AN IDENTITY RELATED TO JORDAN'S INEQUALITY

JIAN-LIN LI

Received 28 July 2006; Revised 29 September 2006; Accepted 2 October 2006

The main purpose of this note is to establish an identity which states that the function $\sin x/x$ is a power series of $(\pi^2 - 4x^2)$ with positive coefficients for all $x \neq 0$. This enable us to obtain a much stronger Jordan's inequality than that obtained before.

1. Introduction

The well-known Jordan's inequality states that

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right], \tag{1.1}$$

with equality holds if and only if $x = \pi/2$ (see [5]). It plays an important role in many areas of pure and applied mathematics. The inequality (1.1) is first extended to the following:

$$\frac{2}{\pi} + \frac{1}{12\pi}\left(\pi^2 - 4x^2\right) \leq \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right], \tag{1.2}$$

and then, it is further extended to the following:

$$\frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) \leq \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right], \tag{1.3}$$

with equality holds if and only if $x = \pi/2$ (see [2, 4, 6]). The inequality (1.3) is slightly stronger than the inequality (1.2) and is sharp in the sense that $1/\pi^3$ cannot be replaced by a larger constant. More recently, the monotone form of L'Hopital's rule (see [1, Lemma 5.1]) has been successfully used by Zhu [9, 10], Wu and Debnath [7, 8] in the sharpening

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 76782, Pages 1–6
DOI 10.1155/IJMMS/2006/76782

Jordan's inequality. For example, it has been shown that if $0 < x \le \pi/2$, then

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \le \frac{\sin x}{x} \le \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \tag{1.4}$$

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5}(\pi^2 - 4x^2)^2 \le \frac{\sin x}{x} \le \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^5}(\pi^2 - 4x^2)^2 \tag{1.5}$$

hold with equality if and only if $x = \pi/2$. Furthermore, the constants $1/\pi^3$ and $(\pi - 2)/\pi^3$ in (1.4) as well as the constants $(12 - \pi^2)/(16\pi^5)$ and $(\pi - 3)/\pi^5$ in (1.5) are the best. Also, in the process of sharpening Jordan's inequality, one can use the same method as did in [7] to introduce a parameter θ $(0 < \theta \le \pi)$ to replace the value $\pi/2$. Unfortunately, the preceding method will become cumbersome to execute in the further generalization of Jordan's inequality.

In this note we establish an identity which states that the function $\sin x/x$ is a power series of $(\pi^2 - 4x^2)$ with positive coefficients for all $x \neq 0$. This enables us to obtain a much better inequality than (1.4) or (1.5) if $0 < x \le \pi/2$.

2. Main result

The main result relating to Jordan's inequality is contained in the following.

Theorem 2.1. *For any $x > 0$, the following identity*

$$\frac{\sin x}{x} = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{(-1)^k R_k}{k!\pi^{2k}}(\pi^2 - 4x^2)^k \tag{2.1}$$

holds, where

$$R_k = \sum_{n=k}^{\infty} \frac{(-1)^n n!}{(2n+1)!(n-k)!}\left(\frac{\pi}{2}\right)^{2n} \tag{2.2}$$

satisfying $(-1)^k R_k > 0$ $(k = 1, 2, 3, \ldots)$.

Proof. Let $x = \sqrt{\pi^2 - t}/2$ and $t = \pi^2 - 4x^2$. It follows from the Taylor expansion for $\sin x$ that

$$\frac{\sin\sqrt{(\pi^2 - t)}/2}{\sqrt{(\pi^2 - t)}/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi^2 - t}{4}\right)^n \quad (-\infty < t < \pi^2). \tag{2.3}$$

Since

$$(\pi^2 - t)^n = \pi^{2n} + \sum_{k=1}^{n} \frac{(-1)^k n!}{k!(n-k)!}\pi^{2(n-k)}t^k \quad (n = 1, 2, 3, \ldots), \tag{2.4}$$

we see that

$$
\begin{aligned}
\frac{\sin\sqrt{(\pi^2-t)}/2}{\sqrt{(\pi^2-t)}/2} &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!4^n}\sum_{k=1}^{n}\frac{(-1)^k n!}{k!(n-k)!}\pi^{2(n-k)}t^k \\
&= \frac{2}{\pi}+\sum_{k=1}^{\infty}\left\{\sum_{n=k}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{n!}{(n-k)!}\left(\frac{\pi}{2}\right)^{2n}\right\}\frac{(-1)^k}{k!\pi^{2k}}t^k \\
&= \frac{2}{\pi}+\sum_{k=1}^{\infty}\frac{(-1)^k R_k}{k!\pi^{2k}}t^k \quad (-\infty<t<\pi^2)
\end{aligned}
\tag{2.5}
$$

which yields (2.1) with R_k given by (2.2).

Let $R_k=\sum_{n=k}^{\infty}(-1)^n c_{n,k}$ with

$$
c_{n,k}=\frac{1}{(n-k)!}\frac{1}{n+1}\frac{\pi^2/4}{n+2}\frac{\pi^2/4}{n+3}\cdots\frac{\pi^2/4}{n+n}\frac{\pi^2/4}{2n+1}. \tag{2.6}
$$

Then for each given k $(k=1,2,3,\ldots)$,

$$
c_{n,k}>c_{n+1,k}, \quad (n\geq k), \qquad \lim_{n\to\infty}c_{n,k}=0. \tag{2.7}
$$

Hence the alternating series $\sum_{n=k}^{\infty}(-1)^n c_{n,k}$ converges, and its sum

$$
R_k=(-1)^k\sum_{j=0}^{\infty}(-1)^j c_{k+j,k} \tag{2.8}
$$

satisfies $(-1)^k R_k>0$ and $|R_k|<c_{k,k}$ for each given k $(k=1,2,3,\ldots)$. This completes the proof of the theorem. □

Next we give a formula of calculating R_k. Let

$$
d_k(x)=\sum_{n=k}^{\infty}\frac{(-1)^n n!}{(2n+1)!(n-k)!}x^{2n}. \tag{2.9}
$$

Then

$$
\begin{gathered}
d_1(x)=\frac{x}{2}\left(\frac{\sin x}{x}\right)', \qquad d_2(x)=\frac{x^3}{2^2}\left(\frac{1}{x}\left(\frac{\sin x}{x}\right)'\right)',\ldots, \\
d_k(x)=\frac{x^{2k-1}}{2^k}\left(\frac{1}{x}\left(\frac{1}{x}\left(\cdots\left(\frac{1}{x}\left(\frac{\sin x}{x}\right)'\right)'\cdots\right)'\right)'\right)'
\end{gathered}
\tag{2.10}
$$

with kth derivative, and

$$
d_{k+1}(x)=-kd_k(x)+\frac{x}{2}d_k'(x) \quad (k=1,2,3,\ldots). \tag{2.11}
$$

Hence $R_k = d_k(\pi/2)$ with

$$\begin{aligned} R_1 = d_1\left(\frac{\pi}{2}\right) = -\frac{1}{\pi}, \qquad R_2 = d_2\left(\frac{\pi}{2}\right) = \frac{12-\pi^2}{8\pi}, \\ R_3 = d_3\left(\frac{\pi}{2}\right) = \frac{-3(10-\pi^2)}{8\pi}, \qquad R_4 = d_4\left(\frac{\pi}{2}\right) = \frac{\pi^4-180\pi^2+1680}{128\pi}. \end{aligned} \tag{2.12}$$

Also the above established identity (2.1) gives

$$\begin{aligned} \frac{\sin x}{x} &= \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2) + \frac{12-\pi^2}{16\pi^5}(\pi^2-4x^2)^2 \\ &\quad + \frac{10-\pi^2}{16\pi^7}(\pi^2-4x^2)^3 + \frac{\pi^4-180\pi^2+1680}{3072\pi^9}(\pi^2-4x^2)^4 \\ &\quad + \sum_{k=5}^{\infty}\frac{(-1)^k R_k}{k!\pi^{2k}}(\pi^2-4x^2)^k \end{aligned} \tag{2.13}$$

for all $x > 0$. Since $(-1)^k R_k > 0$ $(k = 1,2,3,\ldots)$, we have the following corollary.

Corollary 2.2. *If $0 < x \le \pi/2$, then*

$$\begin{aligned} \frac{\sin x}{x} &\ge \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2) + \frac{12-\pi^2}{16\pi^5}(\pi^2-4x^2)^2 \\ &\quad + \frac{10-\pi^2}{16\pi^7}(\pi^2-4x^2)^3 + \frac{\pi^4-180\pi^2+1680}{3072\pi^9}(\pi^2-4x^2)^4 \end{aligned} \tag{2.14}$$

holds with equality if and only if $x = \pi/2$. Furthermore, the constants $1/\pi^3$, $(12-\pi^2)/(16\pi^5)$, $(10-\pi^2)/(16\pi^7)$, and $(\pi^4-180\pi^2+1680)/(3072\pi^9)$ in (2.14) are the best.

The above established inequality (2.14) is much stronger than the left-hand side of inequality (1.5). Also one can add more positive terms to the right-hand side of inequality (2.14) to get higher accuracy.

Finally, it should be pointed out that, in order to give the right-hand side of inequality (1.4) or (1.5), the following Taylor expansion for $x/\sin x$ will play an important role as the above established identity (2.1) or (2.13).

Taylor expansion of $x/\sin x$.

$$\begin{aligned} \frac{x}{\sin x} &= \sum_{n=0}^{\infty}(-1)^{n+1}B_{2n}\frac{2^{2n}-2}{(2n)!}x^{2n} \\ &= 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \cdots \quad (|x| < \pi), \end{aligned} \tag{2.15}$$

where B_{2n} are the Bernoulli numbers satisfying $(-1)^{n+1}B_{2n} > 0$ $(n = 1,2,3,\ldots)$.

Recall that the Bernoulli numbers B_n and the functions $B_n(x)$ are defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n \qquad \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n \quad (|t| < 2\pi). \tag{2.16}$$

It is familiar that they have the following properties (see [3, Section I-13]):

$$
\begin{gathered}
B_0(x) = B_0 = 1, \qquad B_1 = \frac{-1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_4 = \frac{-1}{30}, \qquad B_6 = \frac{1}{42}; \\
B_{2n+1} = 0, \quad (-1)^{n+1}B_{2n} > 0, \quad B_n(0) = B_n \quad (n \geq 1); \\
B_n(1-x) = (-1)^n B_n(x) \quad (n \geq 1); \\
B_{2n}(x) = \frac{2(-1)^{n+1}(2n)!}{(2\pi)^{2n}} \sum_{r=1}^{\infty} \frac{1}{r^{2n}} \cos(2\pi r x) \quad (n \geq 1,\ 0 \leq x \leq 1),
\end{gathered}
\tag{2.17}
$$

from which, we have

$$
B_{2n-1}\left(\frac{1}{2}\right) = 0, \quad 2^{2n}B_{2n}\left(\frac{1}{2}\right) = \left(2 - 2^{2n}\right)B_{2n} \quad (n \geq 1). \tag{2.18}
$$

Therefore, for $|x| < \pi$,

$$
\begin{aligned}
\frac{x}{\sin x} &= \frac{2xie^{xi}}{e^{2xi} - 1} = \sum_{n=0}^{\infty} B_n\left(\frac{1}{2}\right)\frac{2^n i^n}{n!}x^n \\
&= \sum_{n=0}^{\infty} B_{2n}\left(\frac{1}{2}\right)\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}(-1)^{n+1}B_{2n}\frac{2^{2n}-2}{(2n)!}x^{2n}.
\end{aligned}
\tag{2.19}
$$

From (2.15), we have the following type of strengthened right-hand Jordan's inequality:

$$
\frac{\sin x}{x} \leq \left(1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6\right)^{-1} \quad (|x| < \pi). \tag{2.20}
$$

Also one can add more positive terms to the right-hand side of inequality (2.20) to get higher accuracy.

Acknowledgment

The author would like to thank the referees for their valuable suggestions.

References

[1] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen, *Generalized elliptic integrals and modular equations*, Pacific Journal of Mathematics **192** (2000), no. 1, 1–37.

[2] L. Debnath and C.-J. Zhao, *New strengthened Jordan's inequality and its applications*, Applied Mathematics Letters **16** (2003), no. 4, 557–560.

[3] A. Erdélyi, *Higher Transcental Functions, Vol. I*, McGraw-Hill, New York, 1953.

[4] A. McD. Mercer, U. Abel, and D. Caccia, *A sharpening of Jordan's inequality*, The American Mathematical Monthly **93** (1986), 568–569.

[5] D. S. Mitrinović and P. M. Vasić, *Analytic Inequalities*, Springer, New York, 1970.

[6] A. Y. Özban, *A new refined form of Jordan's inequality and its applications*, Applied Mathematics Letters **19** (2006), no. 2, 155–160.

[7] S. Wu and L. Debnath, *A new generalized and sharp version of Jordan's inequality and its application to the improvement of Yang Le inequality*, Applied Mathematics Letters **19** (2006), no. 12, 1378–1384.

[8] ———, *A new generalized and sharp version of Jordan's inequality and its application to the improvement of Yang Le inequality II*, to appear in Applied Mathematics Letters.

[9] L. Zhu, *Sharpening Jordan's inequality and the Yang Le inequality*, Applied Mathematics Letters **19** (2006), no. 3, 240–243.

[10] ———, *Sharpening Jordan's inequality and Yang Le inequality. II*, Applied Mathematics Letters **19** (2006), no. 9, 990–994.

Jian-Lin Li: College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China
E-mail address: jllimath@yahoo.com.cn