[a_{k,k} I_n - EB_{k-1,k-1}\right] \\
&\quad + P_{k-1}(s)\left[a_{k,k-1} I_n - EB_{k-1,k-2} + (A - \beta_{k-1}E - \beta_{n-k} I_n) B_{k-1,k-1}\right] \\
&\quad + \sum_{j=1}^{k-2} P_j(s)\big[a_{k,j} I_n - EB_{k-1,j-1} + (A - \beta_j E - \beta_{n-k} I_n) B_{k-1,j} \\
&\qquad - \gamma_{j+1} EB_{k-1,j+1} - \gamma_{n-k+1} B_{k-2,j}\big] \\
&\quad + P_0(s)\big[a_{k,0} I_n + (A - \beta_0 E - \beta_{n-k} I_n) B_{k-1,0} \\
&\qquad - \gamma_1 EB_{k-1,1} - \gamma_{n-k+1} B_{k-2,0}\big].
\end{aligned} \tag{3.12}$$

Data: $\{\beta_k\}_{k=0}^{n-1}$, $\{\gamma_k\}_{k=1}^{n}$, $\{r_k\}_{k=0}^{n-1}$, $\{s_k\}_{k=1}^{n}$.

Initial Condition: $B_{i,j} = 0$, if $i < j$ or $j < 0$, $a_{0,0} = 1$, $B_{0,0} = I_n$.

For $k = 1, \dots, n-1$

$\alpha_{n-k} = \beta_{n-k} - r_{n-k}$.

$\delta_{n-k+1} = \gamma_{n-k+1} - s_{n-k+1}$.

$A_k = A - \beta_{n-k} I_n$.

For $j = 0, 1, \dots, k$

$$\begin{aligned} a_{k,j} := (1/k)[&\gamma_{j+1}\operatorname{tr}(EB_{k-1,j+1}) + \beta_j \operatorname{tr}(EB_{k-1,j}) + \alpha_{n-k}\operatorname{tr} B_{k-1,j} \\ &+ \operatorname{tr}(EB_{k-1,j-1}) + \delta_{n-k+1}\operatorname{tr} B_{k-2,j} - \operatorname{tr}(AB_{k-1,j})]. \end{aligned}$$

$$\begin{aligned} B_{k,j} := {}& a_{k,j} I_n - EB_{k-1,j-1} + (A_k - \beta_j E)B_{k-1,j} - \gamma_{j+1} EB_{k-1,j+1} \\ & - \gamma_{n-k+1} B_{k-2,j}. \end{aligned}$$

End (For j).

End (For k).

For $j = 0, 1, \dots, n$

$$\begin{aligned} a_{n,j} := (1/n)[&\gamma_{j+1}\operatorname{tr}(EB_{n-1,j+1}) + \beta_j \operatorname{tr}(EB_{n-1,j}) + \beta_0 \operatorname{tr} B_{n-1,j} \\ &+ \operatorname{tr}(EB_{n-1,j-1}) + \gamma_1 \operatorname{tr} B_{n-2,j} - \operatorname{tr}(AB_{n-1,j})]. \end{aligned}$$

End.

Algorithm 3.1

Thus, for $k = 1, 2, \dots, n-1$,

$$\begin{aligned} B_{k,0} &= a_{k,0} I_n + (A - \beta_0 E - \beta_{n-k} I_n) B_{k-1,0} - \gamma_1 EB_{k-1,1} - \gamma_{n-k+1} B_{k-2,0}, \\ &\vdots \\ B_{k,j} &= a_{k,j} I_n - EB_{k-1,j-1} + (A - \beta_j E - \beta_{n-k} I_n) B_{k-1,j} \\ &\quad - \gamma_{j+1} EB_{k-1,j+1} - \gamma_{n-k+1} B_{k-2,j}, \quad j = 1, \dots, k-2, \\ &\vdots \\ B_{k,k-1} &= a_{k,k-1} I_n - EB_{k-1,k-2} + (A - \beta_{k-1} E - \beta_{n-k} I_n) B_{k-1,k-1}, \\ B_{k,k} &= a_{k,k} I_n - EB_{k-1,k-1}. \end{aligned} \tag{3.13}$$

As a conclusion, the algorithm for the computation of the coefficients $a_{i,j}$ in (3.5) and $B_{i,j}$ in (3.6) is as in Algorithm 3.1.

Notice that formula (3.10) in [6] is not right as a simple computation shows. Indeed for a regular pencil it is enough to consider the expression of $a(s)$ and $B(s)$ in the example provided in [6, Section 4].

Next we will give the right result.

THEOREM 3.1. *Let* $A,E \in \mathbb{C}^{n\times n}$, $a(s) = \det(sE - A)$, *and* $B(s) = \operatorname{Adj}(sE - A)$. *Then*

$$\frac{d}{ds}a(s) = \operatorname{tr}\big(EB(s)\big). \tag{3.14}$$

Proof. First, assume that E is a nonsingular matrix. Then $sE - A = (sI_n - AE^{-1})E$ and

$$\begin{aligned}\frac{d}{ds}a(s) &= \det(E)\frac{d}{ds}\big(\det\big(sI_n - AE^{-1}\big)\big)\\ &= \det(E)\operatorname{tr}\big(\operatorname{Adj}\big(sI_n - AE^{-1}\big)\big)\\ &= \det(E)\det(E)^{-1}\det(sE-A)\operatorname{tr}\big(E(sE-A)^{-1}\big)\\ &= \det(sE-A)\operatorname{tr}\big(E(sE-A)^{-1}\big)\\ &= \operatorname{tr}\big(EB(s)\big).\end{aligned} \tag{3.15}$$

Next, if E is a singular matrix, then consider $\varepsilon > 0$, such that $\varepsilon < \min\{|\lambda_i| : \lambda_i$ is an eigenvalue of $E, \lambda_i \neq 0\}$.

Then $E_\varepsilon := E + \varepsilon I_n$ is a nonsingular matrix. Using the first part of the proof,

$$\frac{d}{ds}a_\varepsilon(s) = \operatorname{tr}\big(E_\varepsilon B_\varepsilon(s)\big), \tag{3.16}$$

where $a_\varepsilon(s) = \det(sE_\varepsilon - A)$ and $B_\varepsilon(s) := \operatorname{Adj}(sE_\varepsilon - A)$.

Taking into account $E_\varepsilon \to E$, $a_\varepsilon(s) \to a(s)$, and $B_\varepsilon(s) \to B(s)$, when $\varepsilon \to 0$, we deduce our statement. □

4. Examples

Let $A,E \in \mathbb{C}^{3\times 3}$ given by

$$A = \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{4.1}$$

Notice that $\operatorname{rank} E = 2$. It is straightforward to prove that

$$\begin{gathered} a(s) = \det(sE - A) = -s^2,\\ B(s) = \operatorname{Adj}(sE - A) = \begin{bmatrix} -s & 0 & s\\ 0 & -s & s\\ s & s & s^2 - 2s\end{bmatrix}.\end{gathered} \tag{4.2}$$

Applying the algorithm of the previous section for Hermite polynomials $\{H_k(s)\}_{k=0}^{n}$, we get

$$
\begin{aligned}
&a_{1,0} = -\operatorname{tr} A = -3, \qquad a_{1,1} = \operatorname{tr} E = 2;\\
&B_{1,0} = a_{1,0}I_3 + A = \begin{bmatrix} -2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2 \end{bmatrix}, \qquad B_{1,1} = a_{1,1}I_3 - E = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2 \end{bmatrix};\\
&a_{2,0} = \frac{1}{2}\left[\frac{1}{2}\operatorname{tr}(EB_{1,1}) - \operatorname{tr}(AB_{1,0}) + 3\right] = 2,\\
&a_{2,1} = \frac{1}{2}\left[\operatorname{tr}(EB_{1,0}) - \operatorname{tr}(AB_{1,1})\right] = -4,\\
&a_{2,2} = \frac{1}{2}\operatorname{tr}(EB_{1,1}) = 1;\\
&B_{2,0} = a_{2,0}I_3 + AB_{1,0} - \frac{1}{2}EB_{1,1} - I_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 \end{bmatrix},\\
&B_{2,1} = a_{2,1}I_3 + AB_{1,1} - EB_{1,0} = \begin{bmatrix} -1 & 0 & 1\\ 0 & -1 & 1\\ 1 & 1 & -2 \end{bmatrix},\\
&B_{2,2} = a_{2,2}I_3 - EB_{1,1} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix};\\
&a_{3,0} = \frac{1}{3}\left[\frac{1}{2}\operatorname{tr}(EB_{2,1}) - \operatorname{tr}(AB_{2,0}) + \frac{1}{2}\operatorname{tr} B_{1,0}\right] = -2,\\
&a_{3,1} = \frac{1}{3}\left[\operatorname{tr}(EB_{2,2}) - \operatorname{tr}(AB_{2,1}) + \operatorname{tr}(EB_{2,0}) + \frac{1}{2}\operatorname{tr} B_{1,1}\right] = 1,\\
&a_{3,2} = \frac{1}{3}\left[\operatorname{tr}(EB_{2,1}) - \operatorname{tr}(AB_{2,2})\right] = -1,\\
&a_{3,3} = \frac{1}{3}\operatorname{tr}(EB_{2,2}) = 0.
\end{aligned}
\tag{4.3}
$$

Thus

$$
\begin{aligned}
\hat{a}_1(s) &= a_{1,0}H_0(s) + a_{1,1}H_1(s) = -3H_0(s) + 2H_1(s),\\
\hat{a}_2(s) &= a_{2,0}H_0(s) + a_{2,1}H_1(s) + a_{2,2}H_2(s) = 2H_0(s) - 4H_1(s) + H_2(s),\\
\hat{a}_3(s) &= a_{3,0}H_0(s) + a_{3,1}H_1(s) + a_{3,2}H_2(s) + a_{3,3}H_3(s) = -2H_0(s) + H_1(s) - H_2(s);\\
\widehat{B}_1(s) &= H_0(s)B_{1,0} + H_1(s)B_{1,1},\\
\widehat{B}_2(s) &= H_0(s)B_{2,0} + H_1(s)B_{2,1} + H_2(s)B_{2,2}.
\end{aligned}
\tag{4.4}
$$

Now, the determinant $a(s)$ and the adjoint $B(s)$ of $sE - A$ are given by

$$\begin{aligned} a(s) &= H_3(0) + \hat{a}_1(s)H_2(0) + \hat{a}_2(s)H_1(0) + \hat{a}_3(s)H_0(0) \\ &= -\frac{1}{2}\hat{a}_1(s) + \hat{a}_3(s) = -H_2(s) - \frac{1}{2}H_0(s), \\ B(s) &= H_2(0)\widehat{B}_0(s) + H_1(0)\widehat{B}_1(s) + H_0(0)\widehat{B}_2(s) = -\frac{1}{2}I_3 + \widehat{B}_2(s) \\ &= H_0(s)\left[-\frac{1}{2}I_3 + B_{2,0}\right] + H_1(s)B_{2,1} + H_2(s)B_{2,2}. \end{aligned} \tag{4.5}$$

Next, applying the algorithm for the family $\{L_k^{\alpha}(s)\}_{k=0}^{n}$ (Laguerre polynomials with parameter α), we get

$$a_{1,0} = (1+\alpha)\operatorname{tr}E + 3(3+\alpha) - \operatorname{tr}A = 8 + 5\alpha, \qquad a_{1,1} = \operatorname{tr}E = 2;$$

$$B_{1,0} = (a_{1,0} - 5 - \alpha)I_3 + A - (1+\alpha)E = \begin{bmatrix} 3+3\alpha & 1 & 1 \\ 1 & 3+3\alpha & 1 \\ 1 & 1 & 4+4\alpha \end{bmatrix},$$

$$B_{1,1} = a_{1,1}I_3 - E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

$$\begin{aligned} a_{2,0} &= \frac{1}{2}\left[(1+\alpha)\left(\operatorname{tr}(EB_{1,1}) + \operatorname{tr}(EB_{1,0})\right) + (2+\alpha)\left(\operatorname{tr}B_{1,0} + 6\right) - \operatorname{tr}(AB_{1,0})\right] \\ &= 4(1+\alpha)(3+2\alpha), \\ a_{2,1} &= \frac{1}{2}\left((2+\alpha)\operatorname{tr}B_{1,1} + \operatorname{tr}(EB_{1,0}) + (3+\alpha)\operatorname{tr}(EB_{1,1}) - \operatorname{tr}(AB_{1,1})\right) = 8 + 6\alpha, \\ a_{2,2} &= \frac{1}{2}\operatorname{tr}(EB_{1,1}) = 1; \\ B_{2,0} &= (a_{2,0} - 4 - 2\alpha)I_3 + \left(A - (1+\alpha)E - (3+\alpha)I_3\right)B_{1,0} - (1+\alpha)EB_{1,1} \\ &= (1+\alpha)\begin{bmatrix} 2\alpha & 1 & 2 \\ 1 & 2\alpha & 2 \\ 2 & 2 & 2+4\alpha \end{bmatrix}, \end{aligned}$$

$$B_{2,1} = a_{2,1}I_3 + EB_{1,0} + \left(A - (3+\alpha)(E + I_3)\right)B_{1,1} = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 1 \\ 1 & 1 & 4+4\alpha \end{bmatrix},$$

$$
\begin{aligned}
B_{2,2} &= a_{2,2}I_3 - EB_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \\
a_{3,0} &= \frac{1}{3}\left[(1+\alpha)\left(\operatorname{tr}\left(EB_{2,1}\right) + \operatorname{tr}\left(EB_{2,0}\right) + \operatorname{tr} B_{2,0} + \operatorname{tr} B_{1,0}\right) - \operatorname{tr}\left(AB_{2,0}\right)\right] \\
&= 2\alpha(1+\alpha)(3+2\alpha), \\
a_{3,1} &= \frac{1}{3}\big(2(2+\alpha)\operatorname{tr}\left(EB_{2,2}\right) + (3+\alpha)\operatorname{tr}\left(EB_{2,1}\right) + \operatorname{tr}\left(EB_{2,0}\right) + +(1+\alpha)\operatorname{tr} B_{2,1} \\
&\quad + (1+\alpha)\operatorname{tr} B_{1,1} - \operatorname{tr}\left(AB_{2,1}\right)\big) = 2\alpha(3+2\alpha), \\
a_{3,2} &= \frac{1}{3}\left((1+\alpha)\operatorname{tr} B_{2,2} + \operatorname{tr}\left(EB_{2,1}\right) + (5+\alpha)\operatorname{tr}\left(EB_{2,2}\right) - \operatorname{tr}\left(AB_{2,2}\right)\right) = \alpha, \\
a_{3,3} &= \frac{1}{3}\operatorname{tr}\left(EB_{2,2}\right) = 0.
\end{aligned}
\tag{4.6}
$$

Thus

$$
\begin{aligned}
\hat{a}_1(s) &= a_{1,0}L_0^{\alpha}(s) + a_{1,1}L_1^{\alpha}(s) = (8+5\alpha)L_0^{\alpha}(s) + 2L_1^{\alpha}(s), \\
\hat{a}_2(s) &= a_{2,0}L_0^{\alpha}(s) + a_{2,1}L_1^{\alpha}(s) + a_{2,2}L_2^{\alpha}(s) \\
&= 4(1+\alpha)(3+2\alpha)L_0^{\alpha}(s) + (8+6\alpha)L_1^{\alpha}(s) + L_2^{\alpha}(s), \\
\hat{a}_3(s) &= a_{3,0}L_0^{\alpha}(s) + a_{3,1}L_1^{\alpha}(s) + a_{3,2}L_2^{\alpha}(s) + a_{3,3}L_3^{\alpha}(s) \\
&= 2\alpha(1+\alpha)(3+2\alpha)L_0^{\alpha}(s) + 2\alpha(3+2\alpha)L_1^{\alpha}(s) + \alpha L_2^{\alpha}(s); \\
\hat{B}_1(s) &= L_0^{\alpha}(s)B_{1,0} + L_1^{\alpha}(s)B_{1,1}, \\
\hat{B}_2(s) &= L_0^{\alpha}(s)B_{2,0} + L_1^{\alpha}(s)B_{2,1} + L_2^{\alpha}(s)B_{2,2}.
\end{aligned}
\tag{4.7}
$$

The determinant $a(s)$ and the adjoint $B(s)$ of $sE - A$ are given by

$$
\begin{aligned}
a(s) &= L_3^{\alpha}(0) + \hat{a}_1(s)L_2^{\alpha}(0) + \hat{a}_2(s)L_1^{\alpha}(0) + \hat{a}_3(s)L_0^{\alpha}(0) \\
&= -(1+\alpha)(2+\alpha)L_0^{\alpha}(s) - 2(2+\alpha)L_1^{\alpha}(s) - L_2^{\alpha}(s), \\
B(s) &= L_2^{\alpha}(0)\hat{B}_0(s) + L_1^{\alpha}(0)\hat{B}_1(s) + L_0^{\alpha}(0)\hat{B}_2(s) \\
&= L_0^{\alpha}(s)\left[(1+\alpha)(2+\alpha)I_3 - (1+\alpha)B_{1,0} + B_{2,0}\right] \\
&\quad + L_1^{\alpha}(s)\left[-(1+\alpha)B_{1,1} + B_{2,1}\right] + L_2^{\alpha}(s)B_{2,2}.
\end{aligned}
\tag{4.8}
$$

Finally, if we consider the family $\{T_k(s)\}_{k=0}^{n}$ of the Chebyshev polynomials of first kind, applying the algorithm we get

$$
\begin{aligned}
&a_{1,0} = -\operatorname{tr} A = -3, \qquad a_{1,1} = \operatorname{tr} E = 2;\\
&B_{1,0} = a_{1,0} I_3 + A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \qquad B_{1,1} = a_{1,1} I_3 - E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};\\
&a_{2,0} = \frac{1}{2}\left(\frac{1}{4}\operatorname{tr}(EB_{1,1}) - \operatorname{tr}(AB_{1,0}) + \frac{3}{2}\right) = \frac{5}{4},\\
&a_{2,1} = \frac{1}{2}\left(\operatorname{tr}(EB_{1,0}) - \operatorname{tr}(AB_{1,1})\right) = -4, \qquad a_{2,2} = \frac{1}{2}\left(\operatorname{tr}(EB_{1,1})\right) = 1;\\
&B_{2,0} = a_{2,0} I_3 + AB_{1,0} - \frac{1}{4}EB_{1,1} - \frac{1}{4}I_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\\
&B_{2,1} = a_{2,1} I_3 - EB_{1,0} + AB_{1,1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix},\\
&B_{2,2} = a_{2,2} I_3 - EB_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};\\
&a_{3,0} = \frac{1}{3}\left(\frac{1}{4}\operatorname{tr}(EB_{2,1}) + \frac{1}{2}\operatorname{tr} B_{1,0} - \operatorname{tr}(AB_{2,0})\right) = -2,\\
&a_{3,1} = \frac{1}{3}\left(\frac{1}{4}\operatorname{tr}(EB_{2,2}) + \operatorname{tr}(EB_{2,0}) + \frac{1}{2}\operatorname{tr} B_{1,1} - \operatorname{tr}(AB_{2,1})\right) = 1,\\
&a_{3,2} = \frac{1}{3}\left(\operatorname{tr}(EB_{2,1}) - \operatorname{tr}(AB_{2,2})\right) = -1,\\
&a_{3,3} = \frac{1}{3}\operatorname{tr}(EB_{2,2}) = 0.
\end{aligned}
\tag{4.9}
$$

Thus

$$
\begin{aligned}
\hat{a}_1(s) &= a_{1,0} T_0(s) + a_{1,1} T_1(s) = -3T_0(s) + 2T_1(s),\\
\hat{a}_2(s) &= a_{2,0} T_0(s) + a_{2,1} T_1(s) + a_{2,2} T_2(s) = \frac{5}{4} T_0(s) - 4T_1(s) + T_2(s),\\
\hat{a}_3(s) &= a_{3,0} T_0(s) + a_{3,1} T_1(s) + a_{3,2} T_2(s) + a_{3,3} T_3(s) = -2T_0(s) + T_1(s) - T_2(s);\\
\hat{B}_1(s) &= T_0(s) B_{1,0} + T_1(s) B_{1,1}, \hat{B}_2(s) = T_0(s) B_{2,0} + T_1(s) B_{2,1} + T_2(s) B_{2,2}.
\end{aligned}
\tag{4.10}
$$

The determinant $a(s)$ and the adjoint $B(s)$ of $sE - A$ are given by

$$\begin{aligned} a(s) &= T_3(0) + \hat{a}_1(s)T_2(0) + \hat{a}_2(s)T_1(0) + \hat{a}_3(s)T_0(0) \\ &= -\frac{1}{2}T_0(s) - T_2(s), \\ B(s) &= T_2(0)\hat{B}_0(s) + T_1(0)\hat{B}_1(s) + T_0(0)\hat{B}_2(s) \\ &= T_0(s)\big(B_{2,0} - \frac{1}{2}I_3\big) + T_1(s)B_{2,1} + T_2(s)B_{2,2}. \end{aligned} \tag{4.11}$$

Acknowledgments

We thank the comments and suggestions by the anonymous referees in order to improve the presentation of the manuscript. The work of the second author (Francisco Marcellán) has received the financial support from Dirección General de Investigación (Ministerio de Educación y Ciencia) of Spain, Grant BFM2003-06335-C03-02, and INTAS Research Network NeCCA INTAS 03-51-6637. The work of the first author (Javier Hernández) has been supported by Fundación Universidad Carlos III de Madrid.

References

[1] S. Barnett, *Leverrier's algorithm for orthogonal polynomial bases*, Linear Algebra and Its Applications **236** (1996), 245–263.

[2] J. Hernández, F. Marcellán, and C. Rodríguez, *Leverrier-Fadeev algorithm and classical orthogonal polynomials*, Revista Academia Colombiana de Ciencias Exactas, Físicas y Naturales **28** (2004), no. 106, 39–47.

[3] F. L. Lewis, *Further remarks on the Cayley-Hamilton theorem and Leverrier's method for the matrix pencil $sE - A$*, IEEE Transactions on Automatic Control **31** (1986), no. 9, 869–870.

[4] F. Marcellán, A. Branquinho, and J. Petronilho, *Classical orthogonal polynomials: a functional approach*, Acta Applicandae Mathematicae **34** (1994), no. 3, 283–303.

[5] B. G. Mertzios, *Leverrier's algorithm for singular systems*, IEEE Transactions on Automatic Control **29** (1984), no. 7, 652–653.

[6] G. Wang and L. Qiu, *Leverrier-Chebyshev algorithm for the singular pencils*, Linear Algebra and Its Applications **345** (2002), no. 1–3, 1–8.

Javier Hernández: Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain
E-mail address: jhbenite@math.uc3m.es

Francisco Marcellán: Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain
E-mail address: pacomarc@ing.uc3m.es


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization