UNBOUNDED C^*-SEMINORMS, BIWEIGHTS, AND *-REPRESENTATIONS OF PARTIAL *-ALGEBRAS: A REVIEW

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The notion of (unbounded) C^*-seminorms plays a relevant role in the representation theory of *-algebras and partial *-algebras. A rather complete analysis of the case of *-algebras has given rise to a series of interesting concepts like that of semifinite C^*-seminorm and spectral C^*-seminorm that give information on the properties of *-representations of the given *-algebra $\mathfrak{A}$ and also on the structure of the *-algebra itself, in particular when $\mathfrak{A}$ is endowed with a locally convex topology. Some of these results extend to partial *-algebras too. The state of the art on this topic is reviewed in this paper, where the possibility of constructing unbounded C^*-seminorms from certain families of positive sesquilinear forms, called biweights, on a (partial) *-algebra $\mathfrak{A}$ is also discussed.

1. Introduction

1.1. Motivations. The existence of a C^*-seminorm p defined on a *-algebra $\mathfrak{A}$ (i.e., a seminorm p satisfying the C^*-property $p(a^*a) = p(a)^2$, for every $a \in \mathfrak{A}$) is very closely linked to the existence of nontrivial bounded *-representations of $\mathfrak{A}$ in Hilbert space. For instance, if a bounded *-representation π of $\mathfrak{A}$ exists, then, putting $p(a) = \|\pi(a)\|$, $a \in \mathfrak{A}$, a C^*-seminorm on $\mathfrak{A}$ is immediately obtained. Conversely (and this is much more difficult to prove!) if a C^*-seminorm p on $\mathfrak{A}$ is given, then it is possible to find a bounded *-representation π of $\mathfrak{A}$ such that $p(a) = \|\pi(a)\|$, for every $a \in \mathfrak{A}$. On the other hand, if $\mathfrak{A}$ possesses a C^*-seminorm p, then the family of positive linear functionals on $\mathfrak{A}$ does not reduce to $\{0\}$ and, in turn, a positive linear functional defines, via the Gel'fand-Naĭmark-Segal (GNS) construction, a *-representation of $\mathfrak{A}$, which is however not necessarily bounded, unless a so-called *admissibility* condition is satisfied.

What makes of C^*-seminorms a relevant object to consider is the fact that through them the rich representation theory of C^*-algebras, mostly developed by Gel'fand and

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Naĭmark by the middle of the 20th century (see [27] also for a beautiful historical report) can be invoked. For instance, if $\mathfrak{A}$ is a $*$-algebra and p a C^*-seminorm on $\mathfrak{A}$, then every $*$-representation of the Hausdorff completion of $(\mathfrak{A}, p)$ defines a bounded $*$-representation of $\mathfrak{A}$ in Hilbert space [22]. The existence of a C^*-seminorm guarantees the existence of a rich family of $*$-representations of $\mathfrak{A}$ and this fact clearly reflects on the structure of $\mathfrak{A}$ itself.

This fundamental role of C^*-seminorms in the representation theory of $*$-algebras becomes more evident when the given $*$-algebra $\mathfrak{A}$ is endowed with some topology [27, 37] and it is well known, for instance, that studying the geometric aspects of the collection of all C^*-seminorms gives a deep insight into the structure of Banach $*$-algebras [23, Chapter V.39]. Last, but not least, it is worth reminding that also investigations on more general locally convex $*$-algebras take great advantage from the use of C^*-seminorms (see, e.g., [18, 31, 34]).

The notion of C^*-seminorms has been first considered by Fell [30] and Effros [29] and researches on this topic have been undertaken in several different directions, according to the various situations where they arise.

In many concrete examples, however, a locally convex $*$-algebra $\mathfrak{A}$ does not admit an *everywhere defined* C^*-seminorm, but it is sometimes possible to find a C^*-seminorm p defined only on a $*$-subalgebra $\mathcal{D}(p)$ of $\mathfrak{A}$. These seminorms were first considered by Bhatt et al. [22] who named them *unbounded* C^*-seminorms. On the other hand, *non-everywhere* defined C^*-seminorms had already appeared to be relevant in many mathematical [39, 48] and physical applications [1, 47].

A very natural situation where unbounded C^*-seminorms make their appearance (but they were not named in this way, of course!) was considered by Yood [48]. He studied, in fact, C^*-seminorms on a $*$-algebra $\mathfrak{A}$ that can be defined via a family $\mathcal{F}$ of positive linear functionals on $\mathfrak{A}$. These C^*-seminorms, whose definition is strongly inspired by the Gel'fand seminorm on a Banach $*$-algebra, are, in general, defined only on a $*$-subalgebra of $\mathfrak{A}$. A systematic study of this type of non-everywhere defined C^*-seminorms was then in order and it was actually undertaken by Bhatt et al. [19–22] who obtained a series of deep results in the representation theory of a $*$-algebra, introducing often new concepts or revisiting of old ones (semifinitess, spectrality, stability, etc.).

In the mid 1980's, Antoine and Karwowski [8], retrieving an earlier definition given by Borchers [24], introduced the notion of partial $*$-algebra. This structure appears in a natural way when families of unbounded operators, possessing a common, dense but not necessarily invariant domain, are considered. These studies were also motivated by a number of examples of this type that arise in Quantum theories [28, 32].

Roughly speaking, a *partial $*$-algebra* is a complex vector space $\mathfrak{A}$, with involution $*$ and a multiplication $x \cdot y$ defined only for pairs of *compatible* elements determined by a binary relation Γ on $\mathfrak{A}$. This multiplication is required to be distributive (this makes of the so-called *multiplier* spaces true subspaces of $\mathfrak{A}$) but it is not required to be associative, in general (and, indeed, in many examples it is not!).

After this new object was at hand, Antoine et al. undertook a systematic study of partial $*$-algebras with a special care to partial $*$-algebras of operators (shortly, partial O^*-algebras, see, e.g., [2–5]). A large number of results was found and a long series of papers

appeared. In the meantime other researchers (Mathot, Bagarello, Ekhaguere, etc.) had directed their attention to this subject and all that contributed to make of partial $*$-algebras a rather complete theory. These results are synthesized and sometimes improved in the monograph [7], whose bibliography we refer to.

Coming back to C^*-seminorms, some natural questions can be posed at this point: does the notion of (unbounded) C^*-seminorms extend to the new environment of partial $*$-algebras? Do they play in this case a role as crucial as they do in the theory of representations of $*$-algebras?

A first study in this direction was made in [7, 11], where unbounded C^*-seminorms on partial $*$-algebras were introduced and studied (the usual definition must be adapted, of course, to the lack of an everywhere defined multiplication). As an outcome, some results of representation theory of $*$-algebras (e.g., some of those in [22]) extend to the case of partial $*$-algebras, even if there is a price to pay: the need of a series of sometimes unpleasant technical assumptions.

Also Yood's approach extends to the partial algebraic setting as shown in [44] for the case of quasi $*$-algebras and in [46] for more general partial $*$-algebras. In both cases positive linear functionals are systematically replaced by positive sesquilinear forms, enjoying certain *invariance* properties that make possible to by-pass the partial nature of the multiplication. In particular, in the case of partial $*$-algebras a relevant role is played by a special kind of non-everywhere defined sesquilinear forms called *biweights*: they are exactly the forms that allow a GNS-like contruction for partial $*$-algebras [6, 7]. Of course, one expects that Gel'fand-like seminorms in partial or quasi $*$-algebras can provide useful information on these structures, especially when the latter carry some locally convex topology and a first step in this direction is to consider the case of normed or even Banach partial or quasi $*$-algebras [9, 10, 45].

In this paper we will review the whole subject of (unbounded) C^*-seminorms. We will, in particular, focus our attention both on their interplay with representation theory (Section 2) and on the possibility of constructing (unbounded) C^*-seminorms starting from families of positive linear or sesquilinear forms (Section 3).

Finally, Section 4 is devoted to the construction of Gel'fand-like seminorms on a quasi $*$-algebra $(\mathfrak{A},\mathfrak{A}_0)$. This case exhibits some peculiarities that it is worth mentioning. Indeed, a quasi $*$-algebra $(\mathfrak{A},\mathfrak{A}_0)$ can be easily constructed by completing a locally convex $*$-algebra $\mathfrak{A}_0$ with separately but not jointly continuous multiplications. The topology τ of the completion defines, in a natural way, a *reduced* topology τ_0 on the $*$-algebra $\mathfrak{A}_0$ where the construction started from. If τ is a norm topology and τ_0 can be defined by a complete C^*-norm, then $(\mathfrak{A},\mathfrak{A}_0)$ is called a CQ^*-algebra. The notion of a CQ^*-algebra was introduced by Bagarello and the present author [14] and further studies were developed in [15, 16]. The interest of this structure relies on the fact that CQ^*-algebras exhibit a certain number of analogies with C^*-algebras and this seems to make of them a natural extension of the notion of a C^*-algebra in the partial algebraic setting. As well as the theory of $*$-algebras makes natural to consider C^*-seminorms, the notion of a quasi $*$-algebra leads in the same spirit to consideration of CQ^*-seminorms. This notion, which is different from that of (unbounded) C^*-seminorm considered in the previous sections,

can be used to get, in similar fashion, information on the structure of a locally convex quasi $*$-algebra.

Before closing this introduction, we want to warn the reader that only a few statements are proven here in detail. This choice is essentially due to the will of maintaining the paper as readable as possible, avoiding length and technicalities. Proofs are given only when the statement is presented in a form which differs from the original one or when it is not very easy to find them in the current literature.

1.2. Preliminaries. Before going to the main matter of the paper, we collect below some preliminary definitions.

A *partial $*$-algebra* is a complex vector space $\mathfrak{A}$, endowed with an involution $x \mapsto x^*$ (i.e., a bijection such that $x^{**} = x$) and a partial multiplication defined by a set $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$, for all $\lambda, \mu \in \mathbb{C}$;

(iii) for any $(x, y) \in \Gamma$, a product $x \cdot y$ is defined in $\mathfrak{A}$, which is distributive with respect to the addition and satisfies the relation $(x \cdot y)^* = y^* \cdot x^*$.

We say that a partial $*$-algebra $\mathfrak{A}$ has a unit if there exists an element $e \in \mathfrak{A}$ such that $e^* = e$, $(e, x) \in \Gamma$, for all $x \in \mathfrak{A}$, and $e \cdot x = x \cdot e = x$, for all $x \in \mathfrak{A}$. (If $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial $*$-algebra with unit, in the standard fashion.)

Given the defining set Γ, spaces of multipliers are defined in the obvious way:

$$\begin{aligned} (x, y) \in \Gamma &\Longleftrightarrow x \in L(y) \quad \text{or} \quad x \text{ is a left multiplier of } y \\ &\Longleftrightarrow y \in R(x) \quad \text{or} \quad y \text{ is a right multiplier of } x. \end{aligned} \tag{1.1}$$

If $\mathfrak{M} \subseteq \mathfrak{A}$, we put

$$L\mathfrak{M} = \{x \in \mathfrak{A} : (x, y) \in \Gamma,\ \forall\, y \in \mathfrak{M}\}, \qquad R\mathfrak{M} = \{x \in \mathfrak{A} : (y, x) \in \Gamma,\ \forall\, y \in \mathfrak{M}\}. \tag{1.2}$$

In particular elements of $R\mathfrak{A}$ (resp., $L\mathfrak{A}$) are called *universal* right (resp., left) multipliers.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a *partial $*$-algebra* [7] with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \restriction \mathcal{D}$, and the (*weak*) partial multiplication $X_1 \square X_2 = {X_1}^{\dagger *} X_2$, defined by

$$\begin{gathered} (X_1, X_2) \in \Gamma \Longleftrightarrow X_2 \mathcal{D} \subset D(X_1^{\dagger *}), \quad X_1^\dagger \mathcal{D} \subset D(X_2^*), \\ (X_1 \square X_2)\xi := X_1^{\dagger *} X_2 \xi, \quad \forall \xi \in \mathcal{D}. \end{gathered} \tag{1.3}$$

If $(X_1, X_2) \in \Gamma$, we say that X_2 is a weak right multiplier of X_1 or, equivalently, that X_1 is a weak left multiplier of X_2 (we write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$). When we regard $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ as a partial $*$-algebra with those operations, we denote it by $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$.

A *partial O^*-algebra* on $\mathcal{D}$ is a $*$-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that $X^\dagger \in \mathfrak{M}$ whenever $X \in \mathfrak{M}$ and $X_1 \square X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_2 \in R^{\mathrm{w}}(X_1)$.

Let

$$\mathscr{L}^\dagger(\mathscr{D}) = \{X \in \mathscr{L}^\dagger(\mathscr{D},\mathscr{H}) : X\mathscr{D} \subseteq D,\ X^\dagger\mathscr{D} \subseteq D\}. \tag{1.4}$$

Then $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra with respect to $\square$ and $X_1\square X_2\xi = X_1(X_2\xi)$ for each $\xi \in \mathscr{D}$. A $*$-subalgebra of $\mathscr{L}^\dagger(\mathscr{D})$ is called an O^*-algebra [33, 40].

A $*$-representation of a partial $*$-algebra $\mathfrak{A}$ is a $*$-homomorphism of $\mathfrak{A}$ into $\mathscr{L}^\dagger_{\mathrm{w}}(\mathscr{D}_\pi, \mathscr{H}_\pi)$, for some pair $(\mathscr{D}_\pi,\mathscr{H}_\pi)$, where $\mathscr{D}_\pi$ is a dense subspace of a Hilbert space $\mathscr{H}_\pi$, that is, a linear map $\pi : \mathfrak{A} \to \mathscr{L}^\dagger_{\mathrm{w}}(\mathscr{D}_\pi,\mathscr{H}_\pi)$ such that (i) $\pi(a^*) = \pi(a)^\dagger$ for every $a \in \mathfrak{A}$; (ii) if $a,b \in \mathfrak{A}$ with $a \in L(b)$, then $\pi(a) \in L^{\mathrm{w}}(\pi(b))$ and $\pi(a)\square\pi(b) = \pi(ab)$. If $\mathfrak{A}$ has a unit e, we assume that $\pi(e) = 1$, the identity operator.

If $\mathfrak{A}$ is a $*$-algebra, we may always suppose, without loss of generality, that $\pi(\mathfrak{A}) \subseteq \mathscr{L}^\dagger(\mathscr{D}_\pi)$. Indeed, if π is a $*$-representation of a $*$-algebra $\mathfrak{A}$ into $\mathscr{L}^\dagger_{\mathrm{w}}(\mathscr{D}_\pi,\mathscr{H}_\pi)$, we can define

$$\begin{gathered}\mathscr{D}_{\hat\pi} = \left\{\xi 0 + \sum_{k=1}^n \pi(a_k)\xi_k;\ \xi 0, \xi_k \in \mathscr{D},\ a_k \in \mathfrak{A},\ k = 1,2,\dots,n\right\},\\ \hat\pi(x)\left(\xi 0 + \sum_{k=1}^n \pi(a_k)\xi_k\right) := \pi(x)\xi 0 + \sum_{k=1}^n (\pi(x)\square\pi(a_k))\xi_k.\end{gathered} \tag{1.5}$$

Then $\hat\pi$ is a $*$-representation of $\mathfrak{A}$ on the domain $\mathscr{D}_{\hat\pi}$ having the property $\hat\pi(\mathfrak{A}) \subset \mathscr{L}^\dagger(\mathscr{D}_{\hat\pi})$.

A $*$-representation π of a partial $*$-algebra $\mathfrak{A}$ is called

(i) *cyclic*: if there exists $\eta \in \mathscr{D}_\pi$ such that $\pi(\mathfrak{A})\eta$ is dense in $\mathscr{H}_\pi$;

(ii) *faithful*: if $\pi(a) = 0$ implies $a = 0$.

If π is a $*$-representation of $\mathfrak{A}$ in $\mathscr{L}^\dagger_{\mathrm{w}}(\mathscr{D}_\pi,\mathscr{H}_\pi)$, then the *closure* $\tilde\pi$ of π is defined, for each $x \in \mathfrak{X}$, as the restriction of $\overline{\pi(x)}$ to the domain $\widetilde{\mathscr{D}_\pi}$, which is the completion of $\mathscr{D}_\pi$ under the *graph topology* defined by the seminorms $\xi \in \mathscr{D}_\pi \to \|\pi(x)\xi\|$, $x \in \mathfrak{X}$ [7]. If $\pi = \tilde\pi$, the representation is said to be *closed*.

The adjoint of a $*$-representation π of a partial $*$-algebra $\mathfrak{A}$ is defined as follows:

$$\begin{gathered}\mathscr{D}(\pi^*) = \mathscr{D}^*(\pi(\mathfrak{A})) \equiv \bigcap_{x\in\mathfrak{A}} \mathscr{D}(\pi(x)^*), \quad \pi^*(x) = \pi(x^*)^* \restriction \mathscr{D}(\pi^*), \quad x \in \mathfrak{A},\\ \mathscr{D}(\pi^{**}) = \mathscr{D}^{**}(\pi(\mathfrak{A})) \equiv \bigcap_{x\in\mathfrak{A}} \mathscr{D}(\pi^*(x)^*),\\ \pi^{**}(x) = \pi^*(x^*)^* \restriction \mathscr{D}(\pi^{**}), \quad x \in \mathfrak{A}.\end{gathered} \tag{1.6}$$

In general neither π^* nor π^{**} are $*$-representations of $\mathfrak{A}$. If $\mathfrak{A}$ is a $*$-algebra, then π^* is a representation of $\mathfrak{A}$ and π^{**} is a $*$-representation of $\mathfrak{A}$. If $\pi = \pi^*$, then π is said to be *self-adjoint*.

The set of all $*$-representations of $\mathfrak{A}$ is denoted with $\mathrm{Rep}(\mathfrak{A})$.

The definition of a *quasi $*$-algebra* was originally given by Lassner [35, 36] independently of that of a partial $*$-algebra.

A quasi $*$-algebra [40] is a couple $(\mathfrak{A},\mathfrak{A}_0)$, where $\mathfrak{A}$ is a vector space with involution $*$, $\mathfrak{A}_0$ is a $*$-algebra and a vector subspace of $\mathfrak{A}$, and $\mathfrak{A}$ is an $\mathfrak{A}_0$-bimodule whose module

operations and involution extend those of $\mathfrak{A}_0$. Clearly, a quasi $*$-algebra may be viewed as an instance of a partial $*$-algebra in obvious fashion.

As already mentioned, the most typical instance of a quasi $*$-algebra is provided by the completion $\mathfrak{A} := \widehat{\mathfrak{A}_0}$ of a locally convex $*$-algebra $\mathfrak{A}_0[\tau]$ whose multiplication is not jointly continuous. This situation is rather important for concrete applications: in quantum statistical mechanics, for example, one often takes the completion of the local observable algebra (typically a C^*-algebra) in some locally convex topology suggested by the physical model under consideration [12, 13, 35, 36, 43].

Full details on partial $*$-algebras and their representation theory can be found in the monograph [7].

2. C^*-seminorms and $*$-representations

This section is devoted to unbounded C^*-seminorms on partial $*$-algebras. In particular, the interplay between C^*-seminorms and the representation theory of partial $*$-algebras will be reviewed. This has been discussed in full detail in [11, 46] and so we will only present the main results mostly without proving them.

2.1. Representations originating from a C^*-seminorm

Definition 2.1. A mapping p of a (partial) $*$-subalgebra $\mathfrak{D}(p)$ of a partial $*$-algebra $\mathfrak{A}$ into $\mathbb{R}^+$ is said to be an *unbounded* m^*-(semi)norm on $\mathfrak{A}$ if

(i) p is a (semi) norm on $\mathfrak{D}(p)$;

(ii) $p(x^*) = p(x)$, for all $x \in \mathfrak{D}(p)$;

(iii) $p(xy) \leq p(x)p(y)$, for all $x, y \in \mathfrak{D}(p)$ such that $x \in L(y)$.

An unbounded m^*-(semi)norm p on $\mathfrak{A}$ is said to be an unbounded C^*-(semi)norm if

(iv) $p(x^*x) = p(x)^2$, for all $x \in \mathfrak{D}(p)$ such that $x^* \in L(x)$.

An unbounded m^*-(semi) norm (resp., C^*-(semi)norm) on $\mathfrak{A}$ is said to be an m^**-(semi) norm* (resp., C^**-(semi)norm*) if $\mathfrak{D}(p) = \mathfrak{A}$.

An (unbounded) m^*-seminorm p on $\mathfrak{A}$ is said to have property (D) if it satisfies the following basic density condition:

$$R\mathfrak{A} \cap \mathfrak{D}(p) \text{ is dense in } \mathfrak{D}(p) \text{ with respect to } p. \tag{D}$$

In what follows we will assume that $\mathfrak{A}$ is a *semiassociative* partial $*$-algebra, by which we mean that $y \in R(x)$ implies $yz \in R(x)$ for every $z \in R\mathfrak{A}$ and

$$(xy)z = x(yz). \tag{2.1}$$

Example 2.2. Let π be a $*$-representation of a (partial) $*$-algebra $\mathfrak{A}$. Then, an unbounded C^*-seminorm r_π of $\mathfrak{A}$ is defined by

$$\begin{gathered} \mathfrak{D}(r_\pi) = \mathfrak{A}_b^\pi \equiv \{x \in \mathfrak{A};\ \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\}, \\ r_\pi(x) = \|\overline{\pi(x)}\|, \quad x \in \mathfrak{D}(r_\pi). \end{gathered} \tag{2.2}$$

In the first part of this section we will consider the question as to whether, given an unbounded C^*-seminorm p, there exists a $*$-representation π of $\mathfrak{A}$ such that $p(a) = \|\overline{\pi(a)}\|$, for every $a \in \mathcal{D}(p)$. For a more complete discussion of this problem we refer to [7, 11].

Lemma 2.3 [7, Lemma 8.1.2]. *Let p be an m^*-seminorm on $\mathfrak{A}$ having property (D), that is, $R\mathfrak{A}$ is p-dense in $\mathfrak{A}$. Denote by $\widehat{\mathfrak{A}}$ the set of all Cauchy sequences in $\mathfrak{A}$ with respect to the seminorm p and define an equivalence relation in $\widehat{\mathfrak{A}}$ as follows:*

$\{a_n\} \sim \{b_n\}$ if and only if $\lim_{n\to\infty} p(a_n - b_n) = 0$. Then the following statements hold.

(1) *The quotient space $\widehat{\mathfrak{A}}/\sim$ is a Banach $*$-algebra under the following operations, involution, and norm:*

$$\begin{gathered}
\{a_n\}^\sim + \{b_n\}^\sim \equiv \{a_n + b_n\}^\sim; \qquad \lambda\{a_n\}^\sim \equiv \{\lambda a_n\}^\sim; \\
\{a_n\}^\sim\{b_n\}^\sim \equiv \{x_n y_n\}^\sim, \quad \text{where } \{x_n\}^\sim,\ \{y_n\}^\sim \text{ in } R\mathfrak{A} \text{ s.t.} \\
\{x_n\}^\sim \equiv \{a_n\}^\sim, \qquad \{y_n\}^\sim \equiv \{b_n\}^\sim; \\
\{a_n\}^{\sim *} \equiv \{a_n^*\}^\sim, \qquad \|\{a_n\}^\sim\|_p \equiv \lim_{n\to\infty} p(a_n).
\end{gathered} \tag{2.3}$$

(2) *For each $a \in \mathfrak{A}$, put*

$$\begin{gathered}
\widetilde{a} = \{a_n\}^\sim \quad (a_n = a,\ n \in \mathbb{N}), \\
\widetilde{\mathfrak{A}} = \{\widetilde{a};\ a \in \mathfrak{A}\}.
\end{gathered} \tag{2.4}$$

Then $\widetilde{\mathfrak{A}}$ is a dense $*$-invariant subspace of $\widehat{\mathfrak{A}}/\sim$ satisfying $\widetilde{a}\widetilde{b} = (ab)^\sim$ whenever $a \in L(b)$.

(3) *Suppose p is a C^*-seminorm on $\mathfrak{A}$. Then $\widehat{\mathfrak{A}}/\sim$ is a C^*-algebra.*

The proof that $\widehat{\mathfrak{A}}/\sim$ is a Banach space is made as in the usual construction of the completion of a normed space. The existence of an everywhere defined multiplication in $\widehat{\mathfrak{A}}/\sim$ and the algebra properties depend, in essential way, on the semiassociativity of $\mathfrak{A}$.

The previous lemma is very relevant for our purposes, since a C^*-algebra has plenty of $*$-representations (and also a faithful one, by the Gel'fand-Naĭmark theorem) and they can be used to construct $*$-representations of $\mathfrak{A}$. Indeed if p is an unbounded C^*-seminorm on $\mathfrak{A}$ with property (D), then Lemma 2.3 can be applied to the partial $*$-algebra $\mathcal{D}(p)$. We put $\mathfrak{A}_p := \mathcal{D}(p)/\sim$. Then, for any faithful $*$-representation Π_p of $\mathfrak{A}_p$, we put

$$\pi_p^0(x) = \Pi_p(\widetilde{x}), \quad x \in \mathcal{D}(p), \tag{2.5}$$

where $\widetilde{x}$ denotes any sequence p-converging to x. Then π_p^0 is a bounded $*$-representation of $\mathcal{D}(p)$ on $\mathcal{H}_{\Pi_p}$.

Of course π_p^0 is only a $*$-representation of $\mathcal{D}(p)$ and, in general, it cannot be extended to the whole $\mathfrak{A}$. The possibility of doing this depends, in crucial way, on the *algebra*

$$\mathcal{N}_p = \{x \in \mathcal{D}(p) \cap R\mathfrak{A};\ ax \in \mathcal{D}(p),\ \forall a \in \mathfrak{A}\}. \tag{2.6}$$

We give below an outline of the construction and an account of the main results obtained on this problem. Proofs can be found in the monograph [7].

We remind the notion of nondegeneracy of a $*$-representation π of $\mathfrak{A}$ given in [7]: we put, as in Example 2.2,

$$\mathfrak{A}_b^\pi = \{x \in \mathfrak{A};\ \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\}, \tag{2.7}$$

$$\mathfrak{N}_\pi = \{x \in \mathfrak{A}_b^\pi \cap R\mathfrak{A};\ ax \in \mathfrak{A}_b^\pi,\ \forall a \in \mathfrak{A}\}. \tag{2.8}$$

Definition 2.4. If $\pi(\mathfrak{N}_\pi)\mathfrak{D}(\pi)$ is total in $\mathcal{H}_\pi$, then π is said to be *strongly nondegenerate*.

Now suppose that $\mathcal{N}_p \not\subset \operatorname{Ker} p$. Then we begin with defining the domain $\mathfrak{D}(\pi_p)$ as the linear span of the set

$$\left\{\Pi_p((xy)^\sim)\xi;\ x, y \in \mathcal{N}_p,\ \xi \in \mathcal{H}_{\Pi_p}\right\}, \tag{2.9}$$

and the Hilbert space $\mathcal{H}_{\pi_p}$ as the closure of $\mathfrak{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$. Next we define

$$\begin{aligned} &\pi_p(a)\left(\sum_k \Pi_p(\widetilde{x_k})\xi_k\right) \\ &\quad = \sum_k \Pi_p((ax_k)^\sim)\xi_k, \quad \text{(finite sums) for } a \in \mathfrak{A},\ \{x_k\} \subset \mathcal{N}_p,\ \{\xi_k\} \in \mathcal{H}_{\Pi_p}. \end{aligned} \tag{2.10}$$

Then the following statement holds.

Theorem 2.5. *Let p be an unbounded C^*-seminorm on $\mathfrak{A}$ with property (D). Suppose $\mathcal{N}_p \not\subset \operatorname{Ker} p$. Then, for any $\Pi_p \in \operatorname{Rep}(\mathfrak{A}_p)$, there exists a strongly nondegenerate $*$-representation π_p of $\mathfrak{A}$ such that*

(i) $\|\overline{\pi_p(x)}\| \le p(x)$ *for every* $x \in \mathfrak{D}(p)$;

(ii) $\|\overline{\pi_p(x)}\| = p(x)$ *for every* $x \in \mathcal{N}_p$.

Let p be an unbounded C^*-seminorm with property (D) and such that $\mathcal{N}_p \not\subset \operatorname{Ker} p$. We denote by $\operatorname{Rep}(\mathfrak{A}, p)$ the class of all $*$-representations of $\mathfrak{A}$ constructed as above from $\operatorname{Rep}(\mathfrak{A}_p)$, that is,

$$\operatorname{Rep}(\mathfrak{A}, p) = \{\pi_p;\ \Pi_p \in \operatorname{Rep}(\mathfrak{A}_p)\}. \tag{2.11}$$

Definition 2.6. The unbounded C^*-seminorm p is called

(i) *finite* if $\mathcal{N}_p = \mathfrak{D}(p)$,

(ii) *semifinite* if $\mathcal{N}_p$ is p-dense in $\mathfrak{D}(p)$,

(iii) *weakly semifinite* if it has property (D) and

$$\operatorname{Rep}^{\mathrm{WB}}(\mathfrak{A}, p) \equiv \{\pi_p \in \operatorname{Rep}(\mathfrak{A}, p);\ \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\} \neq \varnothing \tag{2.12}$$

and an element π_p of $\operatorname{Rep}^{\mathrm{WB}}(\mathfrak{A}, p)$ is said to be a *well-behaved* $*$-representation of $\mathfrak{A}$ in $\operatorname{Rep}(\mathfrak{A}, p)$.

We remark that semifinite unbounded C^*-seminorms automatically satisfy property (D) and the condition $\mathcal{N}_p \not\subset \operatorname{Ker} p$.

Theorem 2.7. *Let p be an unbounded C^*-seminorm on $\mathfrak{A}$ with property (D). Then the following statements hold.*

(1) *If p is semifinite, then it is weakly semifinite and*

$$\mathrm{Rep}^{\mathrm{WB}}(\mathfrak{A},p)=\{\pi_p\in\mathrm{Rep}(\mathfrak{A},p);\ \Pi_p \text{ is nondegenerate}\}. \tag{2.13}$$

(2) *Suppose $\pi_p\in\mathrm{Rep}^{\mathrm{WB}}(\mathfrak{A},p)$. Then*

$$\|\overline{\pi_p(x)}\|=p(x),\quad \forall x\in\mathfrak{D}(p). \tag{2.14}$$

Conversely, suppose that $\pi_p\in\mathrm{Rep}(\mathfrak{A},p)$ satisfies condition (2.14) above. Then there exists an element π_p^{WB} of $\mathrm{Rep}^{\mathrm{WB}}(\mathfrak{A},p)$ which is a restriction of π_p.

Remark 2.8. Of course given a *-representation π of a partial *-algebra, it is possible to construct an unbounded C^*-seminorm r_π on $\mathfrak{A}$, as in Example 2.2. Following the same steps as before, we can also build up a *-representation $\pi_{r_\pi}^N$ called *natural*. The relationship between $\pi_{r_\pi}^N$ and the *-representation π where we had started from was investigated in [11].

2.2. Main results on *-algebras. If $\mathfrak{A}$ is a *-algebra, then what we have discussed so far applies, but many simplifications occur and, clearly, a larger amount of results is obtained. This simplification begins with the definition itself. Indeed, Definition 2.1 reads as follows.

Definition 2.9. Let $\mathfrak{A}$ be a *-algebra and $\mathfrak{D}(p)$ a *-subalgebra of $\mathfrak{A}$. A seminorm p on $\mathfrak{D}(p)$ is called an *unbounded C^*-seminorm* if

$$p(a^*a)=p(a)^2,\quad \text{for every } a\in\mathfrak{D}(p). \tag{2.15}$$

In fact, from a beautiful result of Sebestyén [42] it follows that the C^*-condition implies *-preservation and submultiplicativity, that is,

$$p(ab)\le p(a)p(b),\quad \forall a,b\in\mathfrak{D}(p). \tag{2.16}$$

A first question concerns the existence of a well-behaved *-representation. By the definition itself, this is equivalent to the existence of a weakly semifinite unbounded C^*-seminorm.

Proposition 2.10 [18, Proposition 2.4]. *Let $\mathfrak{A}$ be a *-algebra with unit e. The following statements are equivalent.*

(i) *There exists a well-behaved *-representation, that is, there exists a weakly semifinite unbounded C^*-seminorm on $\mathfrak{A}$.*

(ii) *There exists a strongly nondegenerate *-representation of $\mathfrak{A}$.*

(iii) *There exists an unbounded C^*-seminorm on $\mathfrak{A}$ satisfying $\mathcal{N}_p\not\subset\mathrm{Ker}\,p$.*

Remark 2.11. It is worth noticing that a different notion of well-behaved *-representation was given by Schmüdgen [41]. The two notions, that are seemingly unrelated, essentially because the starting points of the two definitions are deeply different, are compared in [7] and they are shown to agree in several situations.

As we have seen, unbounded C^*-seminorms contribute to clarification of the behavior of the family of $*$-representations of a $*$-algebra $\mathfrak{A}$. But they also give information on the structure of $\mathfrak{A}$ itself, when this notion is related to the spectral theory of $\mathfrak{A}$. The most interesting results in this direction are due to Bhatt et al. [22]. Generalizing a notion of Palmer [37], they considered *spectral* unbounded C^*-seminorms.

Assume that $\mathfrak{A}$ has a unit e. Then, as usual, the spectrum $\sigma(a)$ of an element $a \in A$ is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} : (a - \lambda e) \text{ is not invertible in } \mathfrak{A}\} \tag{2.17}$$

and the spectral radius of a as

$$r_{\mathfrak{A}}(a) = \sup\{|\lambda|;\ \lambda \in \sigma(a)\}. \tag{2.18}$$

Definition 2.12. An unbounded m^*-seminorm on $\mathfrak{A}$ is called *spectral* if

$$r_{\mathfrak{D}(p)}(a) \le p(a), \quad \forall a \in \mathfrak{D}(p). \tag{2.19}$$

Proposition 2.13 [22, Lemma 6.1]. *Let p be an unbounded C^*-seminorm on $\mathfrak{A}$. The following statements are equivalent.*

(i) *p is spectral.*
(ii) *For each $a \in D(p)$ such that $p(a) < 1$, $e - a$ has an inverse in $\mathfrak{A}$.*
(iii) *$r_{\mathfrak{D}(p)}(a) \le p(a)$, for all $a \in \mathfrak{D}(p)$.*
(iii) *$r_{\mathfrak{D}(p)}(a) = \lim_{n\to\infty} p(a^n)^{1/n}$, for all $a \in \mathfrak{D}(p)$.*
(iv) *$r_{\mathfrak{D}(p)}(a) = p(a)$, for all $a \in \mathfrak{D}(p)$ such that $a^*a = aa^*$.*

Definition 2.14. An unbounded C^*-seminorm p on $\mathfrak{A}$ is called *hereditary spectral* if for any $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ the restriction $p\restriction \mathfrak{B}$ of p to $\mathfrak{B}$ is spectral.

It is known (see [25, Proposition 2.10.2]) that if $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{B}$ a closed $*$-subalgebra of $\mathfrak{A}$, then for any $*$-representation π of $\mathfrak{B}$ on $\mathcal{H}_\pi$, there exists a $*$-representation $\hat{\pi}$ of $\mathfrak{A}$ in a Hilbert space $\mathcal{H}_{\hat{\pi}}$ such that $\mathcal{H}_\pi$ is a closed subspace of $\mathcal{H}_{\hat{\pi}}$ and $\pi(a) = \hat{\pi}(a)\restriction \mathcal{H}_\pi$ for every $a \in \mathfrak{B}$. This suggests of defining [22] a notion of *stability* for an unbounded C^*-seminorm p.

Let π be a $*$-representation of $\mathfrak{A}$ with domain $\mathfrak{D}_\pi$ in Hilbert space $\mathcal{H}_\pi$. Then the $*$-subalgebra $\mathfrak{A}_b^\pi$, defined in (2.7), is the domain of a natural unbounded C^*-seminorm r_π related to π:

$$r_\pi(a) = \left|\left|\overline{\pi(a)}\right|\right|, \quad a \in \mathfrak{A}_b^\pi. \tag{2.20}$$

Definition 2.15. An unbounded C^*-seminorm p is called *stable* if for any $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and any $*$-representation π of $\mathfrak{B}$ such that

(a) $\mathfrak{B} \cap \mathfrak{D}(p) \subset \mathfrak{B}_b^\pi$;
(b) $\pi(\mathfrak{B} \cap \mathfrak{D}(p))\mathfrak{D}_\pi$ is dense in $\mathcal{H}_\pi$

there exists a $*$-representation ρ of $\mathfrak{A}$ such that

(i) $\mathfrak{D}(p) \subset \mathfrak{A}_b^\rho$;
(ii) $\rho(\mathfrak{D}(p))\mathfrak{D}_\rho$ is dense in $\mathcal{H}_\rho$;

(iii) $\mathcal{H}_p$ is a closed subspace of $\mathcal{H}_\rho$;
(iv) $\overline{\pi(a)} = \overline{\rho(a)} \restriction \mathcal{H}_\pi$ for every $a \in \mathfrak{B} \cap \mathcal{D}(p)$.

The following characterization [22, Theorem 6.10] of the hereditary spectrality of p can then be read as a generalization of the property of C^*-algebras mentioned above.

Theorem 2.16. *Let $\mathfrak{A}$ be a $*$-algebra and p a semifinite unbounded C^*-seminorm on $\mathfrak{A}$. The following statements are equivalent:*
(i) *p is hereditary spectral;*
(ii) *p is spectral and stable.*

Definition 2.17. A $*$-algebra $\mathfrak{A}$ is called an *unbounded (hereditary) C^*-spectral algebra* if it admits an unbounded (hereditary) spectral C^*-seminorm.

Example 2.18. Every GB^*-algebra in the sense of Dixon [26] is an unbounded C^*-spectral algebra [22, Example 7.1].

As the foregoing discussion shows, the notion of spectrality for an unbounded C^*-seminorm on a $*$-algebra has many interesting consequences. For this reason, it would be rather important to have at our disposal fairly good extensions of this notion when dealing with partial $*$-algebras. Unfortunately, a reasonable general definition of spectrum for an element of a partial $*$-algebra still fails: the lack of associativity causes even difficulties of defining a unique inverse of an element!

A complete overview of all the work that has been done for unbounded C^*-seminorms on $*$-algebras goes perhaps beyond the confines of this review paper. But there are however some results that deserve, at least, to be mentioned for the sake of information.

In [18], Bhatt et al. studied the existence of well-behaved $*$-representations for a locally convex $*$-algebra by means of unbounded C^*-seminorms. In this case the interplay between the initial locally convex topology of $\mathfrak{A}$ and an unbounded C^*-seminorm allows to get a characterization of the existence of well-behaved $*$-representations.

Recently Bhatt et al. [17] have studied the existence of *spectral* well-behaved representations of a $*$-algebra. The notion of spectral $*$-representation was introduced in [19]: if $\mathfrak{A}$ is a $*$-algebra and π a (bounded) $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\pi$, then π is called a spectral $*$-representation if

$$\sigma_{\mathfrak{A}}(a) = \sigma_{\mathcal{C}^*(\pi)}(\pi(a)) \cup \{0\}, \quad \forall a \in \mathfrak{A}, \tag{2.21}$$

where $\mathcal{C}^*(\pi)$ is the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ given by the norm-closure $\overline{\pi(\mathfrak{A})}$ of the image $\pi(\mathfrak{A})$ of π in $\mathcal{B}(\mathcal{H}_\pi)$.

Finally, in [34], Inoue and Takeshita have obtained a series of results on the structure of a locally convex $*$-algebra with a topology τ possessing an unbounded C^*-norm p such that the topology τ_p defined by p on $\mathcal{D}(p)$ is finer than $\tau \restriction \mathcal{D}(p)$ and the identity map is closable from $\mathcal{D}(p)[\tau]$ into $\mathcal{D}(p)[\tau_p]$. They have also characterized locally convex $*$-algebras with a normal unbounded C^*-norm that are GB^*-algebras in the sense of Dixon [26].

3. Constructing C^*-seminorms from sesquilinear forms

In the previous section we have discussed the relationship between unbounded C^*-seminorms and $*$-representations of a partial $*$-algebra $\mathfrak{A}$ and the lesson is that under certain circumstances (weakly semifinite unbounded C^*-seminorms, on one side; well-behaved representations, on the other side) this connection is actually very close. But, as it is well known, positive linear functionals play a fundamental role for the existence of $*$-representations of a given $*$-algebra $\mathfrak{A}$ and C^*-seminorms defined by families of positive linear functionals are often used to get information on the structure of a normed or Banach $*$-algebra [23, 37]. Thus it is natural, in the present framework, to try to extend these facts to partial $*$-algebras. To begin with, we summarize in the next example the situation for Banach $*$-algebras. This will help the reader to understand where our work is aimed to.

Example 3.1 (normed $*$-algebras). If $\mathfrak{A}$ is a Banach $*$-algebra with unit e, then every positive linear functional ω is continuous and $\|\omega\| = \omega(e)$. This fact allows us to construct the so-called Gel'fand seminorm p on $\mathfrak{A}$ by putting

$$p(a) = \sup_{\omega\in\mathcal{S}(\mathfrak{A})} \omega(a^*a)^{1/2}, \quad a \in \mathfrak{A}, \tag{3.1}$$

where $\mathcal{S}(\mathfrak{A})$ denotes the set of all positive linear functionals ω on $\mathfrak{A}$ with $\omega(e) = 1$. One has $p(a) \le \|a\|$ for every $a \in \mathfrak{A}$. If ω is a positive linear functional on $\mathfrak{A}$ and $x \in \mathfrak{A}$, then the linear functional ω_x defined by $\omega_x(a) = \omega(x^*ax)$ is also positive and, if $\omega(x^*x) = 1$, then $\omega_x \in \mathcal{S}(\mathfrak{A})$. Using this fact, that, as we will see, is the key for the construction of Gel'fand-like seminorms in general $*$-algebras (cf. Definition 3.5), one can prove that (3.1) actually defines a C^*-seminorm on $\mathfrak{A}$. In general, p is not a norm; that is, there might exist nonzero elements $a \in \mathfrak{A}$ such that $p(a) = 0$. Let

$$\mathcal{K}_p \equiv \operatorname{Ker} p = \{b \in \mathfrak{A} : p(b) = 0\}. \tag{3.2}$$

Then it is easily seen that $\mathcal{K}_p$ is a closed $*$-ideal of $\mathfrak{A}$. The completion of the quotient $\mathfrak{A}/\mathcal{K}_p$ is a C^*-algebra, with norm $\|[a]\|_\sim = p(a)$. The set $\mathcal{K}_p$ is nothing but the $*$-radical $\mathcal{R}^*$ of the Banach $*$-algebra $\mathfrak{A}$ (i.e., the intersection of the kernels of all $*$-representations of $\mathfrak{A}$) and so $*$-semisimplicity of $\mathfrak{A}$ corresponds to $\mathcal{K}_p = \{0\}$. If $\mathfrak{A}$ is a C^*-algebra, for each $a \in \mathfrak{A}$, there exists a positive linear functional ω with $\omega(e) = 1$ such that $\omega(a^*a) = \|a\|^2$. This then leads to the well-known Gel'fand's characterization of the norm of a C^*-algebra:

$$\|a\| = p(a) = \sup_{\omega\in\mathcal{S}(\mathfrak{A})} \omega(a^*a)^{1/2}, \quad \forall a \in \mathfrak{A}. \tag{3.3}$$

3.1. The case of $*$-algebras. Gel'fand-like seminorms on a $*$-algebra where no topology has been given a priori have been considered by Yood [48]. We will now review his main results, proposing a slightly different approach which makes use of the Gel'fand-Naĭmark-Segal (GNS) contruction, that we shortly summarize. For general $*$-algebras the GNS construction was first proved by Powers [38].

If $\mathfrak{A}$ is a $*$-algebra, we denote by $\mathcal{P}(\mathfrak{A})$ the set of all positive linear functionals on $\mathfrak{A}$.

Let $\omega \in \mathcal{P}(\mathfrak{A})$. Then the set

$$N_\omega = \{a \in \mathfrak{A} : \omega(a^* a) = 0\} \tag{3.4}$$

is a left-ideal of $\mathfrak{A}$. The quotient $\mathcal{D}_\omega := \mathfrak{A}/N_\omega$ can be made into a pre-Hilbert space with inner product

$$\langle \lambda_\omega(a) \mid \lambda_\omega(b) \rangle = \omega(b^* a), \quad a, b \in \mathfrak{A}, \tag{3.5}$$

where $\lambda_\omega(a)$, $a \in \mathfrak{A}$, denotes the coset containing a. Let $\mathcal{H}_\omega$ be the Hilbert space obtained by completion of $\mathcal{D}_\omega$.

Then one defines, for $a \in \mathfrak{A}$

$$\pi_\omega^\circ(a)\lambda_\omega(b) = \lambda_\omega(ab), \quad b \in \mathfrak{A}. \tag{3.6}$$

Then π_ω° is a *-representation of $\mathfrak{A}$; we denote with π_ω its closure which is also a *-representation with domain $\widetilde{\mathcal{D}}_\omega$.

Theorem 3.2. *Let $\mathfrak{A}$ be a *-algebra and ω a positive linear functional on $\mathfrak{A}$. Then there exist a dense domain $\widetilde{\mathcal{D}}_\omega$ in a Hilbert space $\mathcal{H}_\omega$ and a closed *-representation π_ω such that*

$$\langle \pi_\omega(a)\lambda_\omega(b) \mid \lambda_\omega(c) \rangle = \omega(c^* ab), \quad \forall a, b, c \in \mathfrak{A}. \tag{3.7}$$

If $\mathfrak{A}$ has a unit e, then π_ω is cyclic with cyclic vector $\xi_\omega := \lambda_\omega(e)$. In this case π_ω is unique, up to unitary equivalence.

Let now $\mathcal{F}$ be a family of positive linear functionals on $\mathfrak{A}$. We put

$$q_{\mathcal{F}}(a) = \sup\{\omega(b^* a^* ab) : \omega \in \mathcal{F},\ b \in \mathfrak{A},\ \omega(b^* b) = 1\} \tag{3.8}$$

on the set

$$\mathcal{D}(q_{\mathcal{F}}) = \{a \in \mathfrak{A} : q_{\mathcal{F}}(a) < \infty\}. \tag{3.9}$$

Lemma 3.3. *The following equalities hold:*

$$\begin{gathered} \mathcal{D}(q_{\mathcal{F}}) = \left\{a \in \mathfrak{A} : \pi_\omega(a) \text{ is bounded}, \ \forall \omega \in \mathcal{F},\ \sup_{\omega \in \mathcal{F}} \|\pi_\omega(a)\| < \infty\right\}. \\ q_{\mathcal{F}}(a) = \sup_{\omega \in \mathcal{F}} \|\pi_\omega(a)\|, \quad \forall a \in \mathcal{D}(q_{\mathcal{F}}). \end{gathered} \tag{3.10}$$

By Lemma 3.3 the following proposition holds.

Proposition 3.4. *$\mathcal{D}(q_{\mathcal{F}})$ is a *-subalgebra of $\mathfrak{A}$ and $q_{\mathcal{F}}$ is an unbounded C^*-seminorm on $\mathfrak{A}$.*

If ω is a positive linear functional on $\mathfrak{A}$ and $x \in \mathfrak{A}$, we put

$$\omega_x(a) = \omega(x^* ax), \quad a \in \mathfrak{A}. \tag{3.11}$$

Clearly ω_x is also a positive linear functional on $\mathfrak{A}$.

Definition 3.5. Let $\mathfrak{A}$ be a $*$-algebra. A family $\mathcal{F}$ of positive linear functionals on $\mathfrak{A}$ is called *balanced* if, for each $x \in \mathfrak{A}$ and $\omega \in \mathcal{F}$, the positive linear functional ω_x also belongs to $\mathcal{F}$.

Remark 3.6. It is clear that $\mathcal{P}(\mathfrak{A})$ itself is balanced, but it is worth reminding the reader that, for a general $*$-algebra, $\mathcal{P}(\mathfrak{A})$ may reduce to $\{0\}$. In what follows, we will suppose that nontrivial balanced families of positive linear functionals do really exist.

If $\mathfrak{A}$ has a unit e and if $\mathcal{F}$ is a balanced family of positive linear functionals on $\mathfrak{A}$, then one can define

$$\mathcal{D}(\mathcal{F}) = \left\{a \in \mathfrak{A} : \sup_{\omega \in \mathcal{F}_s} \omega(a^* a) < \infty\right\}, \tag{3.12}$$

where $\mathcal{F}_s = \{\omega \in \mathcal{F} : \omega(e) = 1\}$ and

$$|a|_{\mathcal{F}}^2 = \sup_{\omega \in \mathcal{F}_s} \omega(a^* a). \tag{3.13}$$

By the definition itself of $|\cdot|_{\mathcal{F}}$ it follows that

$$|a|_{\mathcal{F}} \le q_{\mathcal{F}}(a), \quad \forall a \in \mathfrak{A}. \tag{3.14}$$

On the other hand, if $\omega \in \mathcal{F}$ and $b \in \mathfrak{A}$ with $\omega(b^* b) = 1$, then $\omega_b(e) = 1$ and so $\omega_b \in \mathcal{F}_s$. This implies that

$$q_{\mathcal{F}}(a) \le |a|_{\mathcal{F}}, \quad \forall a \in \mathfrak{A}. \tag{3.15}$$

Then we have the following.

Proposition 3.7. *Let $\mathfrak{A}$ be a $*$-algebra with unit e and $\mathcal{F}$ a balanced family of positive linear functionals on $\mathfrak{A}$. Then*

$$\mathcal{D}(\mathcal{F}) = \mathcal{D}(q_{\mathcal{F}}), \qquad |a|_{\mathcal{F}} = q_{\mathcal{F}}(a), \quad \forall a \in \mathfrak{A}. \tag{3.16}$$

Thus, $\mathcal{D}(\mathcal{F})$ is a $*$-subalgebra of $\mathfrak{A}$ and $|\cdot|_{\mathcal{F}}$ is an unbounded C^*-seminorm on $\mathfrak{A}$.

Remark 3.8. It is instructive to try to prove directly $*$-preservation and C^*-property for $|\cdot|_{\mathcal{F}}$ [48]. Let $a \in \mathcal{D}(\mathcal{F})$ and $\omega \in \mathcal{F}_s$. Then $\omega(aa^*) \le |a|_{\mathcal{F}}^2$. Indeed, if $\omega(aa^*) > 0$ (the case $\omega(aa^*) = 0$ is trivial), the Cauchy-Schwarz inequality implies that

$$\omega(aa^*)^2 \le \omega\left((aa^*)^2\right) = \omega_{a^*}(a^* a). \tag{3.17}$$

Put $b = a^*/\omega(aa^*)^{1/2}$. Then

$$\omega(aa^*)^2 \le \omega_b(a^* a)\omega(aa^*). \tag{3.18}$$

Since $\omega_b(e) = 1$, we have $\omega_b \in \mathcal{F}_s$. Hence

$$\omega(aa^*) \le \omega_b(a^* a) \le |a|_{\mathcal{F}}^2. \tag{3.19}$$

This in turn implies that $a^* \in \mathcal{D}(\mathcal{F})$ and $|a^*|_{\mathcal{F}} \leq |a|_{\mathcal{F}}$. Thus, interchanging the roles of a and a^*, the equality $|a^*|_{\mathcal{F}} = |a|_{\mathcal{F}}$ follows.

Finally we prove the C^*-condition. Let $a \in \mathcal{D}(\mathcal{F})$ and $\omega \in \mathcal{F}_s$. Then, using the Cauchy-Schwarz inequality,

$$\omega(a^*a)^2 \leq \omega((a^*a)^2) \leq |a^*a|_{\mathcal{F}}^2. \tag{3.20}$$

Hence, $|a|_{\mathcal{F}}^2 \leq |a^*a|_{\mathcal{F}}$.

Definition 3.9. A positive linear functional ω on $\mathfrak{A}$ is called *admissible* if, for every $a \in \mathfrak{A}$, there exists $\gamma_a > 0$ such that

$$\omega(x^*a^*ax) \leq \gamma_a \omega(x^*x), \quad \forall x \in \mathfrak{A}. \tag{3.21}$$

We denote with $\mathcal{P}_a(\mathfrak{A})$ the set of all *admissible* positive linear functionals on $\mathfrak{A}$ and, for shortness, we put $\mu(a) = q_{\mathcal{P}_a(\mathfrak{A})}(a)$ for every $a \in \mathfrak{A}$. Then μ is an everywhere defined C^*-seminorm on $\mathfrak{A}$.

Admissibility is a relevant property of positive linear functionals, since it is indeed equivalent to the boundedness of the corresponding GNS representation as we will see below.

Let q be a seminorm on $\mathfrak{A}$ and ω a linear functional on $\mathfrak{A}$. Then ω is said to be continuous with respect to q, or, simply, q-continuous or q-bounded, if there exists $\gamma > 0$ such that

$$|\omega(a)| \leq \gamma q(a), \quad \forall a \in \mathfrak{A}. \tag{3.22}$$

We denote with $\|\omega\|_q$ the infimum of the positive constants for which (3.22) holds.

The positive linear functional ω is said to be relatively q-bounded, if, for every $x \in \mathfrak{A}$, the positive linear functional ω_x, defined in (3.11), is q-bounded (of course, in general, the corresponding constant γ of (3.22) will depend on x). Clearly, each q-bounded positive linear functional is relatively q-bounded. If $\mathfrak{A}$ has a unit e, then the converse also holds.

Admissibility is characterized by the following.

Proposition 3.10. *Let $\mathfrak{A}$ be a $*$-algebra and ω a positive linear functional on $\mathfrak{A}$. The following statements are equivalent.*

(i) *ω is admissible.*

(ii) *π_ω is bounded.*

(iii) *There exists a submultiplicative seminorm q on $\mathfrak{A}$ such that π_ω is q-continuous and $\|\pi_\omega(a)\| \leq q(a)$, for every $a \in \mathfrak{A}$.*

(iv) *There exists a submultiplicative seminorm q on $\mathfrak{A}$ such that ω is relatively q-bounded.*

If $\mathfrak{A}$ has a unit e, then the previous statements are equivalent also to the following.

(iv′) *There exists a submultiplicative seminorm q on $\mathfrak{A}$ such that ω is q-bounded.*

Proof. (i)⇒(ii) Assume that ω is admissible. Then, for each $a \in \mathfrak{A}$ there exists $\gamma_a > 0$ such that

$$\omega(b^*a^*ab) \leq \gamma_a \omega(b^*b), \quad \forall b \in \mathfrak{A}. \tag{3.23}$$

Then we have

$$||\pi_\omega(a)\lambda_\omega(b)||^2 = \omega(b^*a^*ab) \le \gamma_a\omega(b^*b) = \gamma_a||\lambda_\omega(b)||^2, \quad \forall b \in \mathfrak{A}. \tag{3.24}$$

Therefore π_ω is bounded.

(ii)$\Rightarrow$(iii) Assume that π_ω is bounded and define

$$q(a) = ||\pi_\omega(a)||, \quad a \in \mathfrak{A}. \tag{3.25}$$

Then, as it is easily seen, q is a C^*-seminorm that trivially satisfies (iii).

(iii)$\Rightarrow$(iv) Indeed, we have

$$|\,\omega(b^*ab)\,| = |\,\langle \pi_\omega(a)\lambda_\omega(b) \mid \lambda_\omega(b)\rangle\,| \le q(a)||\lambda_\omega(b)||^2, \quad \forall a,b \in \mathfrak{A}. \tag{3.26}$$

Hence ω is relatively q-bounded.

(iv)$\Rightarrow$(i) Assume that ω is relatively q-bounded, where q is a submultiplicative seminorm on $\mathfrak{A}$. Then, for all $a,b \in \mathfrak{A}$ and $n \in \mathbb{N}$, the Kaplansky inequality holds:

$$\omega_b(a^*a) \le \omega(b^*b)^{1-2^n}\left(\omega_b\left((a^*a)^{2^n}\right)\right)^{2^{-n}}. \tag{3.27}$$

Then

$$\omega_b(a^*a) \le \omega(b^*b)^{1-2^n}\left(||\omega_b||_q q\left((a^*a)^{2^n}\right)\right)^{2^{-n}}. \tag{3.28}$$

For $n \to \infty$, we get

$$\omega_b(a^*a) \le q(a^*a)\omega(b^*b). \tag{3.29}$$

Hence ω is admissible. □

Now let us suppose that a C^*-seminorm on $\mathfrak{A}$ is given; it is natural to ask what is the relationship between q and the C^*-seminorm defined by the family of all q-bounded positive linear functionals on $\mathfrak{A}$.

Theorem 3.11. *Let q be a C^*-seminorm on $\mathfrak{A}$. The following statements hold.*

(i) *There exists a bounded $*$-representation π of $\mathfrak{A}$ such that*

$$||\pi(a)|| = q(a), \quad \forall a \in \mathfrak{A}. \tag{3.30}$$

(ii) *Let $\mathcal{F}$ denote the set of all q-bounded positive functionals on $\mathfrak{A}$. Then $\mathcal{D}(\mathcal{F}) = \mathfrak{A}$ and*

$$q(a)^2 = \sup_{\omega\in\mathcal{F}_s} \omega(a^*a), \quad \forall a \in \mathfrak{A}. \tag{3.31}$$

Proof. (i) Let $\mathcal{K}_q = \{a \in \mathfrak{A} : q(a) = 0\}$; then $\mathfrak{A}/\mathcal{K}_q$ is a normed space with norm:

$$||a + \mathcal{K}_q|| = q(a), \quad a \in \mathfrak{A}. \tag{3.32}$$

Its completion $\mathfrak{B}_0$ is a C^*-algebra. Then by the Gel'fand-Naĭmark theorem, there exists an isometric $*$-isomorphism $\hat{\pi}$ of $\mathfrak{B}_0$ onto a C^*-algebra of bounded operators in a Hilbert space $\mathcal{H}$. Now we put

$$\pi(a) = \hat{\pi}(a + \mathcal{K}_q), \quad a \in \mathfrak{A}. \tag{3.33}$$

Then π is a well-defined bounded $*$-representation of $\mathfrak{A}$ and

$$||\pi(a)|| = ||\hat{\pi}(a + \mathcal{K}_q)|| = q(a), \quad \forall a \in \mathfrak{A}. \tag{3.34}$$

Proposition 3.10 implies that each $\omega \in \mathcal{F}$ is admissible and

$$\omega(a^* a) \leq q(a)^2 \omega(e), \quad \forall a \in \mathfrak{A}. \tag{3.35}$$

Therefore $\mathcal{D}(\mathcal{F}) = \mathfrak{A}$ and

$$\sup_{\omega \in \mathcal{F}_s} \omega(a^* a) \leq q(a)^2, \quad \forall a \in \mathfrak{A}. \tag{3.36}$$

We need to prove the converse inequality.

(ii) Let π be the $*$-representation constructed in (i). Let $\xi \in \mathcal{H}$, $\|\xi\| = 1$. We put

$$\omega_\xi(a) = \langle \pi(a)\xi \mid \xi \rangle, \quad a \in \mathfrak{A}. \tag{3.37}$$

Then $\omega_\xi \in \mathcal{F}$ and $\omega_\xi(e) = 1$. Then we have

$$\begin{aligned} q(a) = ||\pi(a)|| &= \sup_{\|\xi\|=1} ||\pi(a)\xi|| \\ &= \sup_{\|\xi\|=1} \omega_\xi(a^* a)^{1/2} \leq \sup_{\omega \in \mathcal{F}_s} \omega(a^* a)^{1/2}, \quad \forall a \in \mathfrak{A}. \end{aligned} \tag{3.38}$$

□

The following statement is now clear.

Theorem 3.12. *Let $\mathfrak{A}$ be a $*$-algebra. The following statements are equivalent.*

(i) *$\mathfrak{A}$ admits a nonzero bounded $*$-representation π.*

(ii) *There exists a nonzero C^*-seminorm q on $\mathfrak{A}$.*

(iii) *There exists a family $\mathcal{F}$ of positive linear functionals such that $q_{\mathcal{F}}$ is an everywhere defined, nonzero C^*-seminorm on $\mathfrak{A}$.*

The set $\mathcal{P}_a(\mathfrak{A})$ of all admissible positive linear functionals is balanced. The corresponding C^*-seminorm μ (which is everywhere defined, as we have already seen) turns out to be the *maximal* C^*-seminorm on $\mathfrak{A}$, in the sense that if p is another C^*-seminorm on $\mathfrak{A}$, then $\mu(a) \geq p(a)$, for every $a \in \mathfrak{A}$ [48, Theorem 2.6].

If $\pi \in \mathrm{Rep}(\mathfrak{A})$ and $\mathcal{D}_\pi$ is its domain in Hilbert space $\mathcal{H}_\pi$, for every $\xi \in \mathcal{D}_\pi$, we can define

$$\omega_\xi(a) = \langle \pi(a)\xi \mid \xi \rangle, \quad a \in \mathfrak{A}. \tag{3.39}$$

If every $\omega \in \mathcal{P}(\mathfrak{A})$ is admissible, then, for every $a \in \mathfrak{A}$, we have

$$\begin{aligned} \|\pi(a)\xi\|^2 &= \langle \pi(a)\xi \mid \pi(a)\xi \rangle = \langle \pi(a^*a)\xi \mid \xi \rangle \\ &= \omega_\xi(a^*a) \le \mu(a)^2\omega(e) = \mu(a)^2\|\xi\|^2. \end{aligned} \tag{3.40}$$

Hence π is bounded and $\|\pi(a)\| \le \mu(a)$, for every $a \in \mathfrak{A}$. This clearly implies that

$$\sup_{\pi \in \mathrm{Rep}(\mathfrak{A})} \|\pi(a)\| \le \mu(a), \quad \forall a \in \mathfrak{A}. \tag{3.41}$$

From these simple facts one can easily deduce the following.

Theorem 3.13. *Let $\mathfrak{A}$ be a $*$-algebra with unit e. The following statements are equivalent.*

(i) *Each positive linear functional ω on $\mathfrak{A}$ is admissible.*

(ii) *Each $*$-representation π of $\mathfrak{A}$ is bounded and*

$$\sup_{\pi \in \mathrm{Rep}(\mathfrak{A})} \|\pi(a)\| < \infty, \quad \forall a \in \mathfrak{A}. \tag{3.42}$$

(iii) *Each GNS representation π_ω, $\omega \in \mathcal{P}(\mathfrak{A})$ is bounded and*

$$\sup_{\omega \in \mathcal{P}(\mathfrak{A})} \|\pi_\omega(a)\| < \infty, \quad \forall a \in \mathfrak{A}. \tag{3.43}$$

3.2. The case of partial $*$-algebras. As we have seen in the case of $*$-algebras, the possibility of constructing C^*-seminorms is closely linked to the GNS representation determined by positive linear functionals. For partial $*$-algebras a GNS contruction is possible starting from a particular class of positive sesquilinear forms called *biweights* that we define below.

Let φ be a positive sesquilinear form on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$, where $\mathcal{D}(\varphi)$ is a subspace of $\mathfrak{A}$. Then we have

$$\varphi(x,y) = \overline{\varphi(y,x)}, \quad \forall x,y \in \mathcal{D}(\varphi), \tag{3.44}$$

$$|\varphi(x,y)|^2 \le \varphi(x,x)\varphi(y,y), \quad \forall x,y \in \mathcal{D}(\varphi). \tag{3.45}$$

We put

$$N_\varphi = \{x \in \mathcal{D}(\varphi) : \varphi(x,x) = 0\}. \tag{3.46}$$

By (3.45) we have

$$N_\varphi = \{x \in \mathcal{D}(\varphi) : \varphi(x,y) = 0,\ \forall y \in \mathcal{D}(\varphi)\}, \tag{3.47}$$

and so N_φ is a subspace of $\mathcal{D}(\varphi)$ and the quotient space $\mathcal{D}(\varphi)/N_\varphi \equiv \{\lambda_\varphi(x) \equiv x + N_\varphi;\ x \in \mathcal{D}(\varphi)\}$ is a pre-Hilbert space with respect to the inner product $(\lambda_\varphi(x) \mid \lambda_\varphi(y)) = \varphi(x,y)$, $x,y \in \mathcal{D}(\varphi)$. We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by the completion of $\mathcal{D}(\varphi)/N_\varphi$.

Definition 3.14. Let φ be a positive sesquilinear form on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$. A subspace $B(\varphi)$ of $\mathcal{D}(\varphi)$ is said to be a *core* for φ if

(i) $B(\varphi) \subseteq R\mathfrak{A}$;

(ii) $\{ax;\ a \in \mathfrak{A},\ x \in B(\varphi)\} \subseteq \mathfrak{D}(\varphi)$;
(iii) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$;
(iv) $\varphi(ax, y) = \varphi(x, a^* y)$, for all $a \in \mathfrak{A}$, for all $x, y \in B(\varphi)$;
(v) $\varphi(a^* x, by) = \varphi(x, (ab)y)$, for all $a \in L(b)$, for all $x, y \in B(\varphi)$.

We denote by $\mathcal{B}_\varphi$ the set of all cores $B(\varphi)$ for φ.

Definition 3.15. A positive sesquilinear form φ on $\mathfrak{D}(\varphi) \times \mathfrak{D}(\varphi)$ such that $\mathcal{B}_\varphi \neq \varnothing$ is called a *biweight* on $\mathfrak{A}$.

For a biweight on $\mathfrak{A}$ a GNS-like construction can be performed as follows.
Let φ be a biweight on $\mathfrak{A}$ with core $B(\varphi)$. We put

$$\pi_\varphi^\circ(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathfrak{A},\ x \in B(\varphi). \tag{3.48}$$

Using (iii) of Definition 3.14 and a simple limit argument, it follows that

$$\varphi(y, y) = 0 \quad \text{for } y \in B(\varphi) \text{ implies } \varphi(ay, ay) = 0, \quad \forall a \in \mathfrak{A}. \tag{3.49}$$

Then $\pi_\varphi^\circ(a)$ is a well-defined linear operator of $\lambda_\varphi(B(\varphi))$ into $\mathcal{H}_\varphi$. Furthermore, it follows from (iv) and (v) of Definition 3.14 that π_φ° is a *-representation of $\mathfrak{A}$. We denote by π_φ^B the closure of π_φ°. Then the triple $(\pi_\varphi^B, \lambda_\varphi, \mathcal{H}_\varphi)$ is called the *GNS construction* for the biweight φ on $\mathfrak{A}$ with the core $B(\varphi)$.

Example 3.16. Very simple examples of biweights can be constructed in L^p-spaces. These spaces can be made into partial *-algebras in a natural way, the partial multiplication being established via Hölder's inequality.

(i) Let $2 \leq p < \infty$. We put

$$\begin{gathered} D(\varphi) = L^p[0,1], \\ \varphi(x, y) = \int_0^1 x(t)\overline{y(t)}dt, \quad x, y \in D(\varphi). \end{gathered} \tag{3.50}$$

Then φ is a biweight on $L^p[0,1]$ with largest core $L^\infty[0,1]$.

(ii) Let $1 \leq p < 2$ and $\mathfrak{A} = L^p[0,1]$. Then the positive sesquilinear form φ on $\mathfrak{D}(\varphi) \times D(\varphi)$ defined by

$$\begin{gathered} D(\varphi) = L^2[0,1], \\ \varphi(x, y) = \int_0^1 x(t)\overline{y(t)}dt, \quad x, y \in D(\varphi), \end{gathered} \tag{3.51}$$

is *not* a biweight on $L^p[0,1]$, because (i) and (ii) in Definition 3.14 are incompatible in this case. Therefore $\mathcal{B}_\varphi = \varnothing$.

It is much more interesting for our purposes to consider the example of *vector forms* of the type $\langle A\xi \mid B\xi \rangle$ defined on a partial O^*-algebra $\mathfrak{M}$ on a pre-Hilbert space $\mathfrak{D}$, with $\xi \in \mathfrak{D}$. We will not discuss here the general case but only the one arising when a *-representation of a partial *-algebra $\mathfrak{A}$ is considered. For full details we refer to [6, 7].

Example 3.17. Let π be a $*$-representation of the partial $*$-algebra $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi,\mathcal{H}_\pi)$ and $\xi \in \mathcal{H}_\pi$. Define

$$\begin{gathered}\mathcal{D}(\varphi_\xi^\pi) = \{a \in \mathfrak{A} : \xi \in D(\pi(a^*)^*)\},\\ \varphi_\xi^\pi(a,b) = \langle \pi(a^*)^*\xi \mid \pi(b^*)^*\xi\rangle.\end{gathered} \tag{3.52}$$

Then φ_ξ^π is a sesquilinear form on $\mathcal{D}(\varphi_\xi^\pi)\times\mathcal{D}(\varphi_\xi^\pi)$.

If $\xi \in \mathcal{D}_\pi$, we put

$$B(\varphi_\xi^\pi) = \{x \in R\mathfrak{A} : \pi(x)\xi \in \mathcal{D}(\pi^{**})\}. \tag{3.53}$$

If $\xi \in \mathcal{H}_\pi \setminus \mathcal{D}_\pi$, we define $B(\varphi_\xi^\pi)$ as the linear span of the set

$$B_0(\varphi_\xi^\pi) = \{x \in R\mathfrak{A} : \xi \in D(\overline{\pi(x)}),\ \overline{\pi(x)}\xi \in \mathcal{D}\}. \tag{3.54}$$

Then, if $\pi(B(\varphi_\xi^\pi))\xi$ is dense in $\overline{\pi(D(\varphi_\xi^\pi))\xi}$, φ_ξ^π is a biweight on $\mathfrak{A}$. A proof of this statement can be done as in [7, Example 9.1.12].

Let now $\mathfrak{B}$ be a subspace of $R\mathfrak{A}$. We denote with $BW(\mathfrak{A},\mathfrak{B})$ the family of biweights of $\mathfrak{A}$ having $\mathfrak{B}$ as a core.

If $\mathcal{F} \subseteq BW(\mathfrak{A},\mathfrak{B})$ we put

$$\begin{gathered}p_{\mathcal{F}}^{\mathfrak{B}}(a) = \sup\{\varphi(ax,ax)^{1/2} : \varphi \in \mathcal{F},\ x \in \mathfrak{B},\ \varphi(x,x) = 1\},\\ \mathcal{D}(p_{\mathcal{F}}^{\mathfrak{B}}) = \{a \in \mathfrak{A} : p_{\mathcal{F}}^{\mathfrak{B}}(a) < \infty\}.\end{gathered} \tag{3.55}$$

Proposition 3.18. *Let $\mathcal{F} \subseteq BW(\mathfrak{A},\mathfrak{B})$. Then*

$$\begin{gathered}\mathcal{D}(p_{\mathcal{F}}^{\mathfrak{B}}) = \left\{a \in \mathfrak{A} : \pi_\varphi^{\mathfrak{B}}(a) \text{ is bounded and } \sup_{\varphi\in\mathcal{F}}\|\pi_\varphi^{\mathfrak{B}}(a)\| < \infty\right\},\\ p_{\mathcal{F}}^{\mathfrak{B}}(a) = \sup_{\varphi\in\mathcal{F}}\|\pi_\varphi^{\mathfrak{B}}(a)\|, \quad \forall a \in \mathcal{D}(p_{\mathcal{F}}^{\mathfrak{B}}).\end{gathered} \tag{3.56}$$

From this equality, one gets that, for any family $\mathcal{F} \subseteq BW(\mathfrak{A},\mathfrak{B})$,

(i) $\mathcal{D}(p_{\mathcal{F}}^{\mathfrak{B}})$ is a partial $*$-algebra in $\mathfrak{A}$;

(ii) $p_{\mathcal{F}}^{\mathfrak{B}}$ is an unbounded C^*-seminorm on $\mathfrak{A}$ with domain $\mathcal{D}(p_{\mathcal{F}}^{\mathfrak{B}})$.

Similarly to the case of positive linear functionals, the notion of admissibility can be introduced for a biweight φ but it is linked to a *core of admissibility*.

Definition 3.19. A biweight φ on the partial $*$-algebra is *admissible* if there exists a core $B(\varphi)$ such that

$$\forall a \in \mathfrak{A}, \quad \exists \gamma_a > 0 : \quad \varphi(ax,ax) \le \gamma_a\varphi(x,x), \quad \forall x \in B(\varphi). \tag{3.57}$$

In principle, if we choose a different core $B_1(\varphi)$, then φ need not satisfy condition (3.57) when $x \in B_1(\varphi)$.

Assume now that $\mathcal{F} \subseteq BW(\mathfrak{A},\mathfrak{B})$. Let $\varphi \in \mathcal{F}$, $a \in \mathfrak{A}$, and $x \in \mathfrak{B}$. If $\varphi(x,x) = 0$, then (3.49) implies $\varphi(ax,ax) = 0$. If $\varphi(x,x) > 0$, then, putting $w = x/\varphi(x,x)^{1/2}$ we have $\varphi(w,w) = 1$ and

$$\varphi(ax,ax) = \varphi(aw,aw)\varphi(x,x). \tag{3.58}$$

Then if $a \in D(p_{\mathcal{F}}^{\mathfrak{B}})$, we obtain

$$\varphi(ax,ax) \leq p_{\mathcal{F}}(a)^2\varphi(x,x). \tag{3.59}$$

Therefore, we have the following proposition.

Proposition 3.20. *Let $\mathcal{F} \subseteq BW(\mathfrak{A},\mathfrak{B})$. If $D(p_{\mathcal{F}}^{\mathfrak{B}}) = \mathfrak{A}$, then each $\varphi \in \mathcal{F}$ is admissible.*

Definition 3.21. Let q be a seminorm defined on a partial *-subalgebra $\mathcal{D}(q)$ of $\mathfrak{A}$ and φ a sesquilinear form on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$, $\mathcal{D}(\varphi) \subseteq \mathfrak{A}$. φ is said to have a q-bounded if

(i) $\mathcal{D}(q) \subseteq \mathcal{D}(\varphi)$;

(ii) $\exists \gamma_\varphi > 0 : |\varphi(a,b)| \leq \gamma_\varphi q(a)q(b)$, for all $a,b \in \mathcal{D}(q)$.

We denote with $\|\varphi\|_q$ the infimum of all positive constants for which (ii) holds.

Let φ be a biweight on $\mathfrak{A}$ with domain $\mathcal{D}(\varphi)$ and core $B(\varphi)$. If $x \in B(\varphi)$, we denote with φ_x the positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ defined by

$$\varphi_x(a,b) = \varphi(ax,bx), \quad a,b \in \mathfrak{A}. \tag{3.60}$$

Definition 3.22. Let φ be a biweight on $\mathfrak{A}$ with domain $\mathcal{D}(\varphi)$ and core $B(\varphi)$ and let q be a seminorm with domain $\mathcal{D}(q)$. φ is said to have a *q-bounded $B(\varphi)$-orbit* if each φ_x, $x \in B(\varphi)$, is q-bounded.

A biweight φ could have a q-bounded $B(\varphi)$-orbit, without φ being q-bounded.

Let now q be a seminorm on $\mathfrak{A}$ with $\mathcal{D}(q) = \mathfrak{A}$ and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$. Let

$$\begin{gathered} CO(q,\mathfrak{B}) = \{\varphi \in BW(\mathfrak{A},\mathfrak{B}) : \varphi \text{ has a } q\text{-bounded } \mathfrak{B}\text{-orbit}\}, \\ CO^e(q,\mathfrak{B}) = \{\varphi \in CO(q,\mathfrak{B}) : \|\varphi_x\|_q \leq \varphi(x,x),\ x \in \mathfrak{B}\}. \end{gathered} \tag{3.61}$$

The following theorem characterizes admissibility of biweights and generalizes Proposition 3.10 to partial *-algebras.

Theorem 3.23. *Let φ be a biweight on $\mathfrak{A}$ with domain $\mathcal{D}(\varphi)$. The following statements are equivalent.*

(i) *φ is admissible.*

(ii) *There exists a core $B(\varphi)$ for φ such that π_φ^B is bounded.*

(iii) *There exist a submultiplicative seminorm q on $\mathfrak{A}$ and core $B(\varphi)$ for φ such that π_φ^B is q-bounded and $\|\pi_\varphi^B(a)\| \leq q(a)$ for every $a \in \mathfrak{A}$.*

(iv) *There exist a submultiplicative seminorm q on $\mathfrak{A}$ and core $B(\varphi)$ for φ such that $\varphi \in CO^e(q,B(\varphi))$.*

Proof. The proofs of (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), and (iii)$\Rightarrow$(iv) are similar to those given in the proof of Proposition 3.10, therefore we omit them.

(iv)⇒(i) Since $\varphi \in CO^e(q, B(\varphi))$, then φ has a q-bounded $B(\varphi)$-orbit and $\|\varphi_x\|_q \leq \varphi(x,x)$, for each $x \in B(\varphi)$. Therefore, for each $a \in \mathfrak{A}$,

$$\varphi(ax, ax) \leq q(a)^2 \varphi(x,x), \quad \forall x \in B(\varphi) \tag{3.62}$$

and so φ is admissible. □

In general, $CO^e(q, \mathfrak{B}) \neq CO(q, \mathfrak{B})$, in contrast with what happens for the corresponding objects in $*$-algebras. A sufficient condition for the equality to hold has been given in [46] and involves the notion of "length" of an element of a partial $*$-algebra [4, 7].

If $\mathfrak{A}$ is a partial $*$-algebra and $a \in \mathfrak{A}$, the *length* $\ell(a)$ of a is the largest $n \in \mathbb{N} \cup \{\infty\}$ such that a^n is well defined in $\mathfrak{A}$.

We put

$$R\mathfrak{A}_\infty = \{x \in R\mathfrak{A} : \ell(x^* x) = \infty\}. \tag{3.63}$$

Then the following statement holds.

Proposition 3.24. *Let q be an m^*-seminorm on $\mathfrak{A}$ (i.e., $\mathfrak{D}(q) = \mathfrak{A}$). Assume that $R\mathfrak{A}_\infty$ is q-dense in $\mathfrak{A}$. Then, for each subspace $\mathfrak{B}$ of $R\mathfrak{A}$, $CO(q, \mathfrak{B}) = CO^e(q, \mathfrak{B})$.*

Remark 3.25. If $\mathfrak{B}$ is a $*$-algebra, q-dense in $\mathfrak{A}$, the assumption of Proposition 3.24 is automatically fulfilled, since $\mathfrak{B} \subseteq R\mathfrak{A}_\infty$; then, in this case, $CO(q, \mathfrak{B}) = CO^e(q, \mathfrak{B})$.

The assumption that $R\mathfrak{A}$ is dense in $\mathfrak{A}$ implies that if $\mathfrak{A}$ is semiassociative, there exist a C^*-algebra $\mathfrak{R}$, with $\|\cdot\|_q$ (cf. Definition 3.21) and a linear map $a \in \mathfrak{A} \to \tilde{a} \in \mathfrak{R}$, preserving the involution and such that $\widetilde{a \cdot b} = \tilde{a} \cdot \tilde{b}$ whenever $a \cdot b$ is well defined (Lemma 2.3). By the Gel'fand-Naĭmark theorem, there exists an isometric $*$-isomorphism Φ of $\mathfrak{R}$ onto a C^*-algebra $\mathfrak{M}$ of bounded operators in Hilbert space $\mathcal{H}$. If $\xi \in \mathcal{H}$, we put

$$\varphi_\xi(a,b) = \langle \Phi(\tilde{a})\xi \mid \Phi(\tilde{b})\xi \rangle, \quad a, b \in \mathfrak{A}. \tag{3.64}$$

Then φ_ξ is a well-defined positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ and it can be shown that it is a biweight with $D(\varphi_\xi) = \mathfrak{A}$ and $B(\varphi_\xi) = \mathfrak{B}$. Moreover $\varphi_\xi \in CO^e(q, \mathfrak{B})$. Now since $e \in \mathfrak{B}$, we have

$$q(a)^2 = \sup\{\varphi_\xi(a,a);\ \|\xi\| = 1\} \leq p_{\mathcal{F}}(a)^2, \quad \forall a \in \mathfrak{A}. \tag{3.65}$$

But $p_{\mathcal{F}}(a) \leq q(a)$, for every $a \in \mathfrak{A}$. Thus we have the following.

Theorem 3.26. *Let $\mathfrak{A}$ be a semiassociative partial $*$-algebra, q a C^*-seminorm on $\mathfrak{A}$ and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$, q-dense in $\mathfrak{A}$ and such that $e \in \mathfrak{B}$. If $\mathcal{F} = CO(q, \mathfrak{B})$, then $\mathfrak{D}(p_{\mathcal{F}}) = \mathfrak{A}$, and*

$$q(a) = p_{\mathcal{F}}(a), \quad \forall a \in \mathfrak{A}. \tag{3.66}$$

The following statement is the analog of Theorem 3.12 in the case of partial $*$-algebras.

Theorem 3.27. *Let $\mathfrak{A}$ be a partial $*$-algebra. The following statements are equivalent.*

(i) *$\mathfrak{A}$ admits a nonzero bounded $*$-representation π.*

(ii) *There exists a nonzero everywhere defined C^*-seminorm q on $\mathfrak{A}$.*
(iii) *There exists a family $\mathcal{F}$ of biweights with core $R\mathfrak{A}$ such that $p_{\mathcal{F}}$ is an everywhere defined, nonzero C^*-seminorm on $\mathfrak{A}$.*

We conclude this section by showing how (unbounded) C^*-seminorms defined on a partial $*$-algebra $\mathfrak{A}$ by families of biweights can be used to get some information on the structure properties of *Banach partial $*$-algebras* (e.g., about the automatic continuity of bounded $*$-representations of $\mathfrak{A}$).

Definition 3.28. A partial $*$-algebra $\mathfrak{A}$ is said to be a *normed partial $*$-algebra* if it carries a norm $\|\cdot\|$ such that
(i) the involution $a \mapsto a^*$ is isometric: $\|a\| = \|a^*\|$, for all $a \in \mathfrak{A}$;
(ii) for every $x \in L\mathfrak{A}$, there exists a constant $\gamma_x > 0$ such that

$$\|xa\| \leq \gamma_x \|a\|, \quad \forall a \in \mathfrak{A}. \tag{3.67}$$

$\mathfrak{A}[\|\cdot\|]$ is called a *Banach partial $*$-algebra* if, in addition,
(iii) $\mathfrak{A}[\|\cdot\|]$ is a Banach space.

Using (i), (ii), and the fact that $R\mathfrak{A} = L\mathfrak{A}^*$, we also have
(ii′) for every $y \in R\mathfrak{A}$, there exists a constant $\gamma_y > 0$ such that

$$\|ay\| \leq \gamma_y \|a\|, \quad \forall a \in \mathfrak{A}. \tag{3.68}$$

Remark 3.29. The first definition of normed partial $*$-algebra was given in [7] where some additional conditions on the topologies of multiplier spaces where required. Recently Antoine and the present author reconsidered the whole question again in [9].

Since we are going to study properties of $*$-representations of a partial $*$-algebra, the vector forms considered in Example 3.17 will play a relevant role in what follows. We refer to the notations introduced there.

Remark 3.30. If π is a *bounded* $*$-representation of $\mathfrak{A}$, then $\mathcal{D}(\varphi_\xi^\pi) = \mathfrak{A}$ and, if $\xi \in \mathcal{D}$, $B(\varphi_\xi^\pi) = R\mathfrak{A}$.

Definition 3.31. A $*$-representation π of a partial $*$-algebra $\mathfrak{A}$ is said to be *regular* if φ_ξ^π is a biweight on $\mathfrak{A}$, for every $\xi \in \mathcal{H}_\pi$.

Proposition 3.32. *Let $\mathfrak{A}$ be a normed partial $*$-algebra and assume that $R\mathfrak{A}$ is dense in $\mathfrak{A}$. Then each continuous bounded representation is regular.*

Proof. If π is a continuous bounded representation and $\xi \in \mathcal{H}$, there exists $\gamma > 0$ such that

$$||\pi(a)|| \leq \gamma \|a\|, \quad \forall a \in \mathfrak{A}. \tag{3.69}$$

For every $a \in \mathfrak{A}$, there exists a sequence $\{x_n\} \subset R\mathfrak{A}$ such that $\|a - x_n\| \to 0$. Then we have

$$\begin{aligned} ||\lambda_{\varphi_\xi^\pi}(a) - \lambda_{\varphi_\xi^\pi}(x_n)||^2 &= \varphi_\xi^\pi(a - x_n, a - x_n) \\ &= ||\pi(a - x_n)\xi||^2 \leq \gamma^2 ||a - x_n||^2 \|\xi\|^2 \longrightarrow 0. \end{aligned} \tag{3.70}$$

□

Theorem 3.33. *Let $\mathfrak{A}[\|\cdot\|]$ be a normed partial $*$-algebra such that every biweight $\varphi \in BW(\mathfrak{A};R\mathfrak{A})$ satisfies the inequality*

$$\varphi(ax,ax) \le \|a\|\varphi(x,x), \quad \forall a \in \mathfrak{A},\ x \in R\mathfrak{A}. \tag{3.71}$$

Then the following hold:

(i) *every regular $*$-representation of $\mathfrak{A}$ is bounded and continuous from $\mathfrak{A}[\|\cdot\|]$ into $\mathfrak{B}(\mathcal{H})[\|\cdot\|]$ and*

$$\|\pi(a)\| \le \|a\|, \quad \forall a \in \mathfrak{A}; \tag{3.72}$$

(ii) *there exists a C^*-seminorm q on $\mathfrak{A}$ such that $q(a) \le \|a\|$, for every $a \in \mathfrak{A}$ and for every regular $*$-representation π one has*

$$\|\pi(a)\| \le q(a), \quad \forall a \in \mathfrak{A}; \tag{3.73}$$

Proof. (i) Let π be a regular $*$-representation of $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi,\mathcal{H}_\pi)$; for $\xi \in \mathcal{D}_\pi$, consider the corresponding vector form φ_ξ^π. By the regularity of π, φ_ξ^π is a biweight with core $R\mathfrak{A}$. Then we have that, for every $a \in \mathfrak{A}$,

$$\|\pi(a)\xi\|^2 = \varphi_\xi^\pi(a,a) \le \|a\|^2\varphi_\xi^\pi(e,e) = \|a\|^2\|\xi\|^2. \tag{3.74}$$

This implies that π is bounded and

$$\|\pi(a)\| \le \|a\|, \quad \forall a \in \mathfrak{A}. \tag{3.75}$$

(ii) The assumption clearly implies that $p_{R\mathfrak{A}}$ is everywhere defined on $\mathfrak{A}$ and

$$p_{R\mathfrak{A}}(a) \le \|a\|, \quad \forall a \in \mathfrak{A}. \tag{3.76}$$

Now, if π is regular, then, by (i), it is bounded. So if $\xi \in \mathcal{D}_\pi$, $\|\xi\| = 1$, the vector form φ_ξ^π is a biweight with domain $\mathfrak{A}$ and core $R\mathfrak{A}$ satisfying $\varphi_\xi^\pi(e,e) = 1$. Then, for every $a \in \mathfrak{A}$, we have

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi \mid \pi(a)\xi\rangle = \varphi_\xi^\pi(a,a) = \varphi_\xi^\pi(ae,ae) \le p_{R\mathfrak{A}}(a)^2 \tag{3.77}$$

which, clearly, implies the statement. □

4. CQ^*-seminorms on quasi $*$-algebras

Among partial $*$-algebras, a distinguished role is played by quasi $*$-algebras. The reason for that is twofold: first, quasi $*$-algebras frequently appear in applications, because one often is led to consider the completion of a locally convex $*$-algebra whose multiplication is separately but not jointly continuous (we refer to [7] for a discussion of a number of physical situations where this structure appears in a natural way); the second reason is that, as a partial $*$-algebra, a quasi $*$-algebra $(\mathfrak{A},\mathfrak{A}_0)$ has a particularly simple structure. Indeed, in this case, the defining set Γ can be taken as

$$\Gamma = \{(a,b) \in \mathfrak{A}\times\mathfrak{A} : a \in \mathfrak{A}_0 \text{ or } b \in \mathfrak{A}_0\} \tag{4.1}$$

and it turns out that, with its natural partial multiplication, $\mathfrak{A}$ may be regarded as a semi-associative partial *-algebra and $\mathfrak{A}_0$ is exactly the set of universal multipliers of $\mathfrak{A}$, that is, $R\mathfrak{A} = L\mathfrak{A} = \mathfrak{A}_0$. The results of the previous sections apply, of course, also to this case, but the central role played by the *-algebra $\mathfrak{A}_0$ makes natural the choice of considering *only* biweights having $\mathfrak{A}_0$ as core. This also leads to a further simplification: if $(\mathfrak{A},\mathfrak{A}_0)$ has a unit e (i.e., $e \in \mathfrak{A}_0$, $e = e^*$ and $xe = ex = x$, for every $x \in \mathfrak{A}$), then condition (ii) of Definition 3.14 implies that a biweight φ is necessarily *everywhere defined*. On the other hand if $(\mathfrak{A},\mathfrak{A}_0)$ has no unit, then it can be embedded in a quasi *-algebra with a unit in a standard fashion. Thus, we can consider only the case where $D(\varphi) = \mathfrak{A}$. For this reason we simplify a little the notations and call $\mathcal{J}(\mathfrak{A}) := BW(\mathfrak{A},\mathfrak{A}_0)$, since sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$, satisfying the conditions (iv) and (v) of Definition 3.14, are often called *invariant* [7].

C^*-seminorms are always connected with C^*-algebras. In this aspect, the following question is of a certain interest: given a quasi *-algebra $(\mathfrak{A},\mathfrak{A}_0)$, could one define seminorms via some families of positive sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$, such that another structure like the one described below be defined on $\mathfrak{A}$?

Assume that $(\mathfrak{A},\mathfrak{A}_0)$ is a quasi *-algebra which is also a normed space with respect to a norm $\|\cdot\|$ enjoying the following properties:

(a) $\|a^*\| = \|a\|$, for all $a \in \mathfrak{A}$;

(b) for every $x \in \mathfrak{A}_0$ there exists $\gamma > 0$ such that

$$\|ax\| \le \gamma\|a\|, \quad \forall a \in \mathfrak{A}; \tag{4.2}$$

(c) $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\|\cdot\|]$.

Then $(\mathfrak{A},\mathfrak{A}_0)$ is said to be a *normed* quasi *-algebra; if $\mathfrak{A}[\|\cdot\|]$ is complete, then $(\mathfrak{A},\mathfrak{A}_0)$ is called a *Banach* quasi *-algebra.

The norm of $\mathfrak{A}$ defines a natural norm on $\mathfrak{A}_0$ by

$$\|x\|_0 = \max\left\{\sup_{\|a\|\le 1}\|ax\|, \sup_{\|a\|\le 1}\|xa\|\right\}. \tag{4.3}$$

If $(\mathfrak{A},\mathfrak{A}_0)$ is a Banach quasi *-algebra and $\mathfrak{A}_0$ is a C^*-algebra with respect to $\|\cdot\|_0$, then $(\mathfrak{A},\mathfrak{A}_0)$ is called a *CQ^*-algebra* [14, 15].

4.1. CQ^*-seminorms. To begin with, following [44], we distinguish some particular types of seminorms on a quasi *-algebra.

Definition 4.1. Let $(\mathfrak{A},\mathfrak{A}_0)$ be a quasi *-algebra with unit e and p a seminorm on $\mathfrak{A}$. p is a *Q^*-seminorm* on $(\mathfrak{A},\mathfrak{A}_0)$ if

(Q^*1) $p(a) = p(a^*)$, for all $a \in \mathfrak{A}$;

(Q^*2) for each $x \in \mathfrak{A}_0$ there exists $\gamma_x > 0$ such that

$$p(ax) \le \gamma_x p(a), \quad \forall a \in \mathfrak{A}; \tag{4.4}$$

(Q^*3) $p(e) = 1$.

If p is a Q*-seminorm, we can define

$$p_0(x) := \max\left\{\sup_{p(a)=1} p(ax),\ \sup_{p(a)=1} p(xa)\right\}; \tag{4.5}$$

then $p(x) \le p_0(x)$ for every $x \in \mathfrak{A}_0$ and

$$p(ax) \le p(a)p_0(x), \quad \forall a \in \mathfrak{A},\ x \in \mathfrak{A}_0. \tag{4.6}$$

We will call p_0 the *reduced* seminorm of p.

Remark 4.2. A quasi $*$-algebra $(\mathfrak{A},\mathfrak{A}_0)$ with unit e which admits a Q^*-norm $\|\cdot\|$ such that $\mathfrak{A}_0$ is dense in $\mathfrak{A}$ is, clearly, a normed quasi $*$-algebra.

On the basis of our previous discussion, it is reasonable to expect that one of the most favorable situations occurs when, in analogy to what happens for CQ^*-algebras [15], p_0 is a C^*-seminorm on $\mathfrak{A}_0$.

Definition 4.3. A Q^*-seminorm p is called a *CQ^*-seminorm* if p_0 is a C^*-seminorm on $\mathfrak{A}_0$.

If p itself satisfies the C^*-condition when restricted to $\mathfrak{A}_0$, then we call it an *extended* C^*-seminorm on $(\mathfrak{A},\mathfrak{A}_0)$.

Definition 4.4. Let $(\mathfrak{A},\mathfrak{A}_0)$ be a quasi $*$-algebra with unit e. A positive sesquilinear form $\varphi \in \mathcal{I}(\mathfrak{A})$ is called *left invariant* if

$$\varphi(xa,b) = \varphi(a,x^*b), \quad \forall a,b \in \mathfrak{A},\ x \in \mathfrak{A}_0. \tag{4.7}$$

We put $\mathcal{I}_\ell(\mathfrak{A}) = \{\varphi \in \mathcal{I}(\mathfrak{A}) : \varphi \text{ is left invariant}\}$.

Let $\varphi \in \mathcal{I}_\ell(\mathfrak{A})$. For each $a \in \mathfrak{A}$, we define

$$\omega_\varphi^a(x) = \varphi(xa,a), \quad x \in \mathfrak{A}_0. \tag{4.8}$$

Then ω_φ^a is a positive linear functional on $\mathfrak{A}_0$.

Let now $\mathcal{F} \subseteq \mathcal{I}_\ell(\mathfrak{A})$ and let

$$\mathcal{F}^0 = \{\omega_\varphi^a : \varphi \in \mathcal{F},\ a \in \mathfrak{A}\}. \tag{4.9}$$

Then $\mathcal{F}^0$ is balanced in the sense of Definition 3.5; thus the set

$$\mathcal{D}(\mathcal{F}^0) = \{x \in \mathfrak{A}_0 : \sup\{\omega_\varphi^a(x^*x) : \varphi \in \mathcal{F},\ a \in \mathfrak{A},\ \varphi(a,a) = 1\} < \infty\} \tag{4.10}$$

is a $*$-subalgebra of $\mathfrak{A}_0$ and

$$|x|_{\mathcal{F}^0} = \sup\left\{\omega_\varphi^a(x^*x)^{1/2} : \varphi \in \mathcal{F},\ a \in \mathfrak{A},\ \varphi(a,a) = 1\right\} \tag{4.11}$$

defines a C^*-seminorm on $\mathcal{D}(p_{\mathcal{F}^0})$.

On the other hand, for $a \in \mathfrak{A}$, we can put

$$p_{\mathcal{F}}(a) = \sup_{\varphi \in \mathcal{F}_s} \varphi(a,a)^{1/2}, \tag{4.12}$$

where $\mathcal{F}_s = \{\varphi \in \mathcal{F} : \varphi(e,e) = 1\}$.

Then we define

$$\begin{gathered} \mathcal{D}(\mathcal{F}) = \{a \in \mathfrak{A} : p_{\mathcal{F}}(a) < \infty,\ p_{\mathcal{F}}(a^*) < \infty\}, \\ p^{\mathcal{F}}(a) = \max\{p_{\mathcal{F}}(a), p_{\mathcal{F}}(a^*)\}. \end{gathered} \tag{4.13}$$

The set $\mathcal{D}(\mathcal{F})$ is a $*$-invariant subspace of $\mathfrak{A}$ but, in general, need not be a quasi $*$-algebra over $\mathcal{D}(\mathcal{F}^0)$. There is, however, some special situation as we will see in Proposition 4.6.

We notice at this point that any family $\mathcal{F} \subseteq \mathcal{I}_\ell(\mathfrak{A})$ defines three seminorms on their respective domains: $p^{\mathcal{F}}$ on $\mathfrak{A}$, its reduced seminorm $p_0^{\mathcal{F}}$ on $\mathfrak{A}_0$ and the C^*-seminorm $|\cdot|_{\mathcal{F}^0}$ also on $\mathfrak{A}_0$. It is then natural to study their properties and their mutual relationships. To begin with, we introduce some notation.

If $\varphi \in \mathcal{I}_\ell(\mathfrak{A})$ and $x \in \mathfrak{A}_0$, we put, as before, $\varphi_x(a,b) := \varphi(ax,bx)$, $a,b \in \mathfrak{A}$. Using the semiassociativity of $(\mathfrak{A},\mathfrak{A}_0)$ it is easily seen that $\varphi_x \in \mathcal{I}_\ell(\mathfrak{A})$.

Definition 4.5. Let $\mathcal{F} \subseteq \mathcal{I}_\ell(\mathfrak{A})$. $\mathcal{F}$ is *regular* if

(i) $\mathcal{F}$ is *balanced*, that is, for each $\varphi \in \mathcal{F}$ and for each $x \in \mathfrak{A}_0$, $\varphi_x \in \mathcal{F}$;

(ii) $p_{\mathcal{F}}(a^*) = p_{\mathcal{F}}(a)$, for all $a \in \mathfrak{A}$.

If $\mathcal{F}$ is regular, then the following inequality holds:

$$p_{\mathcal{F}}(ax) \le |x|_{\mathcal{F}^0} \cdot p_{\mathcal{F}}(a), \quad \forall a \in \mathcal{D}(\mathcal{F}),\ x \in \mathcal{D}(\mathcal{F}^0). \tag{4.14}$$

This implies that $(\mathcal{D}(\mathcal{F}),\mathcal{D}(\mathcal{F}^0))$ is a quasi $*$-algebra, $p^{\mathcal{F}}$ is a $*$-invariant seminorm on $(\mathcal{D}(\mathcal{F}),\mathcal{D}(\mathcal{F}^0))$, and

$$p^{\mathcal{F}}(ax) \le |x|_{\mathcal{F}^0} \cdot p^{\mathcal{F}}(a), \quad \forall a \in \mathcal{D}(\mathcal{F}),\ x \in \mathcal{D}(\mathcal{F}^0). \tag{4.15}$$

Proposition 4.6. *Let $(\mathfrak{A},\mathfrak{A}_0)$ be a quasi $*$-algebra and $\mathcal{F} \subset \mathcal{I}_\ell(\mathfrak{A})$ a regular family of sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$. Then $p_{\mathcal{F}}$ is an extended C^*-seminorm on $(\mathcal{D}(\mathcal{F}),\mathcal{D}(\mathcal{F}^0))$.*

Of course, if $|\cdot|_{\mathcal{F}^0} = p_0^{\mathcal{F}}$, where $p_0^{\mathcal{F}}$ is the reduced seminorm of $p^{\mathcal{F}}$ (see (4.5)), then $p^{\mathcal{F}}$ is automatically a CQ^*-seminorm [44, Proposition 2.9].

Proposition 4.7. *Let $\mathcal{F} \subset \mathcal{I}_\ell(\mathfrak{A})$ be a regular family of sesquilinear forms and $(\mathcal{D}(\mathcal{F}), \mathcal{D}(\mathcal{F}^0))$ the quasi $*$-algebra constructed as above. The following statements are equivalent:*

(i) *$|x|_{\mathcal{F}^0} = p_0^{\mathcal{F}}(x)$, for all $x \in \mathcal{D}(\mathcal{F}^0)$;*

(ii) *$\varphi(xa,xa) \le |x|^2_{\mathcal{F}^0}\varphi(a,a)$, for all $\varphi \in \mathcal{F}$, $x \in \mathcal{D}(\mathcal{F}^0)$, $a \in \mathfrak{A}$;*

(iii) *for each $\varphi \in \mathcal{F}$ and $a \in \mathfrak{A}$, ω_φ^a is $p_0^{\mathcal{F}}$-continuous.*

If any of the previous statements holds, then $p_{\mathcal{F}}$ is a CQ^*-seminorm on $(\mathcal{D}(\mathcal{F}),\mathcal{D}(\mathcal{F}^0))$.

Example 4.8. Let I be a compact interval on the real line and consider the quasi*-algebra [16] of functions $(L^p(I), C(I))$ where $C(I)$ stands for the *-algebra of all continuous functions on I and $L^p(I)$ is the usual L^p-space on I. We assume that $p \geq 2$.

Let $w \in L^{p/(p-2)}(I)$ (we take $1/0 = \infty$) and $w \geq 0$. Then

$$\varphi^{(w)}(f,g) = \int_I f(x)\overline{g(x)}w(x)dx, \quad f,g \in L^p(I), \tag{4.16}$$

defines a left invariant positive sesquilinear form on $L^p(I)$.

If $w \in L^\infty(I)$, then $\varphi^{(w)}$ is admissible.

We put

$$\mathcal{F} = \{\varphi^{(w)} : w \in L^{p/(p-2)}(I),\ w \geq 0\}. \tag{4.17}$$

It is easy to see that $\mathcal{F}$ is strongly balanced and that $\varphi^{(w)} \in \mathcal{F}_s$ if and only if $\|w\|_1 = 1$.

Very easy estimates show that $\mathcal{D}(\mathcal{F}^0) = C(I)$ and $|\phi|_{\mathcal{F}^0} = \|\phi\|_\infty$.

On the other hand,

$$\mathcal{D}(\mathcal{F}) = \left\{ f \in L^p(I) : \sup_{\|w\|_1=1} \int_I |f(x)|^2 w(x)dx < \infty \right\} = L^\infty(I). \tag{4.18}$$

Therefore, the extended C^*-seminorm $p^{\mathcal{F}}$ coincides with the L^∞-norm on $L^\infty(I)$.

Once the seminorm $p_{\mathcal{F}}$ (or $p^{\mathcal{F}}$) is defined, one can consider the set $\mathcal{C}(p_{\mathcal{F}})$ of all elements of $\mathcal{I}_\ell(\mathfrak{A})$ that are $p_{\mathcal{F}}$-bounded. Then $\mathcal{C} := \mathcal{C}(p_{\mathcal{F}})$ is balanced but it is not necessarily regular, since (ii) of Definition 4.5 may fail. Clearly $\mathcal{F} \subseteq \mathcal{C}$ and, as it is easily seen, $p_{\mathcal{F}}(a) = p_{\mathcal{C}}(a)$, for every $a \in \mathcal{D}(\mathcal{F})$. It is therefore natural to ask the question as to whether $\mathcal{F} = \mathcal{C}$ (this equality would mean that the balancedness condition can be satisfied only if $\mathcal{F}$ is large enough). However it is not so, as the next example shows.

Example 4.9. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let us consider a bounded self-adjoint operator A with continuous spectrum $\sigma \subset \mathbb{R}$ and let $C(\sigma)$ denote the *-algebra of all continuous functions on the compact set σ with its usual sup norm $\|\cdot\|_\infty$. Let

$$\mathfrak{A} = \{f(A);\ f \in C(\sigma)\}, \tag{4.19}$$

where $f(A)$ is defined via the functional calculus. As it is known, each $f(A)$ is bounded and $\|f(A)\| = \|f\|_\infty$. Then $\mathfrak{A}$ is a C^*-algebra of bounded operators. We define $\mathcal{F}$ as follows. If $\xi \in \mathcal{H}$, we put

$$\varphi^\xi(A,B) = \langle A\xi \mid B\xi \rangle, \quad A,B \in \mathfrak{A}. \tag{4.20}$$

Then each φ^ξ is left-invariant and the set $\mathcal{F} = \{\varphi^\xi;\ \xi \in \mathcal{H}\}$ is balanced.

In this case $\mathcal{D}(\mathcal{F}) = \mathcal{D}(\mathcal{F}^0) = \mathfrak{A}$ and

$$p^{\mathcal{F}}(A) = \|A\|, \quad \forall A \in \mathfrak{A}. \tag{4.21}$$

The set $\mathscr{C}(p_{\mathscr{F}})$ consists of all sesquilinear forms Φ for which there exists $\gamma > 0$ such that

$$\Phi(A,B) \leq \gamma \|A\| \|B\|, \quad \forall A,B \in \mathfrak{A}. \tag{4.22}$$

We will show that, in general, $\mathscr{C}(p_{\mathscr{F}})$ properly contains $\mathscr{F}$.

Indeed, let $\lambda_0 \in \sigma$ be fixed. We define a sesquilinear form φ_{λ_0} by

$$\varphi_{\lambda_0}(f(A),g(A)) = f(\lambda_0)\overline{g(\lambda_0)}, \quad f,g \in C(\sigma); \tag{4.23}$$

φ_{λ_0} is positive, left invariant, and bounded, that is, $\varphi_{\lambda_0} \in \mathscr{C}(p_{\mathscr{F}})$. Indeed,

$$\begin{aligned} |\varphi_{\lambda_0}(f(A),g(A))| &= |f(\lambda_0)\overline{g(\lambda_0)}| \\ &\leq \|f\|_\infty \|g\|_\infty = \|f(A)\| \|g(A)\|, \quad f,g \in C(\sigma). \end{aligned} \tag{4.24}$$

This implies that $\|\varphi_{\lambda_0}\|_{p_{\mathscr{F}}} \leq 1$ (indeed, the equality holds). Assume that there exists $\eta \in \mathscr{H}$ such that

$$\varphi_{\lambda_0}(f(A),g(A)) = \langle f(A)\eta \mid g(A)\eta \rangle, \quad \forall f,g \in C(\sigma). \tag{4.25}$$

Then, if $E(\cdot)$ denotes the spectral measure of A, we have

$$\begin{aligned} \varphi_{\lambda_0}(f(A),g(A)) &= \langle f(A)\eta \mid g(A)\eta \rangle \\ &= \int_\sigma f(\lambda)\overline{g(\lambda)} d\langle E(\lambda)\eta \mid \eta \rangle \\ &= f(\lambda_0)\overline{g(\lambda_0)}, \quad \forall f,g \in C(\sigma), \end{aligned} \tag{4.26}$$

and this is possible only if λ_0 is an eigenvalue of A. Therefore $\varphi_{\lambda_0} \in \mathscr{C} \setminus \mathscr{F}$.

4.2. The case of Banach quasi *-algebras. Let us consider finally the case where $(\mathfrak{A},\mathfrak{A}_0)$ is a normed quasi *-algebra with norm $\|\cdot\|$ and unit e. For details and proofs we refer to [45].

Let $\mathscr{I}_b(\mathfrak{A})$ denote the subset of all elements of $\mathscr{I}(\mathfrak{A})$ that are $\|\cdot\|$-bounded (it is easily seen that $\mathscr{I}_b(\mathfrak{A}) \subset \mathscr{I}_\ell(\mathfrak{A})$) and $\mathscr{S}(\mathfrak{A}) = \{\varphi \in \mathscr{I}_b(\mathfrak{A}) : \|\varphi\| \leq 1\}$.

To begin with, we put

$$\mathfrak{p}(a) = \sup_{\varphi \in \mathscr{S}(\mathfrak{A})} \varphi(a,a)^{1/2}, \quad a \in \mathfrak{A}, \tag{4.27}$$

then $\mathfrak{p}$ is a seminorm on $\mathfrak{A}$ with $\mathfrak{p}(a) \leq \|a\|$ for every $a \in \mathfrak{A}$.

We also put

$$\mathscr{K}_{\mathfrak{p}} = \{a \in \mathfrak{A} : \mathfrak{p}(a) = 0\}. \tag{4.28}$$

We define

$$\begin{gathered} \mathfrak{q}(a) = \sup\{\varphi(a,a)^{1/2};\ \varphi \in \mathscr{I}_b(\mathfrak{A}),\ \varphi(e,e) = 1\}, \quad a \in \mathfrak{A}, \\ D(\mathfrak{q}) = \{a \in \mathfrak{A} : \mathfrak{q}(a) < \infty\}. \end{gathered} \tag{4.29}$$

It is clear that, if $\varphi \in \mathcal{I}_b(\mathfrak{A})$, then also $\varphi_a \in \mathcal{I}_b(\mathfrak{A})$, for every $a \in \mathfrak{A}_0$, where $\varphi_x(a,b) = \varphi(ax,bx)$ for every $a,b \in \mathfrak{A}$. Thus $\mathcal{I}_b(\mathfrak{A})$ is balanced.

The seminorms $\mathfrak{p}$ and $\mathfrak{q}$ are compared as follows.

Proposition 4.10. *Let $(\mathfrak{A},\mathfrak{A}_0)$ be a normed quasi *-algebra. Then*

(i) $\mathfrak{p}(ax) \le \mathfrak{q}(a)\mathfrak{p}(x)$, *for all $a \in D(\mathfrak{q})$, $x \in \mathfrak{A}_0$;*

(ii) *if $(\mathfrak{A},\mathfrak{A}_0)$ has a unit, then*

$$\mathfrak{p}(a) \le \mathfrak{q}(a), \quad \forall a \in \mathcal{D}(\mathfrak{q}). \tag{4.30}$$

Proposition 4.11. *Let $(\mathfrak{A},\mathfrak{A}_0)$ be a normed quasi *-algebra. The following statements hold:*

(i) $\mathfrak{A}_0 \subseteq D(\mathfrak{q})$ *and* $q(a) \le \|a\|0$, *for all $a \in \mathfrak{A}_0$;*

(ii)

$$\begin{gathered} D(\mathfrak{q}) = \left\{ a \in \mathfrak{A} : \pi_\varphi \text{ bounded}, \ \forall \varphi \in \mathcal{I}_b(\mathfrak{A}), \ \sup_{\varphi \in \mathcal{I}_b(\mathfrak{A})} \|\overline{\pi_\varphi(a)}\| < \infty \right\}, \\ \mathfrak{q}(a) = \sup_{\varphi \in \mathcal{I}_b(\mathfrak{A})} \|\overline{\pi_\varphi(a)}\|, \quad \forall a \in D(\mathfrak{q}); \end{gathered} \tag{4.31}$$

(iii) $\mathfrak{q}$ *is an extended C^*-seminorm on $(\mathfrak{A},\mathfrak{A}_0)$;*

(iv) $\mathfrak{p}(xa) \le \|x\|0\mathfrak{p}(a)$, *for all $a \in \mathfrak{A}$, $x \in \mathfrak{A}_0$.*

Definition 4.12. A Banach quasi *-algebra $(\mathfrak{A},\mathfrak{A}_0)$ is called *regular* if

(i) $\mathcal{S}(\mathfrak{A})$ is sufficient, that is, if $a \in \mathfrak{A}$ and $\varphi(a,a) = 0$, for every $\varphi \in \mathcal{S}(\mathfrak{A})$, then $a = 0$;

(ii) $\mathfrak{p}(a) = \|a\|$, for every $a \in \mathfrak{A}$.

The set of *bounded elements* of a given normed quasi *-algebra has been studied in [45].

Definition 4.13. Let $(\mathfrak{A},\mathfrak{A}_0)$ be a Banach quasi*-algebra. An element $a \in \mathfrak{A}$ is said to be *bounded* if there exists $\gamma_a > 0$ such that

$$\max\{\|ax\|, \|xa\|\} \le \gamma_a \|x\|, \quad \forall x \in \mathfrak{A}_0. \tag{4.32}$$

The set of all bounded elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_{\mathrm{b}}$.

If a is bounded, then the multiplication operators

$$x \in \mathfrak{A}_0 \longmapsto L_a x = ax \qquad x \in \mathfrak{A}_0 \longmapsto R_a x = xa \tag{4.33}$$

have bounded extensions to $\mathfrak{A}$. In $\mathfrak{A}_{\mathrm{b}}$ we define the norm

$$\|a\|_{\mathrm{b}} = \max\{\|\overline{L}_a\|, \|\overline{R}_a\|\}. \tag{4.34}$$

Two new multiplications can be defined. Let $a \in \mathfrak{A}_{\mathrm{b}}$ and $b \in \mathfrak{A}$. Then we put

$$a \blacktriangleright b = \overline{L}_a b. \tag{4.35}$$

Similarly, if $b \in \mathfrak{A}_{\mathrm{b}}$ and $a \in \mathfrak{A}$, we put

$$a_{\blacktriangleleft} b = \overline{R}_b a. \tag{4.36}$$

If $a, b \in \mathfrak{A}_{\mathrm{b}}$, then both $a_{\blacktriangleright} b$ and $a_{\blacktriangleleft} b$ are well defined, but, in general, $a_{\blacktriangleright} b \neq a_{\blacktriangleleft} b$. But for a regular Banach quasi $*$-algebra they always coincide and the set $\mathfrak{A}_{\mathrm{b}}$ is a $*$-algebra.

Theorem 4.14. *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a regular Banach quasi$*$-algebra with unit e. Then $D(\mathfrak{q})$ coincides with the set $\mathfrak{A}_b$ of all bounded elements of $\mathfrak{A}$. Moreover*

$$\mathfrak{q}(a) = \|a\|_b, \quad \forall a \in \mathfrak{A}_b. \tag{4.37}$$

Therefore $(\mathfrak{A}_b, \|\cdot\|_b)$ is a C^*-algebra.

In other words, the previous theorem says that the norm of a regular Banach quasi $*$-algebra is always a CQ^*-norm in the sense of Definition 4.3.

On the other hand, by means of the set of bounded elements one can define a notion of spectrum for elements of a regular Banach quasi $*$-algebra (this notion can actually be introduced for the larger class of *normal* Banach quasi $*$-algebras, but we will not consider this case here).

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a regular Banach quasi $*$-algebra with unit e. An element $a \in \mathfrak{A}$ has a *bounded inverse* if there exists $b \in \mathfrak{A}_{\mathrm{b}}$ such that $\overline{R}_b(a) = \overline{L}_b(a) = e$. If the element b exists, then it is unique. If a has a bounded inverse we denote it with a_{b}^{-1}.

The *resolvent* $\rho(a)$ of $a \in \mathfrak{A}$ is the set

$$\rho(a) = \{\lambda \in \mathbb{C} : (a - \lambda e)_{\mathrm{b}}^{-1} \text{ exists}\}. \tag{4.38}$$

The set $\sigma(a) = \mathbb{C} \setminus \rho(a)$ is called the *spectrum* of a.

Finally, if $a \in \mathfrak{A}$, the nonnegative number

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| \tag{4.39}$$

is called the *spectral radius* of x.

Proposition 4.15. *Let $a \in \mathfrak{A}$. Then $r(a) < \infty$ if and only if $a \in \mathfrak{A}_b$.*

In other words, for regular Banach quasi $*$-algebras, the unbounded C^*-seminorm $\mathfrak{q}$ identifies the elements of finite spectral radius. Thus we could think of $\mathfrak{q}$ as to the smallest *spectral* unbounded C^*-seminorm on $\mathfrak{A}$.

5. Conclusions

As the reader has certainly realized, the extension of the theory of C^*-seminorms to partial or quasi $*$-algebras requires a certain effort. The lack of an everywhere defined multiplication, which is in itself a quite unfamiliar feature, brings with it a series of technical difficulties that it is sometimes impossible to by-pass. There are many differences with the case of $*$-algebras and many (perhaps still too many!) questions are still unsolved also in the Banach case, which should be expected to be more and more regular. A fundamental

question which has still no answer is under which conditions a biweight on a Banach partial $*$-algebra is forced to have continuous orbits with respect to some core. So there is still a lot of work to be done in this area. We hope that the reading of this paper could draw the attention of other mathematicians to this field. This was, indeed, the main motivation when the writing of this paper was undertaken.

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