TAUBERIAN-TYPE THEOREMS WITH APPLICATION TO THE STIELTJES TRANSFORMATION

S. B. GAIKWAD AND M. S. CHAUDHARY

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In the first part, we define the space $L'(r)$ and the modified Stieltjes transformation introduced by Lavoine and Misra (1979) and Marichev (1983), respectively. In the second part of the paper, we extend Tauberian-type theorems for the distributional Stieltjes transformations to the distributional modified Stieltjes transformations.

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1. Introduction

The Abelian and Tauberian-type theorems were introduced by Stanković [7] and Pilipović et al. [5]. In the first part of this paper, we give the definition of the quasiasymptotic expansion at 0^+ and the quasiasymptotic behaviour of distributions at infinity from S'_+ introduced in [1]. In this paper, we give the definition of space $L'(r)$, classical Stieltjes transformation, modified Stieltjes transformation, and generalized modified Stieltjes transformation. This enables us to obtain, in the second part of the paper, the Tauberian-type theorems of quasiasymptotic behaviour of distributions at infinity. We give sufficient conditions under which the behaviour at infinity of the modified Stieltjes transformation $\Gamma(r+1)(T_{r+1}f)(x)$, $r \in \mathbb{R} \setminus (-\mathbb{N})$, $f \in L'(r)$ determines the quasiasymptotic behaviour of f at infinity.

Notation 1.1. As usually $\mathbb{R}, \mathbb{C}, \mathbb{N}$ are the spaces of real, complex, and natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. D is the space of infinitely differentiable functions with compact support. S'_+ denotes the space of tempered distributions with support in the $[0, \infty)$. The space of rapidly decreasing functions is denoted as S , S' is the space of all distributions of slow growth. T_{r+1} denotes the modified Stieltjes transformation with index r .

A positive continuous function L defined on $(0, \infty)$ is called slowly varying function at ∞ if for every $k > 0$,

$$\lim_{k \rightarrow \infty} \frac{L(kx)}{L(k)} = 1. \quad (1.1)$$

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We denote by Σ_∞ the set of all slowly varying functions at ∞ . For the properties of slowly varying functions, we refer the reader to [6].

If L is a slowly varying function at ∞ , then for every $\varepsilon > 0$, there is $A_\varepsilon > 0$ such that

$$x^{-\varepsilon} < L(x) < x^\varepsilon \quad \text{when } x > A_\varepsilon. \quad (1.2)$$

Recall that for $\alpha > -1$, $x_+^\alpha = H(x)x^\alpha$, where H is Heaviside's function. The following scale of distributions from S'_+ has been used in the investigations of the quasiasymptotic behaviour of distributions:

$$f_{\alpha+1} = \begin{cases} \frac{Ht^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1, \\ D^n f_{\alpha+n+1}, & \alpha \leq -1, \alpha+n > -1, \end{cases} \quad (1.3)$$

where D is the distributional derivative.

2. Definitions

2.1. Definition of the quasiasymptotic behaviour (q.a.b.)

Definition 2.1. The quasiasymptotic behaviour of distribution (q.a.b.) at infinity.

If T is a distribution from S'_+ such that the distributional limit

$$\lim_{k \rightarrow \infty} \frac{T(kx)}{\rho(k)} = \gamma(x) \quad (2.1)$$

exists in S' ($\gamma(x) \neq 0$), then T is called the quasiasymptotic behaviour at infinity related to the regular function $\rho(k) = k^\alpha L(k)$ with the limit γ ; write this as

$$T \stackrel{q}{\sim} \gamma \quad \text{in } S' \text{ as } x \rightarrow \infty. \quad (2.2)$$

Here ρ is regularly varying at infinity and the limit γ from S'_+ is of the form

$$\gamma(x) = \mathbb{C} f_{\alpha+1}(x). \quad (2.3)$$

We will repeat in this section some well-known facts about the quasiasymptotic behaviour from [8].

Let $f \in S'_+$. It is said that f has the q.a.b. at ∞ with the limit $g \neq 0$ with respect to

$$k^\alpha L(k), \quad L \in \sum_\infty \left(\left(\frac{1}{k} \right)^\alpha L \left(\frac{1}{k} \right), L \in \sum_0 \right), \quad \alpha \in \mathbb{R}, \quad (2.4)$$

$$\text{if } \lim_{k \rightarrow \infty} \left\langle \frac{f(kt)}{k^\alpha L(k)}, \phi(t) \right\rangle = \langle g(t), \phi(t) \rangle, \quad \phi \in S.$$

Definition 2.2. A function $\rho : (a, \infty) \rightarrow \mathbb{R}$, $a \in \mathbb{R}$, is called regularly varying function at infinity if it is positive, measurable, and there exists a real number α such that for each

$$x > 0, \quad \lim_{k \rightarrow \infty} \frac{\rho(kx)}{\rho(k)} = x^\alpha; \quad (2.5)$$

where the number α is called the index of ρ .

2.2. Space $L'(r)$. We extend the definition of the space $I'(r)$ given in [4] and using the same idea, we give the definition of space $L'(r)$.

$L'(r)$, $r \in \mathbb{R} \setminus (-\mathbb{N})$, the space of all distributions $f \in S'_+(\mathbb{R})$ such that there exist $k \in \mathbb{N}_0$ and locally integrable function F , $\text{supp } F \subset [0, \infty)$, so that f is of the form

$$f = t^{-r} D^k F \quad (2.6)$$

and there exist $C = C(F)$ and $\varepsilon = \varepsilon(F) > 0$ such that

$$|F(x)| \leq C(1+x)^{r+k-\varepsilon}, \quad x \geq 0. \quad (2.7)$$

The *Stieltjes transformation* $S_r(f)(s)$, $r \in \mathbb{R} \setminus (-\mathbb{N})$, is complex-valued function, defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}} dt \quad s \in \mathbb{C} \setminus (-\infty, 0], \quad 0 < t < \infty, \quad r \in \mathbb{R} \setminus (-\mathbb{N}). \quad (2.8)$$

Modified Stieltjes transformation introduced by Marichev is defined as

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(1 + \frac{x}{y}\right)^{-\alpha} \cdot \frac{1}{y} f(y) dy, \quad x \in \mathbb{C} \setminus (-\infty, 0], \quad 0 < y < \infty, \quad r \in \mathbb{R} \setminus (-\mathbb{N}). \quad (2.9)$$

It can be written as

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha} dy, \quad x \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.10)$$

Setting $r = \alpha - 1$, $f(t) = y^{\alpha-1} f(y)$ in (2.8), we get

$$S_{\alpha-1}(y^{\alpha-1} f(y))(x) = \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha} dy, \quad x \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.11)$$

Using (2.10) and (2.11), we obtain the relationship between (2.8) and (2.10),

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} S_{\alpha-1}(y^{\alpha-1} f(y))(x), \quad x \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.12)$$

By changing x by z and α by $r+1$, it follows

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z), \quad z \in \mathbb{C} \setminus (-\infty, 0], \quad r \in \mathbb{R} \setminus (-\mathbb{N}). \quad (2.13)$$

2.3. Modified Stieltjes transformation T_{r+1} . The modified Stieltjes transformation $T_{r+1}(f)$, $r \in \mathbb{R} \setminus (-\mathbb{N})$, is complex-valued function defined by

$$\Gamma(r+1)T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{r+1+k}} dt, \quad r \in \mathbb{R} \setminus (-\mathbb{N}), s \in \mathbb{C} \setminus (-\infty, 0], 0 < t < \infty, \quad (2.14)$$

where $(\alpha)_k = (\alpha) \cdot (\alpha+1) \cdot (\alpha+2) \cdots (\alpha+k-1)$.

2.4. Generalized modified Stieltjes transformation $\tilde{T}_{r+1}$. The $\tilde{T}_{r+1}$ -transformation of a distribution $f \in \mathcal{S}'_+(\mathbb{R})$ is complex-valued function $\tilde{T}_{r+1}(f)$ defined by

$$\begin{aligned} \Gamma(r+1)\tilde{T}_{r+1}(f)(s) \\ = \lim_{w \rightarrow \infty} \langle f(t), \eta(t)(s+t)^{-r-1} \exp(-wt) \rangle, \quad w \in \mathbb{R}, s \in \Lambda \subset (\mathbb{C} \setminus (-\infty, 0]), \eta \in A(s). \end{aligned} \quad (2.15)$$

Here Λ is the set of complex numbers for which this limit exists and $A(s)$ is the family of all smooth functions, defined on $\mathbb{R}$ for which there exists $\varepsilon = \varepsilon_{\eta,s} > 0$ such that $0 \leq \eta(t) \leq 1$, $t \in \mathbb{R}$, $\eta(t) = 1$, if t belongs to the ε -neighbourhood of $\mathbb{R}_+$, $\eta(t) = 0$ or if it belongs to the complement of the 2ε -neighbourhood of $\mathbb{R}_+$, where $\varepsilon > 0$ is arbitrary if $\text{Im } s \neq 0$, and $0 < 2\varepsilon < \max \text{Re } s$, if for some $\text{Im } s = 0$ and $|D^p \eta(t)| \leq C_p$, $t \in \mathbb{R}$. If $\eta(t) \in A(s)$, $s \in (\mathbb{C} \setminus \mathbb{R}_-)$, then $\eta(t)(s+t)^{-r-1} \exp(-wt) \in \mathcal{S}(\mathbb{R})$ for $w \in \mathbb{R}_+$, $r \in \mathbb{R}$.

3. Main results

3.1. Tauberian-type theorems for modified Stieltjes transformation. For the main results of this section, we need the following assertion from [5].

THEOREM 3.1. *Suppose that for some $m > 0$ and*

$$x \longrightarrow \infty, \quad \int_0^\infty \frac{d\phi(\lambda)}{(\phi+x)^{m+1}} \sim \int_0^\infty \frac{d\varphi(\lambda)}{(\phi+x)^{m+1}} \quad (3.1)$$

and the following conditions are satisfied.

- (1) *Functions ϕ and φ are defined for $x > 0$ and are nondecreasing.*
- (2) *$\lim_{x \rightarrow \infty} \phi(x) = \infty$.*
- (3) *For any $C > 1$, there are constants γ and $\mathbb{N}$, $0 < \gamma < m$, $\mathbb{N} > 0$, such that for any $x > y > \mathbb{N}$, $\phi(x)/\phi(y) \leq C(x/y)$. Then for $\lambda \longrightarrow \infty$, $\phi(\lambda) \sim \varphi(\lambda)$.*

(This means $|\phi(\lambda)/\varphi(\lambda) - 1| < \varepsilon$ if $\lambda > \lambda_0(\varepsilon)$, $\lambda \in B$, $\text{meas}((\lambda_0, \infty) \setminus B) = 0$.) Let us note that condition (3) is called as the Keldysh-type condition.

Now we are ready to prove the first Tauberian result.

THEOREM 3.2. *Suppose that $s > 1$, $r + m - s > 0$, $f \in L'(r)$, and F (see (2.6)) is a nondecreasing function. Moreover, let*

$$\Gamma(r+1)(T_{r+1}f)(x) \sim \frac{\Gamma(s)}{\Gamma(r+1)} \cdot \frac{L(s)}{x^s}, \quad x \longrightarrow \infty, \quad (3.2)$$

where L is a slowly varying function at ∞ which is defined in some interval $[A, \infty)$, such that $x^{r=k-s}L(x)$ is a nondecreasing function. Then f has the quasiasymptotic behaviour at ∞ related to $k^{r-s}L(k)$ with the limit Cx^{r-s} , where $C \neq 0$.

Proof. Let us put

$$\phi(x) = \begin{cases} \frac{x^{r+m-s}L(x)}{\Gamma(r+m-s+1)}, & x > A, \\ 0, & x \leq A. \end{cases} \quad (3.3)$$

Then ϕ has the quasiasymptotic behaviour at ∞ related to $k^{r+m-s}L(k)$ with the limit

$$f_{r+m-s+1}. \quad (3.4)$$

Hence

$$\int_0^\infty \frac{d\phi(t)}{(x+t)^{r+m}} = (r+m) \int_0^\infty \frac{\phi(t)dt}{(x+t)^{r+m+1}} \sim \frac{(r+m)\Gamma(s)}{\Gamma(r+m+1)} \cdot \frac{L(x)}{x^s}, \quad x \rightarrow \infty. \quad (3.5)$$

Now, we show that the conditions of Theorem 3.1 hold for ϕ and F . In fact, we have only to show that for some γ , $0 < \gamma < r+m-1$, and every $C > 1$, there exists $\mathbb{N} > 0$, such that

$$\frac{\phi(\lambda y)}{\phi(y)} < C\lambda^\gamma, \quad \text{for } \lambda > 1, y > \mathbb{N}. \quad (3.6)$$

Let us put $\gamma = r+m-s+\varepsilon$, where we choose $\varepsilon > 0$ such that $\gamma > 0$ and $\varepsilon < s-1$. After substituting ϕ in (3.6), we obtain $L(\lambda y) \leq C\lambda^\varepsilon L(y)$ and this inequality is true if $\lambda > 1$ and $y > \mathbb{N}$, where $\mathbb{N}$ depends on C .

From the assumption that $f \in L'(r)$ and from (3.5), we have

$$\begin{aligned} \Gamma(r+1)(T_{r+1}f)(x) &= (r+1)_m \int_0^\infty \frac{F(t)}{(x+t)^{r+m+1}} dt \\ &= (r+1)_{m-1} \int_0^\infty \frac{dF(t)}{(x+t)^{r+m}} \sim \frac{\Gamma(s)}{\Gamma(r+1)} \cdot \frac{L(x)}{x^s} \\ &\sim \int_0^\infty \frac{d\phi(t)}{(x+t)^{r+m}}, \quad x \rightarrow \infty. \end{aligned} \quad (3.7)$$

This implies that

$$\Gamma(r+m+1)(T_{r+m+1}F)(x) \sim \Gamma(r+m+1)(T_{r+m+1}\phi)(x), \quad x \rightarrow \infty, \quad (3.8)$$

and by Theorem 3.1, it follows that $F \sim \phi$, $x \rightarrow \infty$.

Thus we obtain that

$$F(x) \sim \frac{(x^{r+m-s}L(x))}{\Gamma(r+m-s+1)}, \quad x \rightarrow \infty. \quad (3.9)$$

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Since $r + m - s > 0$, it follows that f has the quasiasymptotic behaviour at ∞ related to $k^{r+m-s}L(k)$ with the limit x^{r+m-s} .

Since $f = t^{-r}D^m F$, it easily follows that f has the quasiasymptotic behaviour at ∞ related to $k^{r-s}L(k)$ with the limit Cx^{r-s} , where C is a suitable nonzero constant. This completes the proof of the theorem. $\square$

4. Tauberian-type results related to the quasiasymptotic behaviour

For the quasiasymptotic behaviour of all original f and for the ordinary asymptotic of the corresponding function $\Gamma(r+1)(T_{r+1}f)$, we need the following theorem and [5, Lemmas 1, 2, and 3].

THEOREM 4.1. *Let $a_{n,m}$, $n, m \in \mathbb{N}$, be a matrix of complex numbers.*

(i) *If $a_{n,m}$ converges uniformly in $m \in \mathbb{N}$ to a_m as $n \rightarrow \infty$ and $\lim_{m \rightarrow \infty} a_m$ exists, then*

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{n,m} = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{n,m} = \lim_{m \rightarrow \infty} a_{n,m}. \quad (4.1)$$

(ii) *If $\lim_{n \rightarrow \infty} a_{n,m}$ exists for every $n \in \mathbb{N}$, $\lim_{m \rightarrow \infty} a_{n,m}$ exists for every $n \in \mathbb{N}$, $\lim_{n,m \rightarrow \infty} a_{n,m}$ exists, then $a_{n,m}$ converges uniformly in $n \in \mathbb{N}$ as $m \rightarrow \infty$.*

LEMMA 4.2. *Let $r \in \mathbb{R} \setminus (-\mathbb{N})$, $k \in \mathbb{N}_0$ be given and let $\gamma \in \mathbb{C}$, then for every $n \in \mathbb{N}$,*

$$\begin{aligned} & \sum_{i=1}^{n+1} \binom{n+1}{i} (-1)^i (2n+r+k+3) \cdots (2n+r+k-i) (2n+r+k+\gamma+3-i) \\ & \quad \cdots (r+k+\gamma+i+2) + (2n+r+k+\gamma+3) \cdots (r+k+\gamma+2) \\ & = (-1)^n \gamma(1-\gamma) \cdots (n-\gamma)(r+k+\gamma+2)_{(n+1)}. \end{aligned} \quad (4.2)$$

LEMMA 4.3. *Suppose that $f \in S'_+$ and that f has the quasiasymptotic at ∞ related to $k^\nu L(k)$, where $\nu < r$. Then there exist $k \in \mathbb{N}_0$, $k + \nu > 0$, and a continuous function F , $\text{supp } F \subset [0, \infty)$, such that $f = t^{-r}D^k F$ and $F_1(x) = \int_0^x F(t)dt$, $x \in \mathbb{R}$, $(\angle_{n,r,k+1,x} \Gamma(r+k+2)(T_{r+k+2}F_1)(x) - F_1(x))/(x(1+x)^{\nu+k}L(x))$ converges uniformly to zero in $(0, \infty)$.*

LEMMA 4.4. *Let $f \in L'(r)$ and $\Gamma(r+1)(T_{r+1}f)(x) \sim x^\nu L(x)$, $x \rightarrow \infty$, $\nu > -1$.*

Then $\overline{\Gamma(r+1)(T_{r+1}f)}_+(x)$ has the quasiasymptotic behaviour at ∞ related to $k^\nu L(k)$.

Proof. We have $(\phi \in S)$,

$$\begin{aligned} \frac{1}{k^\nu L(k)} \langle \overline{\Gamma(r+1)(T_{r+1}f)}_+(kx), \phi(x) \rangle &= \frac{1}{k^{\nu+1}L(k)} \left\langle \overline{\Gamma(r+1)(T_{r+1}f)}_+(x), \phi\left(\frac{x}{k}\right) \right\rangle \\ &= \frac{1}{k^{\nu+1}L(k)} \int_0^1 \Gamma(r+1)(T_{r+1}f)(x) \\ & \quad \times \left(\phi\left(\frac{x}{k}\right) - \phi(0) - \cdots - \left(\frac{x}{k}\right)^{l-1} \frac{\phi^{(l-1)}(0)}{(l-1)!} \right) \cdot dx \\ & \quad + \frac{1}{k^{\nu+1}L(k)} \int_1^\infty \Gamma(r+1)(T_{r+1}f)(x) \phi\left(\frac{x}{k}\right) dx. \end{aligned} \quad (4.3)$$

Since the first part on the right-hand side of (4.3) converges to zero as $k \rightarrow \infty$, we have to prove that $(1/k^\nu L(k)) \int_{1/k}^\infty \Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx \rightarrow \int_0^\infty x^\nu L(x)\phi(x)dx$ as $x \rightarrow \infty$.

Let us recall $\Gamma(r+1)(T_{r+1}f)(x) \sim x^\nu L(x)$, $x \rightarrow \infty$.

This implies that for a given $\varepsilon > 0$, there exists $x_0 > 0$, such that

$$|\Gamma(r+1)(T_{r+1}f)(x) - x^\nu L(x)| \leq \varepsilon x^\nu L(x), \quad x \geq x_0 > 1. \quad (4.4)$$

We use the following decomposition:

$$\begin{aligned} & \frac{1}{k^\nu L(k)} \int_{1/k}^\infty \Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx \\ &= \frac{1}{k^\nu L(k)} \left[\int_{1/k}^{x_0/k} \Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx \right. \\ & \quad \left. + \int_{x_0/k}^\infty \Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx \right] S. \end{aligned} \quad (4.5)$$

The first member on the right-hand side of (4.5) tends to zero, when $k \rightarrow \infty$, because

$$\frac{1}{k^\nu L(k)} \int_{1/k}^{x_0/k} |\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)| dx \leq \frac{M}{k^\nu L(k)} \max_{x \in \mathbb{R}} \{|\phi(x)|\} \frac{x_0 - 1}{k}, \quad (4.6)$$

where $M = \max\{|\Gamma(r+1)(T_{r+1}f)(x)| : 1 \leq x \leq x_0\}$. Also, one can prove easily that or a given $x_0 > 1$,

$$\frac{1}{L(k)} \int_0^{x_0/k} |x^\nu L(x)\phi(x)| dx \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4.7)$$

Now by (4.4), (4.6), (4.7), we have

$$\begin{aligned} & \left| \frac{1}{k^\nu L(k)} \int_{1/k}^\infty \Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx - \frac{1}{k^\nu L(k)} \int_0^\infty (kx)^\nu L(kx)\phi(x)dx \right| \\ & \leq \frac{1}{k^\nu L(k)} \int_{1/k}^{x_0/k} |\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)| dx + \frac{1}{k^\nu L(k)} \int_0^{x_0/k} |(kx)^\nu L(kx)\phi(x)| dx \\ & \quad + \frac{1}{k^\nu L(k)} \int_{x_0/k}^\infty |\Gamma(r+1)(T_{r+1}f)(kx) - (kx)^\nu L(kx)| |\phi(x)| dx. \end{aligned} \quad (4.8)$$

Now we completed the proof of the lemma. $\square$

Now we will assume that L satisfies the inequality $L(mx)/L(m) \leq C(1+x)$, $x > 0$, $m > 0$.

Now we are ready to prove the following.

THEOREM 4.5. *If $f \in S'_+$ and f has the quasiasymptotic behaviour at ∞ related to $k^\nu L(k)$, $r-1 < \nu < r$, $r \in \mathbb{R} \setminus (-\mathbb{N})$, then the double sequence*

$$\left\langle \frac{\Gamma(r+k+2)(T_{r+k+2}F_1)_+(xm)}{m^{\nu+k+1}L(m)}, \angle_{n,r,k+1}\phi^{(k+1)}(x) \right\rangle, \quad m, n \in \mathbb{N}, \phi \in S, \quad (4.9)$$

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converges uniformly in $n \in \mathbb{N}$ as $m \rightarrow \infty$, where $k \in \mathbb{N}$, $k + \nu > 0$, and F_1 are defined in Lemma 4.4.

Proof. Let us put

$$a_{n,m} = \left\langle \frac{\widetilde{\angle_{n,r,k,x}} \overline{\Gamma(r+1)(T_{r+1}f)_+}(mx)}{m^\nu L(m)}, \phi(x) \right\rangle. \quad (4.10)$$

Since

$$a_{n,m} = (-1)^{k+1}(r+1)_{k+1} \left\langle \frac{\overline{\Gamma(r+k+2)(T_{r+k+2}F_1)_+}(mx)}{m^{\nu+k+1}L(m)}, \angle_{n,r,k+1,x} \phi^{(k+1)}(x) \right\rangle, \quad (4.11)$$

we have to prove that $a_{n,m}$ converges uniformly in $n \in \mathbb{N}$ as $m \rightarrow \infty$.

First, we will prove that the conditions of Theorem 4.1(i) are satisfied. Then, from Theorem 4.1(ii), the assertion of this theorem will follow

$$a_{n,m} \longrightarrow a_m = \left\langle \frac{f(mx)}{m^\nu L(m)}, \phi(x) \right\rangle, \quad n \longrightarrow \infty, m \in \mathbb{N}. \quad (4.12)$$

Since

$$\begin{aligned} a_{n,m} - a_m &= (-1)^{k+1}(r+1)_{k+1} \int_0^\infty \frac{\angle_{n,r,k+1,x} \Gamma(r+k+2)(T_{r+k+2}F_1)(mx)}{m^{\nu+k+1}L(m)} \phi^{(k+1)}(x) dx \\ &= (-1)^{k+1}(r+1)_{k+1} \int_0^\infty \frac{\angle_{n,r,k+1,x} \Gamma(r+k+2)(T_{r+k+2}F_1)(mx) - F_1(mx)(mx)}{(mx)(1+mx)^{\nu+k}} \\ &\quad \times \frac{(1+mx)^{\nu+k}L(mx)}{m^{\nu+k+1}L(m)} \phi^{(k+1)}(x) dx, \end{aligned} \quad (4.13)$$

from Lemma 4.4 and the fact that

$$\frac{(mx)(1+mx)^{\nu+k}L(mx)}{m^{\nu+k+1}L(m)} \leq 2C(1+x) \frac{mx(1+(mx)^{\nu+k})}{m^{\nu+k+1}} \leq 2C(1+x)(x+x^{\nu+k+1}), \quad x \geq 0, \quad (4.14)$$

we obtain that $a_{n,m} - a_m \rightarrow 0$ uniformly in $n \in \mathbb{N}$ as $n \rightarrow \infty$.

We have $\Gamma(r+1)(T_{r+1}f)(x) \sim (\Gamma(r-\nu)/\Gamma(r+1)) x^{\nu-r+2}L(x)$, $x \rightarrow \infty$. Since $-1 < \nu - r$, we have by Lemma 4.4 that $\overline{\Gamma(r+1)(T_{r+1}f)_+}$ has the quasiasymptotic at ∞ related to $k^{\nu-r}L(k)$ with the limit $(\Gamma(r-\nu)/\Gamma(r+1))x^{\nu-r}$.

By Leibniz formula, we have

$$\begin{aligned}
 & D^{n+1} x^{2n+r+k+3} D^{n+1} \Gamma(r+k+2)(T_{r+k+2} F_1)(x) \\
 &= D^{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \left((x^{2n+r+k+3})^{(i)} \Gamma(r+k+2)(T_{r+k+2} F_1)(x) \right)^{(n+1-i)} \\
 &= \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (2n+r+k+3) \cdots (2n+r+k+4-i) \\
 &\quad \times \left((x^{2n+r+k+3-i}) \left(\Gamma(r+k+2)(T_{r+k+2} F_1)(x) \right)^{(2n+2-i)} \right. \\
 &\quad \left. + (x^{2n+r+k+3}) \Gamma(r+k+3)(T_{r+k+3} F_1)(x) \right)^{(2n+2)}. \tag{4.15}
 \end{aligned}$$

Let $\gamma = \nu - r$. Then $x^{2n+r+k+3-i} \Gamma(r+k+2)(T_{r+k+2} F_1)(x)$ has the quasiasymptotic behaviour at ∞ related to $k^{2n+r+k+\gamma+3-i} L(k)$ with the limit $\Gamma(-\gamma) x^{2n+r+k+\gamma+3-i} / \Gamma(r+1)(r+1)_{k+1}$.

Lemma 4.2 and the properties of quasiasymptotic behaviour imply that $D^{n+1} x^{2n+r+k+3} D^{n+1} \Gamma(r+k+2)(T_{r+k+2} F_1)(x)$ has the quasiasymptotic behaviour at ∞ related to $k^{r+k+\gamma+1} L(k)$ with the limit

$$(-1)^n \gamma(1-\gamma) \cdots (n-\gamma)(r+k+\gamma+2)_{n+1} \frac{\Gamma(-\gamma)}{\Gamma(r+1)(r+1)_{k+1}} x^{r+k+r+1}. \tag{4.16}$$

Thus, we obtain

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} (-1)^{(k+1)} (r+1)_{k+1} \left\langle \frac{\angle_{n,r,k+1,x} \Gamma(r+k+2)(T_{r+k+2} F_1)_+(mx)}{m^{r+k+1} L(m)}, \phi(x) \right\rangle \\
 &= - \lim_{n \rightarrow \infty} \frac{(-1)^{k+1} \Gamma(r+k+2)}{(n+1)! \Gamma(n+r+k+2)} \frac{\Gamma(n-\gamma+1)}{\Gamma(-\gamma)} \frac{\Gamma(r+k+\gamma+n+3)}{\Gamma(r+k+\gamma+2)} \frac{\Gamma(-\gamma)}{\Gamma(r+1)} \\
 &\quad \times \langle x^{r+k+1}, \phi^{(k+1)}(x) \rangle. \tag{4.17}
 \end{aligned}$$

To prove that the last limit exists, we have to use the Stirling formula.

$\Gamma(s+1) \sim \sqrt{2\pi} e^{-s} s^{(s+1)/2}$, $s \rightarrow \infty$. Thus for the double sequence $a_{n,m}$, Theorem 4.1(i) holds and Theorem 4.1(ii) implies the assertion. $\square$

THEOREM 4.6. *Let $f \in L'(r)$ and let $\overline{\Gamma(r+1)(T_{r+1} f)}_+(x)$ have the quasiasymptotic at ∞ related to $k^\alpha L(k)$, $-1 < \alpha < 0$. If for any $\phi \in s$, the double sequence (4.9) converges uniformly in $n \in \mathbb{N}$ as $m \rightarrow \infty$, then f has the quasiasymptotic at ∞ related to $k^{\alpha+r} L(k)$.*

Proof. If $a_{n,m}$ is the double sequence defined in the proof of Theorem 4.5, we have by Theorem 4.1(i) the assertion. $\square$

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References

- [1] J. N. Drožžinov and B. I. Zavalov, *Quasiasymptotic behavior of generalized functions, and Tauberian theorems in the complex domain*, *Matematicheskii-Sbornik.-Novaya-Seriya* **102(144)** (1977), no. 3, 372–390.
- [2] J. Lavoine and O. P. Misra, *Abelian theorems for the distributional Stieltjes transformation*, *Mathematical Proceedings of the Cambridge Philosophical Society* **86** (1979), no. 2, 287–293.
- [3] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions. Theory and Algorithmic Tables*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester; John Wiley & Sons, New York, 1983.
- [4] S. Pilipović, *An inversion theorem for the Stieltjes transform of distributions*, *Proceedings of the Edinburgh Mathematical Society. Series II* **29** (1986), no. 2, 171–185.
- [5] S. Pilipović, B. Stanković, and A. Takači, *Asymptotic Behaviour and Stieltjes Transformation of Distributions*, *Teubner-Texte zur Mathematik*, vol. 116, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1990.
- [6] E. Seneta, *Regularly Varying Functions*, *Lecture Notes in Math*, vol. 508, Springer, Berlin, 1976.
- [7] B. Stanković, *Abelian and Tauberian theorems for the Stieltjes transform of distributions*, *Russian Mathematical Surveys* **40** (1985), no. 4, 99–113.
- [8] B. I. Zavalov, Y. N. Drozinov, and V. S. Vladimirov, *Several Dimension Tauberian Theorems for Generalized Functions in Mathematical Physics*, Nauka, Moscow, 1986.

S. B. Gaikwad: Department of Mathematics, New Arts, Commerce and Science College Parner, Pune University, Pune, District Ahmednagar 414302, Maharashtra, India
E-mail address: shrigaik@yahoo.com

M. S. Chaudhary: Department of Mathematics, Shivaji University, Vidayanagar, Kolhapur 416004, Maharashtra, India
E-mail address: m_s_chaudhary@rediffmail.com