SUMMATION IDENTITIES FOR REPRESENTATION OF CERTAIN REAL NUMBERS

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We present identities used to represent real numbers of the form $xu^m \pm yv^n$ for appropriately chosen real numbers x, y, u, v and nonnegative integers m and n. We present the proofs of the identities by applying Zeilberger's algorithm.

1. Introduction

It is well known that every integer can be written as a sum of integral powers of 2. A somewhat related problem is to find for every positive integer n a positive integer k depending on n with $k(n) < k(n+1)$ and integer coefficients a_i, $i = 1, 2, \ldots, k-1$, such that

$$n = \sum_{i=0}^{k-1} a_i 2^{2i}. \tag{1.1}$$

The background and motivation for studying this problem lies in finding the zeros of the jth-order generalized Fibonacci polynomial defined by $F_j(x) = x^{j-1} - \cdots - x - 1$. Dubeau [1] and Flores [3] have shown that the positive roots of $F_j(x)$ are of the form $2 - O(2^{-j})$. Grossman and Narayan [4] have proven that the single negative zero of $F_j(x)$ has the form $-1 + O(j^{-1})$ for j even and tends to -1 monotonically as $j \to \infty$. Using the fact that

$$F_j(x) = (x - 2 + \varepsilon_j)(x + 1 - \delta_j)(x^{j-2} + a_{j-3}x^{j-3} + \cdots + a_1 x + a_0), \tag{1.2}$$

where δ_j and ε_j are positive, decreasing sequences for $j = 4, 6, \ldots$, Grossman and Zeleke [6] have found an explicit form for the coefficients a_i, where $i = 0, 1, \ldots, j-3$, in terms of δ and ε. The solution was found by solving a nonhomogeneous linear recurrence relation of the form $-a_n + b\, a_{n+1} + c\, a_{n+2} = 1$, where $b = \varepsilon - 1 - \delta$, $c = (1 - \delta)(2 - \varepsilon)$. As byproduct of this solution, several combinatorial identities were formulated by considering even- and odd-indexed terms of a_n.

Some of these identities give representations of positive integers using binomial coefficients. Others are identities for representing certain types of real numbers. In this paper, we call these identities "real number identities." Real number identities are interesting for various reasons. On one side, formulation and proofs of combinatorial identities traditionally have been linked to arguments that have their origin in counting problems. Such identities can then be used to represent positive integers. From this perspective, it is not obvious that one can formulate real number identities. This by itself makes the attempt to represent real numbers in terms of combinatorial identities interesting. Furthermore, by considering some special cases, we can use the real number identities to generate interesting sequences. We give five examples to demonstrate this fact. In the first example, we consider a representation of the integer sequence 2^n, $n \geq 0$. What is different about this representation is that the summand in the identity involves irrational terms. In example two, we discuss an alternate way to generate the Fibonacci sequence F_n with initial values $F_1 = F_2 = 1$. The third example shows how to generate a shifted even-indexed Fibonacci sequence F_{2n+2}, $n \geq 1$. The fourth example shows how to generate from **Theorem 2.1** integer solutions of a Pell equation of the form $x^2 - 3y^2 = 1$. In example five, we give an identity that can be used to represent a certain type of rational numbers.

The real number identities in this paper have been introduced in [6] where the methods of proofs involve induction, previously known combinatorial identities, and algebraic manipulations. The goal of this paper is to present computer-generated proofs using Zeilberger's algorithm. For a superb exposition of Zeilberger's algorithm see, among others, the books $A = B$ [8] and *Hypergeometric Summation* [7] which are devoted to this and other methods. The paper is organized as follows. In Section 2 we present several real number identities. Computer-generated proofs of these identities are presented in Section 3. Throughout this paper, we denote the set $\{k, k+1, k+2, \ldots\}$ for $k \in \mathbb{Z}$ by $\mathbb{N}_k$.

2. Main theorems

THEOREM 2.1. *For $n \in \mathbb{N}_0$, $b \in \mathbb{R} \setminus \{-2, -1\}$,*

$$\frac{1}{(b+1)^{n+2}} \sum_{k=0}^{n} \frac{b^{2k+1}}{(b+1)^k} \binom{n+k+1}{2k+1} = \frac{1}{b+2} \left(1 - \frac{1}{(b+1)^{2(n+1)}}\right). \tag{2.1}$$

THEOREM 2.2. *Let $n \in \mathbb{N}_1$ and let $b, c \in \mathbb{R} \setminus \{0\}$ such that $b^2 + 4c > 0$. Then*

$$\sum_{k=1}^{n} \frac{1}{2^n b^2} \left(\frac{b}{2c}\right)^{2n} \left(1 + \frac{4c}{b^2}\right)^{k-1} \binom{2n}{2k-1} = \frac{\alpha\beta}{\alpha - \beta} (\alpha^n - \beta^n), \tag{2.2}$$

where $\alpha = 1/(b^2 + 2c - \sqrt{b^2(b^2 + 4c)})$ and $\beta = 1/(b^2 + 2c + \sqrt{b^2(b^2 + 4c)})$.

THEOREM 2.3. *For $n \in \mathbb{N}_{-1}$, $b \in \mathbb{R} \setminus \{-2, -1\}$,*

$$\sum_{k=0}^{n+1} \frac{b^{2k}}{(b+1)^{n+k+1}} \binom{n+k+1}{2k} = \frac{b+1}{b+2} \left(1 + \frac{1}{(b+1)^{2n+3}}\right). \tag{2.3}$$

THEOREM 2.4. *Let $n \in \mathbb{N}_1$ and let $b, c \in \mathbb{R}$ such that $b \neq 0$ and $c > 0$. Then*

$$\sum_{k=0}^{n} \left(\frac{b}{2^{n-1}}\right) \left(\frac{b}{2c}\right)^{2n+1} \left(1 + \frac{4c}{b^2}\right)^{k+1} \binom{2n+1}{2k+1} = \frac{\sqrt{\alpha} + \sqrt{\beta}}{\sqrt{\alpha\beta}} \left(\beta^{n+1/2} - \alpha^{n+1/2}\right), \qquad (2.4)$$

where $\alpha = 1/(b^2 + 2c - \sqrt{b^2(b^2 + 4c)})$ and $\beta = 1/(b^2 + 2c + \sqrt{b^2(b^2 + 4c)})$.

THEOREM 2.5. *For $n \in \mathbb{N}_1$, $b \in \mathbb{R}\{-2, -1\}$,*

$$\sum_{k=0}^{n-1} \sum_{m=2k+2}^{n+1+k} \frac{b^m}{(b+1)^{n+k+1}} \binom{n+k+1}{m} = \frac{b(n+1)}{b+2} - \frac{(b+1)^2}{(b+2)^2} \left(1 - \frac{1}{(b+1)^{2n+2}}\right). \qquad (2.5)$$

Remark 2.6. By substituting specific values for the parameters in the above theorems, one can get interesting identities. We consider five examples.

Example 2.7 (identities for 2^n). By substituting $b = -1 + \sqrt{2}/2$ in Theorem 2.1, we get the following identity for the integer sequence 2^n:

$$\frac{1}{2} + \frac{2^{n/2}}{2} \sum_{k=0}^{n} \left(\frac{3}{\sqrt{2}} - 2\right)^k \binom{n+k+1}{2k+1} = 2^n. \qquad (2.6)$$

If we let $b = -1 - \sqrt{2}/2$, we get a different identity for the sequence 2^n:

$$\frac{1}{2} + \frac{2^{n/2}}{2} \sum_{k=0}^{n} (-1)^{n+k} \left(\frac{3}{\sqrt{2}} + 2\right)^k \binom{n+k+1}{2k+1} = 2^n. \qquad (2.7)$$

These two identities are different from other representations of 2^n in the sense that they involve irrational terms to generate an integer sequence. They are also interesting from this perspective.

Example 2.8 (generating the Fibonacci sequence.). In Theorem 2.2, choose $c = \pm 1/2$ and $b = \pm\sqrt{(\sqrt{5} - 2)/2}$. This makes $\alpha = (\sqrt{5} + 1)/2$ and $\beta = (\sqrt{5} - 1)/2$. Note that since $\alpha\beta = \alpha - \beta = 1$, the right-hand side of Theorem 2.2 is the sequence

$$G_n = \left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n, \quad n \geq 1. \qquad (2.8)$$

By setting $F_0 = G_0 = 0$, we get the following relation between G_n and F_n for $n \geq 0$:

$$G_{2n} = \sqrt{5}F_{2n}, \qquad G_{2n+1} = F_{2n} + F_{2n+2}. \qquad (2.9)$$

From this relation, we see that Theorem 2.2 provides a combinatorial identity to generate the Fibonacci sequence.

Example 2.9 (generating a shifted even-indexed Fibonacci sequence). In **Theorem 2.1**, if we let $b = (1 \pm \sqrt{5})/2$, then $b^2/(b+1) = 1$ and **Theorem 2.1** takes the form

$$\sum_{k=0}^{n} \binom{n+k+1}{2k+1} = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n. \tag{2.10}$$

One can show that the right-hand side of (2.10) is F_{2n+2}.

Example 2.10. By substituting $b = 1 \pm \sqrt{3}$ in **Theorem 2.1**, we get the following identity that generates the sequence $a_0 = 1$, $a_1 = 4$, and $a_{n+2} = 4a_{n+1} - a_n$ for $n \geq 0$:

$$\sum_{k=0}^{n} 2^k \binom{n+k+1}{2k+1} = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right)(2+\sqrt{3})^n + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right)(2-\sqrt{3})^n. \tag{2.11}$$

The first few terms of the sequence a_n are 1, 4, 15, 56, 209, 780, ... One interesting property of these terms is that $3a_n^2 + 1$ is a perfect square.

Example 2.11. By substituting $b = 1$ in **Theorem 2.1** and $b = -1$, $c = 2$ in **Theorem 2.2**, we get the identities discovered in [6] for real numbers of the form $(1/6)(1 - 1/4^n)$.

3. Proofs of the theorems

Proof of Theorem 2.1. Since $\binom{n+k+1}{2k+1} = 0$ for $k > n+1$ and $k < 0$, the identity in **Theorem 2.1** can be rewritten as

$$\frac{1}{(b+1)^{n+2}} \sum_{k \in \mathbb{Z}} \frac{b^{2k+1}}{(b+1)^k} \binom{n+k+1}{2k+1} = \frac{1}{b+2}\left(1 - \frac{1}{(b+1)^{2(n+1)}}\right). \tag{3.1}$$

This is also equivalent to

$$\sum_{k \in \mathbb{Z}} \frac{(2+b)b^{2\,k+1}}{(b+1)^{k-n}} \binom{n+k+1}{2k+1} = (1+b)^{2(n+1)} - 1. \tag{3.2}$$

Let us denote the left-hand side of (3.2) by $S_b(n)$. Let $F_b(n,k)$ denote the summand of $S_b(n)$, that is,

$$F_b(n,k) = \frac{(2+b)b^{2k+1}}{(b+1)^{k-n}} \binom{n+k+1}{2k+1}. \tag{3.3}$$

Then F_b satisfies the recurrence equation (the recurrence equation is automatically generated by the Maple package ekhad which is available (for free) from http://www.math.rutgers.edu/~zeilberg/):

$$-(b+1)^2 F_b(n,k) + (b^2 + 2b + 2)F_b(n+1,k) - F_b(n+2,k) = G_b(n,k+1) - G_b(n,k),$$

$$\text{where } G_b(n,k) = (2+b)b^{2k+1}(b+1)^{n-k+2}\binom{n+k+1}{2k-1}.$$

$$\tag{3.4}$$

By summing both sides of (3.4) with respect to k, we get

$$-(b+1)^2 S_b(n) + (b^2 + 2b + 2) S_b(n+1) - S_b(n+2) = 0. \tag{3.5}$$

Moreover, $S_b(0) = (b+1)^2 - 1$ and $S_b(1) = (1+b)^4 - 1$. Also $(1+b)^{2(n+1)} - 1$ satisfies (3.5), and hence $S_b(n) = (1+b)^{2(n+1)} - 1$. $\qquad\square$

Remarks 3.1. (1) In the sequel, we will use the same proof style. Compute a recurrence relation of order d for the summand and sum the summand recurrence with respect to the summation index to get a recurrence for the sum $S(n)$. Check that the right-hand side also satisfies the same recurrence and that it agrees with $S(n)$ for the first d initial values of n. Then the identity follows.

(2) We would like to note that one can also get the right-hand side of (3.2) by directly solving the recurrence (3.5) which has constant coefficients. For more details, see for instance [8, 9].

To make the proofs more compact, we introduce the following operator notations.

Defintion 3.2. Given a function $F(n,k)$ of the (discrete) variables n and k, define the *forward shift operator* E_n by $E_n F(n,k) = F(n+1,k)$.

With this notation, (3.4) becomes

$$\left(-(b+1)^2 + (b^2 + 2b + 2) E_n - E_n^2 \right) F_b(n,k) = G_b(n,k+1) - G_b(n,k). \tag{3.6}$$

From the proof of Theorem 2.1, we observe that to certify the truth of the identity, it would suffice to display the pair (P, G_b) (called a "Z-pair"), where $P = -(b+1)^2 + (b^2 + 2b + 2)E_n - E_n{}^2$.

In the sequel for a given function $F(n,k)$, a Z-pair (P, G) means that $PF(n,k) = G(n, k+1) - G(n,k)$. Since the Z-pair actually certifies the truth of the identity, it is called the *proof certificate*. Therefore, in the proofs of Theorems 2.2, 2.4, we only give the proof certificates.

Proof of Theorem 2.2. Since $\binom{2n}{2k-1} = 0$ for $k > n$ and $k < 1$, the identity in Theorem 2.2 can be rewritten as

$$\sum_{k \in \mathbb{Z}} \frac{1}{2^n b^2} \left(\frac{b}{2c} \right)^{2n} \left(1 + \frac{4c}{b^2} \right)^{k-1} \binom{2n}{2k-1} = \frac{\alpha^n - \beta^n}{\alpha\beta(\alpha - \beta)}. \tag{3.7}$$

Let us denote the summand of the left-hand side of (3.7) by $F_{b,c}(n,k)$, that is,

$$F_{b,c}(n,k) = \frac{1}{2^n b^2} \left(\frac{b}{2c} \right)^{2n} \left(1 + \frac{4c}{b^2} \right)^{k-1} \binom{2n}{2k-1}. \tag{3.8}$$

The required Z-pair $(P, G_{b,c})$ is $P = 1 - (2b^2 + 4c)E_n + 4c^2 E_n^2$ and

$$G_{b,c}(n,k) = \left(\frac{1}{2^{n+1}b^2}\right)\left(\frac{b}{2c}\right)^{2n+2}\left(1+\frac{4c}{b^2}\right)^{k-1}\left(-\frac{(3n-2k+6)b^2}{2k-3}\binom{2n+1}{2k-4}+2c\binom{2n}{2k-3}\right).$$

$$(3.9)$$

$\square$

Proof of Theorem 2.3. Since $\binom{n+k+1}{2k} = 0$ for $k > n+1$ and $k < 0$, the identity in Theorem 2.3 can be rewritten as

$$\sum_{k \in \mathbb{Z}} \frac{b^{2k}}{(b+1)^{n+k+1}}\binom{n+k+1}{2k} = \frac{b+1}{b+2}\left(1+\frac{1}{(b+1)^{2n+3}}\right). \qquad (3.10)$$

The Z-pair (P, G_b) associated with the summand $F_b(n,k)$ of the left-hand side of (3.10) is

$$P = 1 - (b^2 + 2b + 2)E_n + (b+1)^2 E_n^2, \qquad G_b(n,k) = -\frac{b^{2k}}{(b+1)^{n+k+1}}\binom{n+k+1}{2k-2}. $$

$$(3.11)$$

$\square$

Proof of Theorem 2.4. Since $\binom{2n+1}{2k+1} = 0$ for $k > n$ and $k < 0$, the identity in Theorem 2.4 can be rewritten as

$$\sum_{k \in \mathbb{Z}} \left(\frac{b}{2^{n-1}}\right)\left(\frac{b}{2c}\right)^{2n+1}\left(1+\frac{4c}{b^2}\right)^{k+1}\binom{2n+1}{2k+1} = \frac{\sqrt{\alpha}+\sqrt{\beta}}{\sqrt{\alpha\beta}}\left(\beta^{n+1/2} - \alpha^{n+1/2}\right). \qquad (3.12)$$

The Z-pair $(P, G_{b,c})$ associated with the summand $F_{b,c}(n,k)$ of the left-hand side of (3.12) is

$$P = 1 - (2b^2 + 4c)E_n + 4c^2 E_n^2,$$

$$G_{b,c}(n,k) = \left(\frac{b}{2^{n-1}}\right)\left(\frac{b}{2c}\right)^{2n+3}\left(1+\frac{4c}{b^2}\right)^{k+1}\left(-\frac{(6n-4k+11)b^2}{8k-4}\binom{2n+2}{2k-2}+\binom{2n+1}{2k-1}c\right).$$

$$(3.13)$$

$\square$

Proof of Theorem 2.5. Reversing the order of summation, the identity can be rewritten as

$$\sum_{m=2}^{2n}\sum_{k=0}^{\lfloor (m-2)/2\rfloor} \frac{b^m}{(b+1)^{n+k+1}}\binom{n+k+1}{m} = \frac{b(n+1)}{b+2} - \frac{(b+1)^2}{(b+2)^2}\left(1-\frac{1}{(b+1)^{2n+2}}\right). \qquad (3.14)$$

Let us denote the left-hand side of (3.14) by $S_b(n)$ and its summand by $F_b(n,m,k)$, that is,

$$F_b(n,m,k) = \frac{b^m}{(b+1)^{n+k+1}}\binom{n+k+1}{m}. \qquad (3.15)$$

Then $F_b(n,m,k) = F_b(k,m,n)$ and $F_b(n+1,m,k) = F_b(n,m,k+1)$. This implies that

$$F_b(n+1,m,k) - F_b(n,m,k) = F_b(n,m,k+1) - F_b(n,m,k). \tag{3.16}$$

Summing both sides of (3.16) with respect to k and with respect to m, we get

$$
\begin{aligned}
& S_b(n+1) - \frac{b^{2n+1}}{(b+1)^{2n+1}} - \frac{b^{2n+2}}{(b+1)^{2n+2}} - S_b(n) \\
& = \sum_{m=2}^{2n} \binom{n+\left\lfloor \frac{m}{2} \right\rfloor + 1}{m} \frac{b^m}{(b+1)^{n+\lfloor m/2 \rfloor + 1}} - \sum_{m=2}^{2n} \binom{n+1}{m} \frac{b^m}{(b+1)^{n+1}}.
\end{aligned}
\tag{3.17}
$$

But

$$
\begin{aligned}
\sum_{m=2}^{2n} \binom{n+1}{m} \frac{b^m}{(b+1)^{n+1}} &= \frac{1}{(b+1)^{n+1}} \left(\sum_{m=0}^{n+1} \binom{n+1}{m} b^m - 1 - (n+1)b \right) \\
&= \frac{1}{(b+1)^{n+1}} \left((b+1)^{n+1} - 1 - (n+1)\, b \right) \\
&= 1 - \frac{1}{(b+1)^{n+1}} - \frac{(n+1)b}{(b+1)^{n+1}},
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{m=2}^{2n} \binom{n+\left\lfloor \frac{m}{2} \right\rfloor + 1}{m} \frac{b^m}{(b+1)^{n+\lfloor m/2 \rfloor + 1}} \\
& = \sum_{m=1}^{n} \binom{n+m+1}{2m} \frac{b^{2m}}{(b+1)^{n+m+1}} + \sum_{m=1}^{n-1} \binom{n+m+1}{2m+1} \frac{b^{2m+1}}{(b+1)^{n+m+1}} \\
& = \sum_{m=0}^{n+1} \binom{n+m+1}{2m} \frac{b^{2m}}{(b+1)^{n+m+1}} - \frac{1}{(b+1)^{n+1}} - \frac{b^{2n+2}}{(b+1)^{2n+2}} \\
& \quad + \sum_{m=0}^{n} \binom{n+m+1}{2m+1} \frac{b^{2m+1}}{(b+1)^{n+m+1}} - \frac{(n+1)b}{(b+1)^{n+1}} - \frac{b^{2n+1}}{(b+1)^{2n+1}} \\
& = \frac{b+1}{b+2} \left(1 + \frac{1}{(b+1)^{2n+3}} \right) - \frac{1}{(b+1)^{n+1}} - \frac{b^{2n+2}}{(b+1)^{2n+2}} \\
& \quad + \frac{b+1}{b+2} \left(1 - \frac{1}{(b+1)^{2n+2}} \right) - \frac{(n+1)b}{(b+1)^{n+1}} - \frac{b^{2n+1}}{(b+1)^{2n+1}}.
\end{aligned}
\tag{3.18}
$$

From (3.17) and (3.18), we get that

$$S_b(n+1) - S_b(n) = \frac{b}{b+2} \left(1 - \frac{1}{(b+2)^{2n+2}} \right) \tag{3.19}$$

and $S_b(1) = b^2/(b+1)^2$. The right-hand side of (3.14) also satisfies this recurrence relation. Hence

$$S_b(n) = \frac{b(n+1)}{b+2} - \frac{(b+1)^2}{(b+2)^2}\left(1 - \frac{1}{(b+1)^{2n+2}}\right) \quad \forall n \in \mathbb{N}. \qquad (3.20)$$

$\square$

Remark 3.3. We would like to mention that the recurrence (3.16) can also be automatically generated by MultiSum [10], a Mathematica package which is available from the website http://www.risc.uni-linz.ac.at/research/combinat/software/.
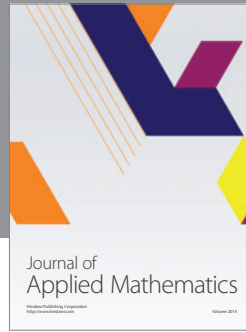
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