INVERSION FORMULAS FOR RIEMANN-LIOUVILLE TRANSFORM AND ITS DUAL ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATORS

C. BACCAR, N. B. HAMADI, AND L. T. RACHDI

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We define Riemann-Liouville transform \Re_{α} and its dual ${}^{t}\Re_{\alpha}$ associated with two singular partial differential operators. We establish some results of harmonic analysis for the Fourier transform connected with \Re_{α} . Next, we prove inversion formulas for the operators \Re_{α} , ${}^{t}\Re_{\alpha}$ and a Plancherel theorem for ${}^{t}\Re_{\alpha}$.

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1. Introduction

The mean operator is defined for a continuous function f on \mathbb{R}^2 , even with respect to the first variable by

$$\mathfrak{R}_0(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r\sin\theta, x + r\cos\theta) d\theta, \tag{1.1}$$

which means that $\mathfrak{R}_0(f)(r,x)$ is the mean value of f on the circle centered at (0,x) and radius r. The dual of the mean operator ${}^t\mathfrak{R}_0$ is defined by

$${}^{t}\mathfrak{R}_{0}(f)(r,x) = \frac{1}{\pi} \int_{\mathbb{R}} f\left(\sqrt{r^{2} + (x-y)^{2}}, y\right) dy.$$
(1.2)

The mean operator \Re_0 and its dual ${}^t\Re_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [11, 12] or in the linearized inverse scattering problem in acoustics [6].

Our purpose in this work is to define and study integral transforms which generalize the operators \mathfrak{R}_0 and ${}^t\mathfrak{R}_0$. More precisely, we consider the following singular partial differential operators:

$$\Delta_{1} = \frac{\partial}{\partial x},$$

$$\Delta_{2} = \frac{\partial^{2}}{\partial r^{2}} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^{2}}{\partial x^{2}}, \quad (r, x) \in]0, +\infty[\times\mathbb{R}, \alpha \ge 0.$$
(1.3)

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We associate to Δ_1 and Δ_2 the Riemann-Liouville transform \mathfrak{R}_{α} , defined on $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on \mathbb{R}^2 , even with respect to the first variable) by

$$\Re_{\alpha}(f)(r,x) = \begin{cases} \frac{\alpha}{\pi} \iint_{-1}^{1} f\left(rs\sqrt{1-t^{2}}, x+rt\right) \\ \times (1-t^{2})^{\alpha-1/2} (1-s^{2})^{\alpha-1} dt \, ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} f\left(r\sqrt{1-t^{2}}, x+rt\right) \frac{dt}{\sqrt{1-t^{2}}}, & \text{if } \alpha = 0. \end{cases}$$
(1.4)

The dual operator ${}^{t}\mathfrak{R}_{\alpha}$ is defined on the space $\mathscr{G}_{*}(\mathbb{R}^{2})$ (the space of infinitely differentiable functions on \mathbb{R}^{2} , rapidly decreasing together with all their derivatives, even with respect to the first variable) by

$${}^{t}\mathfrak{R}_{\alpha}(f)(r,x) = \begin{cases} \frac{2\alpha}{\pi} \int_{r}^{+\infty} \int_{-\sqrt{u^{2}-r^{2}}}^{\sqrt{u^{2}-r^{2}}} f(u,x+v) (u^{2}-v^{2}-r^{2})^{\alpha-1} u \, du \, dv, & \text{if } \alpha > 0, \\ \\ \frac{1}{\pi} \int_{\mathbb{R}} f\left(\sqrt{r^{2}+(x-y)^{2}}, y\right) dy, & \text{if } \alpha = 0. \end{cases}$$

$$(1.5)$$

For more general fractional integrals and fractional differential equations, we can see the works of Debnath [3, 4] and Debnath with Bhatta [5].

We establish for the operators \Re_{α} and ${}^{t}\Re_{\alpha}$ the same results given by Helgason, Ludwig, and Solmon for the classical Radon transform on \mathbb{R}^{2} [10, 14, 17] and we find the results given in [15] for the spherical mean operator. Especially

- (i) we give some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville transform \mathfrak{R}_{α} ;
- (ii) we define and characterize some spaces of the functions on which \mathfrak{R}_{α} and ${}^{t}\mathfrak{R}_{\alpha}$ are isomorphisms;
- (iii) we give the following inversion formulas for \mathfrak{R}_{α} and ${}^{t}\mathfrak{R}_{\alpha}$:

$$f = \mathfrak{R}_{\alpha} K_{\alpha}^{1t} \mathfrak{R}_{\alpha}(f), \qquad f = K_{\alpha}^{1t} \mathfrak{R}_{\alpha} \mathfrak{R}_{\alpha}(f),$$

$$f = {}^{t} \mathfrak{R}_{\alpha} K_{\alpha}^{2} \mathfrak{R}_{\alpha}(f), \qquad f = K_{\alpha}^{2} \mathfrak{R}_{\alpha} {}^{t} \mathfrak{R}_{\alpha}(f),$$
(1.6)

where K^1_{α} and K^2_{α} are integro-differential operators;

- (iv) we establish a Plancherel theorem for ${}^{t}\mathfrak{R}_{\alpha}$;
- (v) we show that \mathfrak{R}_{α} and ${}^{t}\mathfrak{R}_{\alpha}$ are transmutation operators.

This paper is organized as follows. In Section 2, we show that for $(\mu, \lambda) \in \mathbb{C}^2$, the differential system

$$\Delta_1 u(r, x) = -i\lambda u(r, x),$$

$$\Delta_2 u(r, x) = -\mu^2 u(r, x),$$
(1.7)

$$u(0,0) = 1, \qquad \frac{\partial u}{\partial r}(0,x) = 0, \quad \forall x \in \mathbb{R},$$

admits a unique solution $\varphi_{\mu,\lambda}$ given by

$$\varphi_{\mu,\lambda}(r,x) = j_{\alpha} \left(r \sqrt{\mu^2 + \lambda^2} \right) \exp(-i\lambda x), \qquad (1.8)$$

where j_{α} is the modified Bessel function defined by

$$j_{\alpha}(s) = 2^{\alpha} \Gamma(\alpha+1) \frac{J_{\alpha}(s)}{s^{\alpha}}, \qquad (1.9)$$

and J_{α} is the Bessel function of first kind and index α . Next, we prove a Mehler integral representation of $\varphi_{\mu,\lambda}$ and give some properties of \Re_{α} .

In Section 3, we define the Fourier transform \mathfrak{F}_{α} connected with \mathfrak{R}_{α} , and we establish some harmonic analysis results (inversion formula, Plancherel theorem, Paley-Wiener theorem) which lead to new properties of the operator \mathfrak{R}_{α} and its dual ${}^{t}\mathfrak{R}_{\alpha}$.

In Section 4, we characterize some subspaces of $\mathscr{G}_*(\mathbb{R}^2)$ on which \mathfrak{R}_{α} and ${}^t\mathfrak{R}_{\alpha}$ are isomorphisms, and we prove the inversion formulas cited below where the operators K^1_{α} and K^2_{α} are given in terms of Fourier transforms. Next, we introduce fractional powers of the Bessel operator,

$$\ell_{\alpha} = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r},\tag{1.10}$$

and the Laplacian operator,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2},\tag{1.11}$$

that we use to simplify K_{α}^1 and K_{α}^2 .

Finally, we prove the following Plancherel theorem for ${}^t\mathfrak{R}_{\alpha}$:

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \left| f(r,x) \right|^{2} r^{2\alpha+1} dr \, dx = \int_{\mathbb{R}} \int_{0}^{+\infty} \left| K_{\alpha}^{3} \binom{t}{\mathfrak{R}_{\alpha}}(f)(r,x) \right|^{2} dr \, dx, \tag{1.12}$$

where K_{α}^{3} is an integro-differential operator.

In Section 5, we show that \Re_{α} and ${}^{t}\Re_{\alpha}$ satisfy the following relations of permutation:

$${}^{t}\mathfrak{R}_{\alpha}(\Delta_{2}f) = \frac{\partial^{2}}{\partial r^{2}}{}^{t}\mathfrak{R}_{\alpha}(f), \qquad {}^{t}\mathfrak{R}_{\alpha}(\Delta_{1}f) = \Delta_{1}{}^{t}\mathfrak{R}_{\alpha}(f),$$

$$\Delta_{2}\mathfrak{R}_{\alpha}(f) = \mathfrak{R}_{\alpha}\left(\frac{\partial^{2}f}{\partial r^{2}}\right), \qquad \Delta_{1}\mathfrak{R}_{\alpha}(f) = \mathfrak{R}_{\alpha}(\Delta_{1}f).$$
(1.13)

2. Riemann-Liouville transform and its dual associated with the operators Δ_1 and Δ_2

In this section, we define the Riemann-Liouville transform \mathfrak{R}_{α} and its dual ${}^{t}\mathfrak{R}_{\alpha}$, and we give some properties of these operators. It is well known [21] that for every $\lambda \in \mathbb{C}$, the system

$$\ell_{\alpha}v(r) = -\lambda^2 v(r);$$

 $v(0) = 1; \quad v'(0) = 0,$
(2.1)

where ℓ_{α} is the Bessel operator, admits a unique solution, that is, the modified Bessel function $r \mapsto j_{\alpha}(r\lambda)$. Thus, for all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}$, the system

$$\Delta_1 u(r,x) = -i\lambda u(r,x),$$

$$\Delta_2 u(r,x) = -\mu^2 u(r,x),$$

$$u(0,0) = 1, \qquad \frac{\partial u}{\partial r}(0,x) = 0, \quad \forall x \in \mathbb{R},$$
(2.2)

admits the unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_{\alpha} \left(r \sqrt{\mu^2 + \lambda^2} \right) \exp(-i\lambda x).$$
(2.3)

The modified Bessel function j_{α} has the Mehler integral representation, (we refer to [13, 21])

$$j_{\alpha}(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^{1} (1-t^2)^{\alpha-1/2} \exp(-ist) dt.$$
(2.4)

In particular,

$$\forall k \in \mathbb{N}, \ \forall s \in \mathbb{R}, \quad \left| j_{\alpha}^{(k)}(s) \right| \leqslant 1.$$
(2.5)

On the other hand,

$$\sup_{r \in \mathbb{R}} |j_{\alpha}(r\lambda)| = 1 \quad \text{iff } \lambda \in \mathbb{R}.$$
(2.6)

This involves that

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1 \quad \text{iff } (\mu,\lambda)\in\Gamma,$$
(2.7)

where Γ is the set defined by

$$\Gamma = \mathbb{R}^2 \cup \{ (i\mu, \lambda); \ (\mu, \lambda) \in \mathbb{R}^2, \ |\mu| \leqslant |\lambda| \}.$$
(2.8)

PROPOSITION 2.1. The eigenfunction $\varphi_{\mu,\lambda}$ given by (2.3) has the following Mehler integral representation:

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \frac{\alpha}{\pi} \iint_{-1}^{1} \cos\left(\mu r s \sqrt{1-t^{2}}\right) \exp\left(-i\lambda(x+rt)\right) \left(1-t^{2}\right)^{\alpha-1/2} \left(1-s^{2}\right)^{\alpha-1} dt \, ds, \\ if \, \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} \cos\left(r \mu \sqrt{1-t^{2}}\right) \exp\left(-i\lambda(x+rt)\right) \frac{dt}{\sqrt{1-t^{2}}}, & if \, \alpha = 0. \end{cases}$$
(2.9)

Proof. From the following expansion of the function j_{α} :

$$j_{\alpha}(s) = 2^{\alpha} \Gamma(\alpha+1) \frac{J_{\alpha}(s)}{s^{\alpha}} = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha+k+1)} \left(\frac{s}{2}\right)^{2k},$$
(2.10)

we deduce that

$$j_{\alpha}\left(r\sqrt{\mu^{2}+\lambda^{2}}\right) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^{k}}{k!\Gamma(k+\alpha+1)}\left(\frac{r\mu}{2}\right)^{2k}j_{\alpha+k}(r\lambda),$$
(2.11)

and from the equality (2.4), we obtain

$$j_{\alpha}\left(r\sqrt{\mu^{2}+\lambda^{2}}\right) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_{-1}^{1} j_{\alpha-1/2}\left(r\mu\sqrt{1-t^{2}}\right) \exp(-ir\lambda t) \left(1-t^{2}\right)^{\alpha-1/2} dt.$$
(2.12)

Then, the results follow by using again the relation (2.4) for $\alpha > 0$, and from the fact that

$$j_{-1/2}(s) = \cos s$$
, for $\alpha = 0$. (2.13)

Definition 2.2. The Riemann-Liouville transform \mathfrak{R}_{α} associated with the operators Δ_1 and Δ_2 is the mapping defined on $\mathscr{C}_*(\mathbb{R}^2)$ by the following. For all $(r, x) \in \mathbb{R}^2$,

$$\mathfrak{R}_{\alpha}(f)(r,x) = \begin{cases} \frac{\alpha}{\pi} \iint_{-1}^{1} f\left(rs\sqrt{1-t^{2}}, x+rt\right) \\ \times (1-t^{2})^{\alpha-1/2} (1-s^{2})^{\alpha-1} dt \, ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} f\left(r\sqrt{1-t^{2}}, x+rt\right) \frac{dt}{\sqrt{1-t^{2}}}, & \text{if } \alpha = 0. \end{cases}$$
(2.14)

Remark 2.3. (i) From Proposition 2.1 and Definition 2.2, we have

$$\varphi_{\mu,\lambda}(r,x) = \Re_{\alpha}(\cos(\mu)\exp(-i\lambda))(r,x).$$
(2.15)

(ii) We can easily see, as in [2], that the transform \mathfrak{R}_{α} is continuous and injective from $\mathscr{C}_*(\mathbb{R}^2)$ (the space of infinitely differentiable functions on \mathbb{R}^2 , even with respect to the first variable) into itself.

LEMMA 2.4. For $f \in \mathcal{C}_*(\mathbb{R}^2)$, f bounded, and $g \in \mathcal{G}_*(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \mathfrak{R}_{\alpha}(f)(r,x)g(r,x)r^{2\alpha+1}dr\,dx = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r,x)\,{}^{t}\mathfrak{R}_{\alpha}(g)(r,x)dr\,dx, \qquad (2.16)$$

where ${}^{t}\mathfrak{R}_{\alpha}$ is the dual transform defined by

$${}^{t}\mathfrak{R}_{\alpha}(g)(r,x) = \begin{cases} \frac{2\alpha}{\pi} \int_{r}^{+\infty} \int_{-\sqrt{u^{2}-r^{2}}}^{\sqrt{u^{2}-r^{2}}} g(u,x+v) (u^{2}-v^{2}-r^{2})^{\alpha-1} u \, du \, dv, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{\mathbb{R}} g\left(\sqrt{r^{2}+(x-y)^{2}}, y\right) dy, & \text{if } \alpha = 0. \end{cases}$$

$$(2.17)$$

To obtain this lemma, we use Fubini's theorem and an adequate change of variables. *Remark 2.5.* By a simple change of variables, we have

$$\mathfrak{R}_0(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r\sin\theta, x + r\cos\theta) d\theta.$$
(2.18)

3. Fourier transform associated with Riemann-Liouville operator

In this section, we define the Fourier transform associated with the operator \Re_{α} , and we give some results of harmonic analysis that we use in the next sections.

We denote by

(i) $d\nu(r,x)$ the measure defined on $[0, +\infty[\times\mathbb{R} \text{ by }$

$$d\nu(r,x) = \frac{1}{\sqrt{2\pi}2^{\alpha}\Gamma(\alpha+1)}r^{2\alpha+1}dr \otimes dx,$$
(3.1)

(ii) $L^1(d\nu)$ the space of measurable functions f on $[0, +\infty[\times \mathbb{R} \text{ satisfying }$

$$\|f\|_{1,\nu} = \int_{\mathbb{R}} \int_{0}^{+\infty} |f(r,x)| \, d\nu(r,x) < +\infty.$$
(3.2)

Definition 3.1. (i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(d\nu)$ by the following. For all $(r, x), (s, y) \in [0, +\infty[\times\mathbb{R},$

$$\mathcal{T}_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^{\pi} f\left(\sqrt{r^2 + s^2 + 2rs\cos\theta}, x+y\right) \sin^{2\alpha}\theta d\theta.$$
(3.3)

(ii) The convolution product associated with the Riemann-Liouville transform of f, $g \in L^1(d\nu)$ is defined by the following. For all $(r, x) \in [0, +\infty[\times\mathbb{R},$

$$f #g(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} \mathcal{T}_{(r,-x)}\check{f}(s,y)g(s,y)d\nu(s,y),$$
(3.4)

where $\check{f}(s, y) = f(s, -y)$.

We have the following properties.

(i) Since

$$\forall r, s \ge 0, \quad j_{\alpha}(r\lambda) j_{\alpha}(s\lambda) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{0}^{\pi} j_{\alpha} \left(\lambda \sqrt{r^{2}+s^{2}+2rs\cos\theta}\right) \sin^{2\alpha}\theta \, d\theta, \quad (3.5)$$

(we refer to [21]) we deduce that the eigenfunction $\varphi_{\mu,\lambda}$ defined by the relation (2.3) satisfies the product formula

$$\mathcal{T}_{(r,x)}\varphi_{\mu,\lambda}(s,y) = \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y).$$
(3.6)

(ii) If $f \in L^1(d\nu)$, then for all $(r,x) \in [0, +\infty[\times\mathbb{R}, \mathcal{T}_{(r,x)}f$ belongs to $L^1(d\nu)$, and we have

$$\|\mathcal{T}_{(r,x)}f\|_{1,\nu} \leqslant \|f\|_{1,\nu}.$$
 (3.7)

(iii) For $f,g \in L^1(d\nu)$, f #g belongs to $L^1(d\nu)$, and the convolution product is commutative and associative.

(iv) For $f,g \in L^1(d\nu)$,

$$\|f \# g\|_{1,\nu} \leqslant \|f\|_{1,\nu} \|g\|_{1,\nu}.$$
(3.8)

Definition 3.2. The Fourier transform associated with the Riemann-Liouville operator is defined by

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathfrak{F}_{\alpha}(f)(\mu, \lambda) = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu(r, x), \tag{3.9}$$

where Γ is the set defined by the relation (2.8).

We have the following properties.

(i) Let *f* be in $L^1(d\nu)$. For all $(r, x) \in [0, +\infty[\times\mathbb{R}, we have$

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathfrak{F}_{\alpha} \big(\mathcal{T}_{(r, -x)} f \big) (\mu, \lambda) = \varphi_{\mu, \lambda}(r, x) \mathfrak{F}_{\alpha}(f)(\mu, \lambda). \tag{3.10}$$

(ii) For $f,g \in L^1(d\nu)$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathfrak{F}_{\alpha}(f \# g)(\mu, \lambda) = \mathfrak{F}_{\alpha}(f)(\mu, \lambda)\mathfrak{F}_{\alpha}(g)(\mu, \lambda). \tag{3.11}$$

(iii) For $f \in L^1(d\nu)$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathfrak{F}_{\alpha}(f)(\mu, \lambda) = B \circ \widetilde{\mathfrak{F}}_{\alpha}(f)(\mu, \lambda), \tag{3.12}$$

where

$$\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \widetilde{\mathfrak{F}}_{\alpha}(f)(\mu, \lambda) = \int_{\mathbb{R}} \int_0^{+\infty} f(r, x) j_{\alpha}(r\mu) \exp(-i\lambda x) d\nu(r, x),$$

$$\forall (\mu, \lambda) \in \Gamma, \quad Bf(\mu, \lambda) = f(\sqrt{\mu^2 + \lambda^2}, \lambda).$$

$$(3.13)$$

3.1. Inversion formula and Plancherel theorem for \mathfrak{F}_{α} . We denote by (see [15])

(i) $\mathcal{G}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on \mathbb{R}^2 rapidly decreasing together with all their derivatives, even with respect to the first variable;

(ii) $\mathscr{G}_*(\Gamma)$ the space of functions $f : \Gamma \to \mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that is, for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\Gamma} \left(1+|\mu|^2+|\lambda|^2\right)^{k_1} \left| \left(\frac{\partial}{\partial\mu}\right)^{k_2} \left(\frac{\partial}{\partial\lambda}\right)^{k_3} f(\mu,\lambda) \right| < +\infty, \tag{3.14}$$

where

$$\frac{\partial f}{\partial \mu}(\mu,\lambda) = \begin{cases} \frac{\partial}{\partial r}(f(r,\lambda)), & \text{if } \mu = r \in \mathbb{R}, \\ \frac{1}{i}\frac{\partial}{\partial t}(f(it,\lambda)), & \text{if } \mu = it, \ |t| \leqslant |\lambda|. \end{cases}$$
(3.15)

Each of these spaces is equipped with its usual topology:

(i) $L^2(d\nu)$ the space of measurable functions on $[0, +\infty[\times\mathbb{R} \text{ such that}$

$$\|f\|_{2,\nu} = \left(\int_{\mathbb{R}} \int_{0}^{+\infty} |f(r,x)|^{2} d\nu(r,x)\right)^{1/2} < +\infty;$$
(3.16)

(ii) $d\gamma(\mu, \lambda)$ the measure defined on Γ by

$$\iint_{\Gamma} f(\mu,\lambda) d\gamma(\mu,\lambda) = \frac{1}{\sqrt{2\pi} 2^{\alpha} \Gamma(\alpha+1)} \left\{ \int_{\mathbb{R}} \int_{0}^{+\infty} f(\mu,\lambda) (\mu^{2}+\lambda^{2})^{\alpha} \mu \, d\mu \, d\lambda + \int_{\mathbb{R}} \int_{0}^{|\lambda|} f(i\mu,\lambda) (\lambda^{2}-\mu^{2})^{\alpha} \mu \, d\mu \, d\lambda \right\};$$
(3.17)

(iii) $L^p(dy)$, p = 1, p = 2, the space of measurable functions on Γ satisfying

$$\|f\|_{p,\gamma} = \left(\iint_{\Gamma} |f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\right)^{1/p} < +\infty.$$
(3.18)

Remark 3.3. It is clear that a function f belongs to $L^1(d\nu)$ if, and only if, the function Bf belongs to $L^1(d\gamma)$, and we have

$$\iint_{\Gamma} Bf(\mu,\lambda)d\gamma(\mu,\lambda) = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r,x)d\nu(r,x).$$
(3.19)

PROPOSITION 3.4 (inversion formula for \mathfrak{F}_{α}). Let $f \in L^1(d\nu)$ such that $\mathfrak{F}_{\alpha}(f)$ belongs to $L^1(d\nu)$, then for almost every $(r, x) \in [0, +\infty[\times\mathbb{R},$

$$f(r,x) = \iint_{\Gamma} \mathfrak{F}_{\alpha}(f)(\mu,\lambda)\overline{\varphi}_{\mu,\lambda}(r,x)d\gamma(\mu,\lambda).$$
(3.20)

 \Box

Proof. From [9, 19], one can see that if $f \in L^1(d\nu)$ is such that $\widetilde{\mathfrak{F}}_{\alpha}(f) \in L^1(d\nu)$, then for almost every $(r, x) \in [0, +\infty[\times\mathbb{R},$

$$f(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} \widetilde{\mathfrak{F}}_{\alpha}(f)(\mu,\lambda) j_{\alpha}(r\mu) \exp(i\lambda x) d\nu(\mu,\lambda).$$
(3.21)

Then, the result follows from the relation (3.12) and Remark 3.3.

THEOREM 3.5. (i) The Fourier transform \mathfrak{F}_{α} is an isomorphism from $\mathscr{G}_*(\mathbb{R}^2)$ onto $\mathscr{G}_*(\Gamma)$. (ii) (Plancherel formula) for $f \in \mathscr{G}_*(\mathbb{R}^2)$,

$$\|\mathfrak{F}_{\alpha}(f)\|_{2,\nu} = \|f\|_{2,\nu}.$$
(3.22)

(iii) (Plancherel theorem) the transform \mathfrak{F}_{α} can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

Proof. This theorem follows from the relation (3.12), Remark 3.3, and the fact that $\tilde{\mathfrak{F}}_{\alpha}$ is an isomorphism from $\mathscr{G}_*(\mathbb{R}^2)$ onto itself, satisfying that for all $f \in \mathscr{G}_*(\mathbb{R}^2)$,

$$\||\widetilde{\mathfrak{F}}_{\alpha}(f)||_{2,\nu} = \|f\|_{2,\nu}.$$
(3.23)

LEMMA 3.6. For $f \in \mathcal{G}_*(\mathbb{R}^2)$,

$$\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \mathfrak{F}_{\alpha}(f)(\mu, \lambda) = \Lambda_{\alpha} \circ {}^t \mathfrak{R}_{\alpha}(f)(\mu, \lambda), \tag{3.24}$$

where ${}^{t}\mathfrak{R}_{\alpha}$ is the dual transform of the Riemann-Liouville operator, and Λ_{α} is a constant multiple of the classical Fourier transform on \mathbb{R}^{2} defined by

$$\Lambda_{\alpha}(f)(\mu,\lambda) = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r,x) \cos(r\mu) \exp(-i\lambda x) dm(r,x), \qquad (3.25)$$

where dm(r,x) is the measure defined on $[0, +\infty[\times \mathbb{R} by]]$

$$dm(r,x) = \frac{1}{\sqrt{2\pi}2^{\alpha}\Gamma(\alpha+1)}dr \otimes dx.$$
(3.26)

This lemma follows from the relation (2.15) and Lemma 2.4.

Using the relation (3.12) and the fact that the mapping *B* is continuous from $\mathscr{G}_*(\mathbb{R}^2)$ into itself, we deduce that the Fourier transform \mathfrak{F}_{α} is continuous from $\mathscr{G}_*(\mathbb{R}^2)$ into itself. On the other hand, Λ_{α} is an isomorphism from $\mathscr{G}_*(\mathbb{R}^2)$ onto itself. Then, Lemma 3.6 implies that the dual transform ${}^t\mathfrak{R}_{\alpha}$ maps continuously $\mathscr{G}_*(\mathbb{R}^2)$ into itself.

PROPOSITION 3.7. (i) ${}^{t}\mathfrak{R}_{\alpha}$ is not injective when applied to $\mathscr{G}_{*}(\mathbb{R}^{2})$. (ii) ${}^{t}\mathfrak{R}_{\alpha}(\mathscr{G}_{*}(\mathbb{R}^{2})) = \mathscr{G}_{*}(\mathbb{R}^{2})$.

Proof. (i) Let $g \in \mathcal{G}_*(\mathbb{R}^2)$ such that supp $g \subset \{(r,x) \in \mathbb{R}^2, |r| \leq |x|\}, g \neq 0$.

Since $\widetilde{\mathfrak{F}}_{\alpha}$ is an isomorphism from $\mathscr{G}_*(\mathbb{R}^2)$ onto itself, there exists $f \in \mathscr{G}_*(\mathbb{R}^2)$ such that $\widetilde{\mathfrak{F}}_{\alpha}(f) = g$. From the relation (3.12) and Lemma 3.6, we deduce that ${}^t\mathfrak{R}_{\alpha}(f) = 0$.

(ii) We obtain the result by the same way as in [1].

3.2. Paley-Wiener theorem. We denote by

(i) $\mathfrak{D}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on \mathbb{R}^2 , even with respect to the first variable, and with compact support;

(ii) $\mathbb{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \to \mathbb{C}$, even with respect to the first variable rapidly decreasing of exponential type, that is, there exists a positive constant M, such that for all $k \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2} \left(1+|\mu|^2+|\lambda|^2\right)^k \left|f(\mu,\lambda)\right| \exp\left(-M\left(|\operatorname{Im}\mu|+|\operatorname{Im}\lambda|\right)\right) < +\infty;$$
(3.27)

(iii) $\mathbb{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathbb{H}_*(\mathbb{C}^2)$, consisting of functions $f : \mathbb{C}^2 \to \mathbb{C}$, such that for all $k \in \mathbb{N}$,

$$\sup_{\substack{(\mu,\lambda)\in\mathbb{R}^2\\|\mu|\leq|\lambda|}} (1-\mu^2+2\lambda^2)^k \left| f(i\mu,\lambda) \right| <+\infty;$$
(3.28)

(iv) $\mathscr{C}'_*(\mathbb{R}^2)$ the space of distributions on \mathbb{R}^2 , even with respect to the first variable, and with compact support;

(v) $\mathcal{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \to \mathbb{C}$, even with respect to the first variable, slowly increasing of exponential type, that is, there exist a positive constant M and an integer k, such that

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2} \left(1+|\mu|^2+|\lambda|^2\right)^{-k} \left| f(\mu,\lambda) \right| \exp\left(-M\left(|\operatorname{Im}\mu|+|\operatorname{Im}\lambda|\right)\right) < +\infty;$$
(3.29)

(vi) $\mathcal{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathcal{H}_*(\mathbb{C}^2)$, consisting of functions $f : \mathbb{C}^2 \to \mathbb{C}$, such that there exists an integer k, satisfying

$$\sup_{\substack{(\mu,\lambda)\in\mathbb{R}^2\\|\mu|\leq|\lambda|}} \left(1-\mu^2+2\lambda^2\right)^{-k} \left|f(i\mu,\lambda)\right| < +\infty.$$
(3.30)

Each of these spaces is equipped with its usual topology.

Definition 3.8. The Fourier transform associated with the Riemann-Liouville operator is defined on $\mathscr{C}'_*(\mathbb{R}^2)$ by

$$\forall (\mu, \lambda) \in \mathbb{C}^2, \quad \mathfrak{F}_{\alpha}(T)(\mu, \lambda) = \langle T, \varphi_{\mu, \lambda} \rangle. \tag{3.31}$$

PROPOSITION 3.9. For every $T \in \mathscr{C}'_*(\mathbb{R}^2)$,

$$\forall (\mu, \lambda) \in \mathbb{C}^2, \quad \mathfrak{F}_{\alpha}(T)(\mu, \lambda) = B \circ \widetilde{\mathfrak{F}}_{\alpha}(T)(\mu, \lambda), \tag{3.32}$$

where

$$\forall (\mu, \lambda) \in \mathbb{C}^2, \quad \widetilde{\mathfrak{F}}_{\alpha}(T)(\mu, \lambda) = \langle T, j_{\alpha}(\mu) \exp(-i\lambda) \rangle, \tag{3.33}$$

and *B* is the transform defined by the relation (3.12).

Using [7, Lemma 2] (see also [15]) and the fact that $\widetilde{\mathfrak{F}}_{\alpha}$ is an isomorphism from $\mathfrak{D}_*(\mathbb{R}^2)$ (resp., $\mathscr{C}'_*(\mathbb{R}^2)$) onto $\mathbb{H}_*(\mathbb{C}^2)$ (resp., $\mathscr{H}_*(\mathbb{C}^2)$), we deduce the following theorem.

THEOREM 3.10 (of Paley-Wiener). The Fourier transform \mathfrak{F}_{α} is an isomorphism (i) from $\mathfrak{D}_{*}(\mathbb{R}^{2})$ onto $\mathbb{H}_{*,0}(\mathbb{C}^{2})$;

(ii) from $\mathscr{E}'_*(\mathbb{R}^2)$ onto $\mathscr{H}_{*,0}(\mathbb{C}^2)$.

From Lemma 3.6, Theorem 3.10, and the fact that Λ_{α} is an isomorphism from $\mathfrak{D}_*(\mathbb{R}^2)$ onto $\mathbb{H}_*(\mathbb{C}^2)$, we have the following corollary.

COROLLARY 3.11. (i) ${}^{t}\mathfrak{R}_{\alpha}$ maps injectively $\mathfrak{D}_{*}(\mathbb{R}^{2})$ into itself. (ii) ${}^{t}\mathfrak{R}_{\alpha}(\mathfrak{D}_{*}(\mathbb{R}^{2})) \neq \mathfrak{D}_{*}(\mathbb{R}^{2}).$

4. Inversion formulas for \mathfrak{R}_{α} and ${}^{t}\mathfrak{R}_{\alpha}$ and Plancherel theorem for ${}^{t}\mathfrak{R}_{\alpha}$

In this section, we will define some subspaces of $\mathscr{G}_*(\mathbb{R}^2)$ on which \mathfrak{R}_{α} and ${}^t\mathfrak{R}_{\alpha}$ are isomorphisms, and we give their inverse transforms in terms of integro-differential operators. Next, we establish Plancherel theorem for ${}^t\mathfrak{R}_{\alpha}$.

We denote by

(i) \mathcal{N} the subspace of $\mathcal{G}_*(\mathbb{R}^2)$, consisting of functions f satisfying

$$\forall k \in \mathbb{N}, \ \forall x \in \mathbb{R}, \quad \left(\frac{\partial}{\partial r^2}\right)^k f(0, x) = 0,$$
(4.1)

where

$$\frac{\partial}{\partial r^2} = \frac{1}{r} \frac{\partial}{\partial r}; \tag{4.2}$$

(ii) $\mathcal{G}_{*,0}(\mathbb{R}^2)$ the subspace of $\mathcal{G}_*(\mathbb{R}^2)$, consisting of functions f, such that

$$\forall k \in \mathbb{N}, \ \forall x \in \mathbb{R}, \quad \int_0^{+\infty} f(r, x) r^{2k} dr = 0; \tag{4.3}$$

(iii) $\mathcal{G}^0_*(\mathbb{R}^2)$ the subspace of $\mathcal{G}_*(\mathbb{R}^2)$, consisting of functions f, such that

$$\operatorname{supp}\widetilde{\mathfrak{F}}_{\mathfrak{a}}(f) \subset \{(\mu, \lambda) \in \mathbb{R}^2; \ |\mu| \ge |\lambda|\}.$$

$$(4.4)$$

LEMMA 4.1. (i) The mapping Λ_{α} is an isomorphism from $\mathcal{G}_{*,0}(\mathbb{R}^2)$ onto \mathcal{N} . (ii) The subspace \mathcal{N} can be written as

$$\mathcal{N} = \left\{ f \in \mathcal{G}_*(\mathbb{R}^2); \ \forall k \in \mathbb{N}, \forall x \in \mathbb{R}; \ \left(\frac{\partial}{\partial r}\right)^{2k} f(0, x) = 0 \right\}.$$
(4.5)

Proof. Let $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$.

(i) For $\nu > -1$, we have

$$\left(\frac{\partial}{\partial\mu^2}\right)^k (j_\nu(r\mu)) = \frac{\Gamma(\nu+1)}{2^k \Gamma(\nu+k+1)} (-r^2)^k j_{\nu+k}(r\mu), \tag{4.6}$$

thus, from the expression of Λ_{α} , given in Lemma 3.6, and the fact that $j_{-1/2}(s) = \cos s$, we obtain

$$\left(\frac{\partial}{\partial\mu^2}\right)^k (\Lambda_{\alpha}(f))(0,\lambda) = \frac{\sqrt{\pi}}{2^k \Gamma(k+1/2)} (-1)^k \int_{\mathbb{R}} \int_0^{+\infty} f(r,x) r^{2k} \exp(-i\lambda x) dm(r,x),$$
(4.7)

which gives the result.

(ii) The proof of (ii) is immediate.

THEOREM 4.2. (i) For all real numbers y, the mappings

(i) $f \mapsto (r^2 + x^2)^{\gamma} f$

(ii) $f \mapsto |r|^{\gamma} f$

are isomorphisms from ${\mathcal N}$ onto itself.

(ii) For $f \in \mathcal{N}$, the function g defined by

$$g(r,x) = \begin{cases} f\left(\sqrt{r^2 - x^2}, x\right) & \text{if } |r| \ge |x|, \\ 0 & \text{otherwise,} \end{cases}$$
(4.8)

belongs to $\mathcal{G}_*(\mathbb{R}^2)$.

Proof. (i) Let $f \in \mathcal{N}$, by Leibnitz formula, we have

$$\left(\frac{\partial}{\partial r}\right)^{k_1} \left(\frac{\partial}{\partial x}\right)^{k_2} \left[\left(r^2 + x^2\right)^{\gamma} f\right](r, x)$$

$$= \sum_{j=0}^{k_1} \sum_{i=0}^{k_2} C_{k_1}^j C_{k_2}^i P_j(r) P_i(x) \left(r^2 + x^2\right)^{\gamma - i - j} \frac{\partial^{k_1 + k_2 - i - j}}{\partial r^{k_1 - j} \partial x^{k_2 - i}} f(r, x),$$
(4.9)

where P_i and P_j are real polynomials.

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Let $n \in \mathbb{N}$ such that $\gamma - k_1 - k_2 + n > 0$. By Taylor formula and the fact that $f \in \mathcal{N}$, we have

$$\left(\frac{\partial}{\partial r}\right)^{k_1-j}(f)(r,x) = \frac{r^{2n}}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \left(\frac{\partial}{\partial r}\right)^{k_1-j+2n}(f)(rt,x)dt = -\frac{r^{2n}}{(2n-1)!} \int_1^{+\infty} (1-t)^{2n-1} \left(\frac{\partial}{\partial r}\right)^{k_1-j+2n}(f)(rt,x)dt,$$
(4.10)

$$\left(\frac{\partial}{\partial x}\right)^{k_2 - i} \left(\frac{\partial}{\partial r}\right)^{k_1 - j} f(r, x) = \frac{r^{2n}}{(2n - 1)!} \int_0^1 (1 - t)^{2n - 1} \left(\frac{\partial}{\partial x}\right)^{k_2 - i} \left(\frac{\partial}{\partial r}\right)^{k_1 - j + 2n} f(rt, x) dt$$

$$= -\frac{r^{2n}}{(2n - 1)!} \int_1^{+\infty} (1 - t)^{2n - 1} \left(\frac{\partial}{\partial x}\right)^{k_2 - i} \left(\frac{\partial}{\partial r}\right)^{k_1 - j + 2n} f(rt, x) dt.$$

$$(4.11)$$

The relations (4.9) and (4.11) imply that the function

$$(r,x) \longmapsto (r^2 + x^2)^{\gamma} f(r,x)$$
 (4.12)

belongs to ${\mathcal N}$ and that the mapping

$$f \longmapsto (r^2 + x^2)^{\gamma} f \tag{4.13}$$

is continuous from $\mathcal N$ onto itself. The inverse mapping is given by

$$f \longmapsto (r^2 + x^2)^{-\gamma} f. \tag{4.14}$$

By the same way, we show that the mapping

$$f \longmapsto |r|^{\gamma} f \tag{4.15}$$

is an isomorphism from $\mathcal N$ onto itself.

(ii) Let $f \in \mathcal{N}$, and

$$g(r,x) = \begin{cases} f(\sqrt{r^2 - x^2}, x) & \text{if } |r| \ge |x|, \\ 0 & \text{if } |r| \le |x|, \end{cases}$$
(4.16)

we have

$$\left(\frac{\partial}{\partial x}\right)^{k_2} \left(\frac{\partial}{\partial r}\right)^{k_1}(g)(r,x) = \sum_{j=0}^{k_1} P_j(r) \left(\sum_{p,q=0}^{k_2} Q_{p,q}(x) \left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial r^2}\right)^{q+j}(f) \left(\sqrt{r^2 - x^2}, x\right)\right),\tag{4.17}$$

where P_j and $Q_{p,q}$ are real polynomials. This equality, together with the fact that f belongs to \mathcal{N} , implies that g belongs to $\mathcal{G}_*(\mathbb{R}^2)$.

THEOREM 4.3. The Fourier transform \mathfrak{F}_{α} associated with Riemann-Liouville transform is an isomorphism from $\mathscr{F}^0_*(\mathbb{R}^2)$ onto \mathscr{N} .

Proof. Let $f \in \mathcal{G}^0_*(\mathbb{R}^2)$. From the relation (3.12), we get

$$\left(\frac{\partial}{\partial\mu^2}\right)^k \mathfrak{F}_{\alpha}(f)(0,\lambda) = \left(\frac{\partial}{\partial\mu^2}\right)^k \left(B \circ \widetilde{\mathfrak{F}}_{\alpha}(f)\right)(0,\lambda)$$
$$= B\left(\left(\frac{\partial}{\partial\mu^2}\right)^k \widetilde{\mathfrak{F}}_{\alpha}(f)\right)(0,\lambda)$$
$$= \left(\frac{\partial}{\partial\mu^2}\right)^k \widetilde{\mathfrak{F}}_{\alpha}(f)(\lambda,\lambda) = 0,$$
(4.18)

because supp $\widetilde{\mathfrak{F}}_{\alpha}(f) \subset \{ (\mu, \lambda) \in \mathbb{R}^2, |\mu| \ge |\lambda| \}$, this shows that \mathfrak{F}_{α} maps injectively $\mathscr{G}^0_*(\mathbb{R}^2)$ into \mathscr{N} . On the other hand, let $h \in \mathscr{N}$ and

$$g(r,x) = \begin{cases} h(\sqrt{r^2 - x^2}, x) & \text{if } |r| \ge |x|, \\ 0 & \text{if } |r| \le |x|. \end{cases}$$
(4.19)

From Theorem 4.2(ii), g belongs to $\mathscr{G}_*(\mathbb{R}^2)$, so there exists $f \in \mathscr{G}_*(\mathbb{R}^2)$ satisfying $\widetilde{\mathfrak{F}}_{\alpha}(f) = g$. Consequently, $f \in \mathscr{G}^0_*(\mathbb{R}^2)$ and $\mathfrak{F}_{\alpha}(f) = h$.

From Lemmas 3.6, 4.1, and Theorem 4.3, we deduce the following result.

COROLLARY 4.4. The dual transform ${}^{t}\mathfrak{R}_{\alpha}$ is an isomorphism from $\mathscr{G}^{0}_{*}(\mathbb{R}^{2})$ onto $\mathscr{G}_{*,0}(\mathbb{R}^{2})$.

4.1. Inversion formula for \Re_{α} **and** ${}^{t}\Re_{\alpha}$

THEOREM 4.5. (i) The operator K^1_{α} defined by

$$K^{1}_{\alpha}(f)(r,x) = \Lambda^{-1}_{\alpha} \left(\frac{\pi}{2^{2\alpha+1} \Gamma^{2}(\alpha+1)} \left(\mu^{2} + \lambda^{2} \right)^{\alpha} |\mu| \Lambda_{\alpha}(f) \right)(r,x)$$

$$(4.20)$$

is an isomorphism from $\mathcal{G}_{*,0}(\mathbb{R}^2)$ onto itself.

(ii) The operator K^2_{α} defined by

$$K_{\alpha}^{2}(g)(r,x) = \mathfrak{F}_{\alpha}^{-1} \left(\frac{\pi}{2^{2\alpha+1}\Gamma^{2}(\alpha+1)} \left(\mu^{2} + \lambda^{2} \right)^{\alpha} |\mu| \mathfrak{F}_{\alpha}(g) \right)(r,x)$$
(4.21)

is an isomorphism from $\mathcal{G}^0_*(\mathbb{R}^2)$ onto itself.

This theorem follows from Lemma 4.1, Theorems 4.2 and 4.3.

THEOREM 4.6. (i) For $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$ and $g \in \mathcal{G}^0_*(\mathbb{R}^2)$, there exists the inversion formula for \mathfrak{R}_{α} :

$$g = \mathfrak{R}_{\alpha} K_{\alpha}^{1 t} \mathfrak{R}_{\alpha}(g), \qquad f = K_{\alpha}^{1 t} \mathfrak{R}_{\alpha} \mathfrak{R}_{\alpha}(f).$$
(4.22)

(ii) For $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$ and $g \in \mathcal{G}^0_*(\mathbb{R}^2)$, there exists the inversion formula for ${}^t\mathfrak{R}_{\alpha}$:

$$f = {}^{t}\mathfrak{R}_{\alpha}K_{\alpha}^{2}\mathfrak{R}_{\alpha}(f), \qquad g = K_{\alpha}^{2}\mathfrak{R}_{\alpha}{}^{t}\mathfrak{R}_{\alpha}(g).$$
(4.23)

Proof. (i) Let $g \in \mathcal{G}^0_*(\mathbb{R}^2)$. From the relation (2.15), Proposition 3.4, Lemma 3.6, and Theorem 4.3, we have

$$g(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} (\mu^{2} + \lambda^{2})^{\alpha} \mu \Lambda_{\alpha} \circ^{t} \mathfrak{R}_{\alpha}(g)(\mu,\lambda) \mathfrak{R}_{\alpha}(\cos(\mu)\exp(i\lambda.))(r,x) dm(\mu,\lambda)$$

$$= \mathfrak{R}_{\alpha} \bigg(\int_{\mathbb{R}} \int_{0}^{+\infty} (\mu^{2} + \lambda^{2})^{\alpha} \mu \Lambda_{\alpha} \circ^{t} \mathfrak{R}_{\alpha}(g)(\mu,\lambda) \cos(\mu.)\exp(i\lambda.) dm(\mu,\lambda))(r,x)$$

$$= \mathfrak{R}_{\alpha} \bigg(\Lambda_{\alpha}^{-1} \bigg(\frac{\pi}{2^{2\alpha+1}\Gamma^{2}(\alpha+1)} (\mu^{2} + \lambda^{2})^{\alpha} |\mu| \Lambda_{\alpha} \circ^{t} \mathfrak{R}_{\alpha}(g) \bigg) \bigg)(r,x)$$

$$= \mathfrak{R}_{\alpha} K_{\alpha}^{1} {}^{t} \mathfrak{R}_{\alpha}(g)(r,x).$$

$$(4.24)$$

This relation, together with Corollary 4.4 and Theorem 4.5(i), implies that \mathfrak{R}_{α} is an isomorphism from $\mathscr{G}_{*,0}(\mathbb{R}^2)$ onto $\mathscr{G}^0_*(\mathbb{R}^2)$, and that $K^{It}_{\alpha}\mathfrak{R}_{\alpha}$ is its inverse; in particular for $f \in \mathscr{G}_{*,0}(\mathbb{R}^2)$, we have

$$K^{1t}_{\alpha}\mathfrak{R}_{\alpha}\mathfrak{R}_{\alpha}(f) = f. \tag{4.25}$$

(ii) Let $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$. From (i), we have

$$K^{1t}_{\alpha}\mathfrak{R}_{\alpha}\mathfrak{R}_{\alpha}(f) = f. \tag{4.26}$$

Let us put $g = \mathfrak{R}_{\alpha}(f)$, then $g \in \mathcal{G}^{0}_{*}(\mathbb{R}^{2})$, and we have

$$\mathfrak{R}_{\alpha}^{-1}(g) = K_{\alpha}^{1t} \mathfrak{R}_{\alpha}(g), \qquad (4.27)$$

and from Lemma 3.6, it follows that

$$\mathfrak{R}_{\alpha}^{-1}(g) = \Lambda_{\alpha}^{-1} \left(\frac{\pi}{2^{2\alpha+1}\Gamma^{2}(\alpha+1)} \left(\mu^{2} + \lambda^{2} \right)^{\alpha} |\mu| \mathfrak{F}_{\alpha}(g) \right),$$

$${}^{t}\mathfrak{R}_{\alpha}^{-1}\mathfrak{R}_{\alpha}^{-1}(g) = \mathfrak{F}_{\alpha}^{-1} \left(\frac{\pi}{2^{2\alpha+1}\Gamma^{2}(\alpha+1)} \left(\mu^{2} + \lambda^{2} \right)^{\alpha} |\mu| \mathfrak{F}_{\alpha}(g) \right) = K_{\alpha}^{2}(g),$$

$$(4.28)$$

which gives

$$f = {}^{t}\mathfrak{R}_{\alpha}K_{\alpha}^{2}\mathfrak{R}_{\alpha}(f). \tag{4.29}$$

4.2. The expressions of the operators K_{α}^1 and K_{α}^2 . In the previous subsection, we have defined the operators K_{α}^1 and K_{α}^2 in terms of Fourier transforms Λ_{α} and \mathfrak{F}_{α} . Here, we will give nice expressions of these operators using fractional powers of partial differential operators. For this, we need the following inevitable notations.

(i) $\mathscr{C}_*(\mathbb{R})$ is the space of even infinitely differentiable functions on \mathbb{R} .

(ii) $\mathscr{G}_*(\mathbb{R})$ is the subspace of $\mathscr{C}_*(\mathbb{R})$, consisting of functions rapidly decreasing together with all their derivatives.

(iii) $\mathcal{G}'_*(\mathbb{R})$ is the space of even tempered distributions on \mathbb{R} .

(iv) $\mathcal{G}'_*(\mathbb{R}^2)$ is the space of tempered distributions on \mathbb{R}^2 , even with respect to the first variable.

Each of these spaces is equipped with its usual topology.

(i) For $a \in \mathbb{R}$, $a \ge -1/2$, $d\omega_a(r)$ is the measure defined on $[0, +\infty)$ by

$$d\omega_a(r) = \frac{1}{2^a \Gamma(a+1)} r^{2a+1} dr.$$
 (4.30)

(ii) ℓ_a is the Bessel operator defined on $]0, +\infty[$ by

$$\ell_a = \frac{d^2}{dr^2} + \frac{2a+1}{r}\frac{d}{dr}, \quad a \ge -\frac{1}{2}.$$
(4.31)

(iii) For an even measurable function f on \mathbb{R} , $T_f^{\omega_a}$ is the element of $\mathscr{G}'_*(\mathbb{R})$, defined by

$$\langle T_f^{\omega_a}, \varphi \rangle = \int_0^{+\infty} f(r)\varphi(r)d\omega_a(r), \quad \varphi \in \mathcal{G}_*(\mathbb{R}).$$
(4.32)

(iv) For a measurable function g on \mathbb{R}^2 , even with respect to the first variable, T_g^{ν} (resp., T_g^m) is the element of $\mathcal{G}'_*(\mathbb{R}^2)$, defined by

$$\langle T_{g}^{\nu}, \varphi \rangle = \int_{\mathbb{R}} \int_{0}^{+\infty} g(r, x) \varphi(r, x) d\nu(r, x),$$

$$(\text{resp.}, \langle T_{g}^{m}, \varphi \rangle = \int_{\mathbb{R}} \int_{0}^{+\infty} g(r, x) \varphi(r, x) dm(r, x) \Big), \quad \varphi \in \mathcal{G}_{*}(\mathbb{R}^{2}),$$

$$(4.33)$$

where dv and dm are the measures defined by the relations (3.1) and (3.26).

Definition 4.7. (i) The translation operator τ_r^a , $r \in \mathbb{R}$, associated with Bessel operator ℓ_a is defined on $\mathcal{G}_*(\mathbb{R})$ by the following. For all $s \in \mathbb{R}$,

$$\tau_r^a f(s) = \begin{cases} \frac{\Gamma(a+1)}{\sqrt{\pi}\Gamma(a+1/2)} \int_0^{\pi} f\left(\sqrt{r^2 + s^2 + 2rs\cos\theta}\right) \sin^{2a}\theta \, d\theta & \text{if } a > -\frac{1}{2}, \\ \frac{f(r+s) + f\left(|r-s|\right)}{2} & \text{if } a = -\frac{1}{2}. \end{cases}$$
(4.34)

(ii) The convolution product of $f \in \mathcal{G}_*(\mathbb{R})$ and $T \in \mathcal{G}'_*(\mathbb{R})$ is defined by

$$\forall r \in \mathbb{R}, \quad T *_a f(r) = \langle T, \tau_r^a f \rangle. \tag{4.35}$$

(iii) The Fourier Bessel transform is defined on $\mathcal{G}_*(\mathbb{R})$ by

$$\forall \mu \in \mathbb{R}, \quad F_a(f)(\mu) = \int_0^{+\infty} f(r) j_a(r\mu) d\omega_a(r), \tag{4.36}$$

and on $\mathscr{G}'_*(\mathbb{R})$ by

$$\forall \varphi \in \mathcal{G}_*(\mathbb{R}), \quad \langle F_a(T), \varphi \rangle = \langle T, F_a(\varphi) \rangle.$$
(4.37)

We have the following properties (we refer to [19]). (i) F_a is an isomorphism from $\mathcal{G}_*(\mathbb{R})$ (resp., $\mathcal{G}'_*(\mathbb{R})$) onto itself, and we have

$$F_a^{-1} = F_a. (4.38)$$

(ii) For $f \in \mathcal{G}_*(\mathbb{R})$, and $r \in \mathbb{R}$, $\tau_r^a f$ belongs to $\mathcal{G}_*(\mathbb{R})$, and we have

$$F_a(\tau_r^a f)(\mu) = j_a(r\mu)F_a(f)(\mu).$$
(4.39)

(iii) For $f \in \mathcal{G}_*(\mathbb{R})$ and $T \in \mathcal{G}'_*(\mathbb{R})$, the function $T *_a f$ belongs to $\mathscr{C}_*(\mathbb{R})$, and is slowly increasing, moreover

$$F_a\left(T^{\omega_a}_{T*_af}\right) = F_a(f)F_a(T). \tag{4.40}$$

In the following, we will define the fractional powers of Bessel operator and the Laplacian operator defined on \mathbb{R}^2 by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2} \tag{4.41}$$

that we use to give simple expressions of K^1_{α} and K^2_{α} .

In [16], the author has proved that the mappings

$$z \longmapsto T^{\omega_a}_{|r|^z}, \qquad z \longmapsto T^{\omega_a}_{(2^{z+a+1}\Gamma(z/2+a+1)/\Gamma(-z/2))|r|^{-z-2a-2}}, \tag{4.42}$$

defined initially for $-2(a+1) < \Re e(z) < 0$, can be extended to a valued functions on $\mathcal{G}'_*(\mathbb{R})$, analytic on $\mathbb{C} \setminus \{-2(k+a), k \in \mathbb{N}^*\}$, and we have

$$T^{\omega_a}_{|r|^z} = F_a \Big(T^{\omega_a}_{(2^{z+a+1}\Gamma(z/2+a+1)/\Gamma(-z/2))|r|^{-z-2a-2}} \Big).$$
(4.43)

Definition 4.8. For $z \in \mathbb{C} \setminus \{-(k+a), k \in \mathbb{N}^*\}$, the fractional power of Bessel operator ℓ_a is defined on $\mathcal{G}_*(\mathbb{R})$ by

$$\left(-\ell_{a}\right)^{z}f(r) = \left(T_{(2^{2z+a+1}\Gamma(z+a+1)/\Gamma(-z))|s|^{-2z-2a-2}}^{\omega_{a}}\right) *_{a}f(r).$$
(4.44)

From the relations (4.40) and (4.43), we deduce that for $f \in \mathcal{G}_*(\mathbb{R})$ and $z \in \mathbb{C} \setminus \{-(k + a), k \in \mathbb{N}^*\}$, we have

$$F_a(T^{\omega_a}_{(-\ell_a)^z f}) = F_a(f) T^{\omega_a}_{|r|^{2z}}.$$
(4.45)

On the other hand, from [8, 10], we deduce that the mappings

$$z \mapsto T^m_{(r^2+x^2)^z}, \qquad T^m_{\sqrt{2/\pi}(2^{2z+\alpha+1}\Gamma(z+1)\Gamma(\alpha+1)/\Gamma(-z))(r^2+x^2)^{-z-1}},$$
 (4.46)

defined initially for $-1 < \Re e(z) < 0$, can be extended to a valued functions in $\mathscr{G}'_*(\mathbb{R}^2)$, analytic on $\mathbb{C} \setminus \{-k, k \in \mathbb{N}^*\}$, and we have

$$T^{m}_{(r^{2}+x^{2})^{z}} = \Lambda_{\alpha} \Big(T^{m}_{\sqrt{2/\pi}(2^{2z+\alpha+1}\Gamma(z+1)\Gamma(\alpha+1)/\Gamma(-z))(r^{2}+x^{2})^{-z-1}} \Big),$$
(4.47)

where Λ_{α} is defined on $\mathscr{G}'_{*}(\mathbb{R}^{2})$ by

$$\langle \Lambda_{\alpha}(T), \varphi \rangle = \langle T, \Lambda_{\alpha}(\varphi) \rangle, \quad \varphi \in \mathcal{G}_{*}(\mathbb{R}^{2}),$$

$$(4.48)$$

and $\Lambda_{\alpha}(\varphi)$ is given in Lemma 3.6.

Definition 4.9. For $z \in \mathbb{C} \setminus \{-k, k \in \mathbb{N}^*\}$, the fractional power of the Laplacian operator Δ is defined on $\mathcal{G}_*(\mathbb{R}^2)$ by

$$(-\Delta)^{z} f(\mathbf{r}, \mathbf{x}) = \left(T^{m}_{(1/\pi)(2^{2z+1}\Gamma(z+1)/\Gamma(-z))(s^{2}+y^{2})^{-z-1}*f} \right)(\mathbf{r}, \mathbf{x}),$$
(4.49)

where

(i) * is the usual convolution product defined by

$$T * f(r,x) = \langle T, \sigma_{(r,x)}\check{f} \rangle, \quad T \in \mathcal{G}'_*(\mathbb{R}^2), \ f \in \mathcal{G}_*(\mathbb{R}^2);$$
(4.50)

(ii)

$$\sigma_{(r,x)}f(s,y) = \frac{1}{2}[f(r+s,y-x) + f(r-s,y-x)], \quad f \in \mathcal{G}_*(\mathbb{R}^2).$$
(4.51)

It is well known that for $f \in \mathcal{G}_*(\mathbb{R}^2)$ and $T \in \mathcal{G}'_*(\mathbb{R}^2)$, the function T * f belongs to $\mathscr{C}_*(\mathbb{R}^2)$ and is slowly increasing, and we have

$$\Lambda_{\alpha}(T^{m}_{T*f}) = \Lambda_{\alpha}(f)\Lambda_{\alpha}(T), \qquad (4.52)$$

thus from the relations (4.47) and (4.52), we deduce that for $f \in \mathcal{G}_*(\mathbb{R}^2)$ and $z \in \mathbb{C} \setminus \{-k, k \in \mathbb{N}^*\},\$

$$\Lambda_{\alpha}\left(T^{m}_{\sqrt{2\pi}2^{\alpha}\Gamma(\alpha+1)(-\Delta)^{z}f}\right) = \Lambda_{\alpha}(f)T^{m}_{(r^{2}+x^{2})^{z}}.$$
(4.53)

THEOREM 4.10. The operator K^1_{α} defined in Theorem 4.5 can be written as

$$K^{1}_{\alpha}(f) = \frac{\pi}{2^{2\alpha+1}\Gamma^{2}(\alpha+1)} \left(-\frac{\partial^{2}}{\partial r^{2}}\right)^{1/2} (-\Delta)^{\alpha} f, \qquad (4.54)$$

where

$$\left(-\frac{\partial^2}{\partial r^2}\right)^{1/2} f(r,x) = \left(-\ell_{-1/2}\right)^{1/2} (f(\cdot,x))(r).$$
(4.55)

Proof. Let $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$. Using Fubini's theorem, we get for every $\varphi \in \mathcal{G}_*(\mathbb{R}^2)$ the following:

$$\left\langle \Lambda_{\alpha} \left(T^{m}_{(-\partial^{2}/\partial r^{2})^{1/2} f} \right), \varphi \right\rangle$$

$$= \frac{1}{2^{2\alpha+2} \Gamma^{2}(\alpha+1)} \int_{\mathbb{R}} \int_{0}^{+\infty} \left\langle T^{\omega_{-1/2}}_{(-\ell_{-1/2})^{1/2}(f(\cdot,x))}, F_{-1/2}(\varphi(\cdot,y)) \right\rangle \times \exp(-ixy) dx dy$$

$$(4.56)$$

and by the relation (4.45), we obtain

$$\left\langle \Lambda_{\alpha} \left(T^{m}_{(-\partial^{2}/\partial r^{2})^{1/2} f} \right), \varphi \right\rangle$$

= $\frac{1}{2^{2\alpha+2}\Gamma^{2}(\alpha+1)} \int_{\mathbb{R}} \int_{0}^{+\infty} \left\langle F_{-1/2}(f(\cdot,x)) T^{\omega_{-1/2}}_{|r|}, \varphi(\cdot,y) \right\rangle \times \exp(-ixy) dx dy,$
(4.57)

which involves that

$$\left\langle \Lambda_{\alpha} \Big(T^m_{(-\partial^2/\partial r^2)^{1/2} f} \Big), \varphi \right\rangle = \int_{\mathbb{R}} \int_0^{+\infty} r \Lambda_{\alpha}(f)(r, y) \varphi(r, y) dm(r, y),$$
(4.58)

this shows that

$$\Lambda_{\alpha} \Big(T^m_{(-\partial^2/\partial r^2)^{1/2} f} \Big) = T^m_{|r|\Lambda_{\alpha}(f)}.$$
(4.59)

Now, from Lemma 4.1, we deduce that the function

$$(\mu,\lambda) \longmapsto |\mu| \Lambda_{\alpha}(f)(\mu,\lambda) \tag{4.60}$$

belongs to the subspace \mathcal{N} . Then, from the relation (4.59), it follows that the function $(-\partial^2/\partial r^2)^{1/2}f$ belongs to the subspace $\mathcal{G}_{*,0}(\mathbb{R}^2)$, and we have

$$\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \Lambda_{\alpha} \left(\left(-\frac{\partial^2}{\partial r^2} \right)^{1/2} f \right) (\mu, \lambda) = |\mu| \Lambda_{\alpha}(\mu, \lambda). \tag{4.61}$$

By the same way, and using the relation (4.53), we deduce that for every $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$, the function $(-\Delta)^{\alpha} f$ belongs to the subspace $\mathcal{G}_{*,0}(\mathbb{R}^2)$, and we have that for all $(\mu, \lambda) \in \mathbb{R}^2$,

$$\Lambda_{\alpha}(\sqrt{2\pi}2^{\alpha}\Gamma(\alpha+1)(-\Delta)^{\alpha}f)(\mu,\lambda) = (\mu^{2}+\lambda^{2})^{\alpha}\Lambda_{\alpha}(f)(\mu,\lambda).$$
(4.62)

Hence, the theorem follows from the relations (4.61) and (4.62).

Definition 4.11. Let $a, b \in \mathbb{R}$, $b \ge a \ge -1/2$.

(i) The Sonine transform is the mapping defined on $\mathscr{C}_*(\mathbb{R})$ by the following. For all $r \in \mathbb{R}$,

$$S_{b,a}(f)(r) = \begin{cases} \frac{2\Gamma(b+1)}{\Gamma(b-a)\Gamma(a+1)} \int_0^1 (1-t^2)^{b-a-1} f(rt) t^{2a+1} dt & \text{if } b > a, \\ f(r) & \text{if } b = a. \end{cases}$$
(4.63)

(ii) The dual transform ${}^{t}S_{b,a}$ is the mapping defined on $\mathcal{G}_{*}(\mathbb{R})$ by the following. For all $r \in \mathbb{R}$,

$${}^{t}S_{b,a}(f)(r) = \begin{cases} \frac{2\Gamma(b+1)}{\Gamma(b-a)\Gamma(a+1)} \int_{r}^{+\infty} (t^{2}-r^{2})^{b-a-1} f(t)t \, dt & \text{if } b > a, \\ f(r) & \text{if } b = a. \end{cases}$$
(4.64)

Then, we have the following.

- (i) The Sonine transform is an isomorphism from $\mathscr{E}_*(\mathbb{R})$ onto itself.
- (ii) The dual Sonine transform is an isomorphism from $\mathcal{G}_*(\mathbb{R})$ onto itself.

(iii) For $f \in \mathscr{E}_*(\mathbb{R})$, f bounded, and $g \in \mathscr{G}_*(\mathbb{R})$, we have

$$\int_{0}^{+\infty} S_{b,a}(f)(r)g(r)r^{2b+1}dr = \int_{0}^{+\infty} f(r)^{t}S_{b,a}(g)(r)r^{2a+1}dr.$$
(4.65)

(iv) $j_b = S_{b,a}(j_a)$. (v)

$$F_{b} = \frac{\Gamma(a+1)}{2^{b-a}\Gamma(b+1)} F_{a} \circ {}^{t}S_{b,a}.$$
(4.66)

For more details, we refer to [18, 20, 21].

We denote the following.

(i) For $T \in \mathcal{G}'_*(\mathbb{R}^2)$, $\varphi \in \mathcal{G}_*(\mathbb{R}^2)$,

$$\langle S_{a,0}(T), \varphi \rangle = \langle T, \psi \rangle,$$
(4.67)

with $\psi(r,x) = {}^{t}S_{a,0}(\varphi(\cdot,x))(r)$. (ii) For all $(r,x) \in \mathbb{R}^{2}$,

$$T # \varphi(r, x) = \langle T, \mathcal{T}_{(r, -x)} \check{\varphi} \rangle, \qquad (4.68)$$

where $\mathcal{T}_{(r,x)}$ is the translation operator given by Definition 3.1.

(iii) \mathfrak{F}_{α} is the mapping defined on $\mathscr{G}'_*(\mathbb{R}^2)$ by

$$\forall \varphi \in \mathscr{G}_*(\mathbb{R}^2), \quad \big\langle \widetilde{\mathfrak{F}}_{\alpha}(T), \varphi \big\rangle = \big\langle T, \widetilde{\mathfrak{F}}_{\alpha}(\varphi) \big\rangle. \tag{4.69}$$

(iv) L_{α} is the operator defined on $\mathcal{G}_*(\mathbb{R}^2)$ by

$$L_{\alpha}f(r,x) = \left(-\ell_{\alpha}\right)^{2\alpha} \left(f(\cdot,x)\right)(r), \qquad (4.70)$$

where $(-\ell_{\alpha})^{z}$ is the fractional power of Bessel given by Definition 4.8.

THEOREM 4.12. The operator K^2_{α} , defined in Theorem 4.5, is given by

$$K_{\alpha}^{2}(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^{4}(\alpha+1)} S_{\alpha,0}(T) \# (-\Delta_{2}) L_{\alpha}(\check{f})(r,-x), \quad f \in \mathcal{G}_{*}^{0}(\mathbb{R}^{2}),$$
(4.71)

where

(i) *T* is the distribution defined by

$$\langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi(y, y) dy;$$
 (4.72)

(ii) Δ_2 is the operator defined in Section 2.

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Proof. By the definition of K^2_{α} , and the relation (3.12), we have that for $f \in \mathcal{G}^0_*(\mathbb{R}^2)$,

$$K_{\alpha}^{2}(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^{3}(\alpha+1)} \int_{\mathbb{R}} \int_{0}^{+\infty} \mu^{2} (\mu^{2}+\lambda^{2})^{2\alpha} \widetilde{\mathfrak{F}}_{\alpha}(f) (\sqrt{\mu^{2}+\lambda^{2}},\lambda) j_{\alpha}(r\sqrt{\mu^{2}+\lambda^{2}}) \exp(i\lambda x) d\mu d\lambda.$$

$$(4.73)$$

By a change of variables, and using Fubini's theorem, we get

$$K_{\alpha}^{2}(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^{3}(\alpha+1)} \int_{0}^{+\infty} \int_{-\nu}^{\nu} \nu^{4\alpha} (\nu^{2} - \lambda^{2}) \widetilde{\mathfrak{F}}_{\alpha}(f)(\nu,\lambda) \frac{\exp(i\lambda x)}{\sqrt{\nu^{2} - \lambda^{2}}} j_{\alpha}(r\nu)\nu d\nu d\lambda.$$

$$(4.74)$$

On the other hand, for $f \in \mathcal{G}^0_*(\mathbb{R}^2)$, the function $L_{\alpha}f$ belongs to $\mathscr{C}_*(\mathbb{R}^2)$, and is slowly increasing. Moreover, we have

$$\widetilde{\mathfrak{F}}_{\alpha}\left(T_{L_{\alpha}f}^{\nu}\right) = T_{|\mu|^{4\alpha}\widetilde{\mathfrak{F}}_{\alpha}(f)}^{\nu}.$$
(4.75)

But, for $f \in \mathcal{G}^0_*(\mathbb{R}^2)$, the function $\widetilde{\mathfrak{F}}_{\alpha}(f)$ belongs to the subspace \mathcal{N} ; according to Theorem 4.2, we deduce that the function $L_{\alpha}f$ belongs to $\mathcal{G}_*(\mathbb{R}^2)$, and we have

$$\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \widetilde{\mathfrak{F}}_{\alpha}(L_{\alpha}f)(\mu, \lambda) = |\mu|^{4\alpha} \widetilde{\mathfrak{F}}_{\alpha}(f)(\mu, \lambda).$$
(4.76)

This involves that

$$K_{\alpha}^{2}(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^{3}(\alpha+1)} \int_{0}^{+\infty} \int_{-\nu}^{\nu} (\nu^{2} - \lambda^{2}) \widetilde{\mathfrak{F}}_{\alpha}(L_{\alpha}f)(\nu,\lambda) \frac{\exp(i\lambda x)}{\sqrt{\nu^{2} - \lambda^{2}}} j_{\alpha}(r\nu)\nu d\nu d\lambda$$
$$= \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^{3}(\alpha+1)} \int_{0}^{+\infty} \int_{-\nu}^{\nu} \widetilde{\mathfrak{F}}_{\alpha}((-\Delta_{2})L_{\alpha}f)(\nu,\lambda) \frac{\exp(i\lambda x)}{\sqrt{\nu^{2} - \lambda^{2}}} j_{\alpha}(r\nu)\nu d\nu d\lambda.$$
(4.77)

Since for every $f \in \mathcal{G}_*(\mathbb{R}^2)$, we have that

$$\forall (r,x), (\mu,\lambda) \in \mathbb{R}^2, \quad \widetilde{\mathfrak{F}}_{\alpha}(\mathcal{T}_{(r,x)}f)(\nu,\lambda) = j_{\alpha}(r\nu)\exp(i\lambda x)\widetilde{\mathfrak{F}}_{\alpha}(f)(\nu,\lambda), \tag{4.78}$$

we get

$$K_{\alpha}^{2}(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^{3}(\alpha+1)} \int_{0}^{+\infty} \int_{-\nu}^{\nu} \widetilde{\mathfrak{F}}_{\alpha}(\mathcal{T}_{(r,x)}(-\Delta_{2})L_{\alpha}f)(\nu,\lambda) \frac{\nu d\nu d\lambda}{\sqrt{\nu^{2}-\lambda^{2}}}.$$
 (4.79)

Using the expression of $\widetilde{\mathfrak{F}}_{\alpha}$, we obtain

$$K_{\alpha}^{2}(f)(r,x) = \frac{1}{2^{4\alpha+2}\Gamma^{4}(\alpha+1)} \int_{0}^{+\infty} \int_{-\nu}^{\nu} \left[\int_{\mathbb{R}} \int_{0}^{+\infty} \left(\mathcal{T}_{(r,x)}(-\Delta_{2})L_{\alpha}f \right)(s,y) \times j_{\alpha}(s\nu) \exp(-i\lambda y) s^{2\alpha+1} ds dy \right] \frac{d\lambda}{\sqrt{\nu^{2}-\lambda^{2}}} \nu d\nu.$$
(4.80)

From the fact that

$$\int_{-\nu}^{\nu} \frac{\exp(-i\lambda y)}{\sqrt{\nu^2 - \lambda^2}} d\lambda = \pi j_0(\nu y), \tag{4.81}$$

and using Fubini's theorem, we deduce that

$$K_{\alpha}^{2}(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^{4}(\alpha+1)} \int_{\mathbb{R}} \left\{ \iint_{0}^{+\infty} \left(\mathcal{T}_{(r,x)}(-\Delta_{2})L_{\alpha}f\right)(s,y)j_{\alpha}(sv) \times j_{0}(vy)s^{2\alpha+1}dsvdv \right\} dy \\ = \frac{\pi}{2^{3\alpha+2}\Gamma^{3}(\alpha+1)} \int_{\mathbb{R}} \left\{ \int_{0}^{+\infty} F_{\alpha}(\left(\mathcal{T}_{(r,x)}(-\Delta_{2})L_{\alpha}f\right)(\cdot,y))(v)j_{0}(vy)vdv \right\} dy,$$

$$(4.82)$$

and from the relation (4.66), we have

$$K_{\alpha}^{2}(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^{4}(\alpha+1)} \int_{\mathbb{R}} \left\{ \int_{0}^{+\infty} F_{0} \circ {}^{t}S_{\alpha,0} \left(\left(\mathcal{T}_{(r,x)}(-\Delta_{2})L_{\alpha}f\right)(\cdot,y) \right)(v) \times j_{0}(vy)v dv \right\} dy,$$

$$(4.83)$$

and the relation (4.38) implies that

$$K_{\alpha}^{2}(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^{4}(\alpha+1)} \int_{\mathbb{R}} {}^{t} S_{\alpha,0}((\mathcal{T}_{(r,x)}(-\Delta_{2})L_{\alpha}f)(\cdot,y))(y)dy.$$
(4.84)

4.3. Plancherel theorem for ${}^t\mathfrak{R}_{\alpha}$

PROPOSITION 4.13. The operator K^3_{α} defined by

$$K_{\alpha}^{3}(f) = \pi \left(-\frac{\partial^{2}}{\partial r^{2}}\right)^{1/4} (-\triangle)^{\alpha/2} f$$
(4.85)

is an isomorphism from $\mathcal{G}_{*,0}(\mathbb{R}^2)$ onto itself, where

$$\left(-\frac{\partial^2}{\partial r^2}\right)^{1/4} f(r,x) = \left(-\ell_{-1/2}\right)^{1/4} (f(\cdot,x))(r).$$
(4.86)

Proof. Let $f \in \mathcal{G}_{*,0}(\mathbb{R}^2)$. From the relations (4.45) and (4.53), we deduce that for all $(\mu, \lambda) \in \mathbb{R}^2$,

$$\sqrt{|\mu|} \left(\mu^2 + \lambda^2\right)^{\alpha/2} \Lambda_{\alpha}(f)(\mu, \lambda) = \Lambda_{\alpha} \left(\sqrt{2\pi} 2^{\alpha} \Gamma(\alpha + 1) \left(-\frac{\partial^2}{\partial r^2}\right)^{1/4} (-\Delta)^{\alpha/2} f\right)(\mu, \lambda),$$
(4.87)

 \Box

which implies that for all $(\mu, \lambda) \in \mathbb{R}^2$,

$$\Lambda_{\alpha}(K_{\alpha}^{3})(f)(\mu,\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \sqrt{|\mu|} \left(\mu^{2} + \lambda^{2}\right)^{\alpha/2} \Lambda_{\alpha}(f)(\mu,\lambda).$$
(4.88)

Then, the result follows from Lemma 4.1 and Theorem 4.2. PROPOSITION 4.14. For $g \in \mathcal{G}^0_*(\mathbb{R}^2)$, there exists the Plancherel formula

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \left| g(r,x) \right|^{2} d\nu(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} \left| K_{\alpha}^{3} \binom{t}{\mathfrak{R}_{\alpha}(g)}(r,x) \right|^{2} dm(r,x).$$
(4.89)

Proof. Let $g \in \mathcal{G}^0_*(\mathbb{R}^2)$, from Theorem 3.5 (Plancherel formula), we have

$$\int_{\mathbb{R}} \int_{0}^{+\infty} |g(r,x)|^{2} d\nu(r,x) = \iint_{\Gamma} |\mathfrak{F}_{\alpha}(g)(\mu,\lambda)|^{2} d\gamma(\mu,\lambda).$$
(4.90)

From the relation (3.12), Lemma 3.6, and the fact that

$$\operatorname{supp}\widetilde{\mathfrak{F}}_{\alpha}(g) \subset \{(\mu, \lambda) \in \mathbb{R}^2 / |\mu| \ge |\lambda|\},\tag{4.91}$$

we get

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \left| g(r,x) \right|^{2} d\nu(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} \left| \sqrt{\mu} (\mu^{2} + \lambda^{2})^{\alpha/2} \Lambda_{\alpha} \circ {}^{t} \mathfrak{R}_{\alpha}(g)(\mu,\lambda) \right|^{2} dm(\mu,\lambda).$$
(4.92)

We complete the proof by using the formula (4.88), and the fact that for every $f \in \mathcal{G}_*(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \left| \Lambda_{\alpha}(f)(\mu,\lambda) \right|^{2} dm(\mu,\lambda) = \frac{\pi}{2^{2\alpha+1}\Gamma^{2}(\alpha+1)} \int_{\mathbb{R}} \int_{0}^{+\infty} \left| f(\mu,\lambda) \right|^{2} dm(\mu,\lambda).$$
(4.93)

We denote by

(i) $L_0^2(d\nu)$ the subspace of $L^2(d\nu)$ consisting of functions g such that

$$\operatorname{supp}\widetilde{\mathfrak{F}}_{\alpha}(g) \subset \{(\mu, \lambda) \in \mathbb{R}^2 / |\mu| \ge |\lambda|\};$$

$$(4.94)$$

(ii) $L^2(dm)$ the space of square integrable functions on $[0, +\infty[\times\mathbb{R} \text{ with respect to the measure } dm(r, x)$.

THEOREM 4.15. The operator $K^3_{\alpha} \circ {}^t\mathfrak{R}_{\alpha}$ can be extended to an isometric isomorphism from $L^2_0(d\nu)$ onto $L^2(dm)$.

Proof. The theorem follows from Propositions 4.13, 4.14, and the density of $\mathscr{G}_{*,0}(\mathbb{R}^2)$ (resp., $\mathscr{G}^0_*(\mathbb{R}^2)$) in $L^2(dm)$ (resp., $L^2_0(d\nu)$).

5. Transmutation operators

PROPOSITION 5.1. The Riemann-Liouville transform and its dual satisfy the following permutation properties.

(i) For all $f \in \mathcal{G}_*(\mathbb{R}^2)$,

$${}^{t}\mathfrak{R}_{\alpha}(\Delta_{2}f) = \frac{\partial^{2}}{\partial r^{2}} {}^{t}\mathfrak{R}_{\alpha}(f), \qquad {}^{t}\mathfrak{R}_{\alpha}(\Delta_{1}f) = \Delta_{1} {}^{t}\mathfrak{R}_{\alpha}(f).$$
(5.1)

(ii) For all $f \in \mathscr{C}_*(\mathbb{R}^2)$,

$$\Delta_2 \mathfrak{R}_{\alpha}(f) = \mathfrak{R}_{\alpha} \left(\frac{\partial^2 f}{\partial r^2} \right), \qquad \Delta_1 \mathfrak{R}_{\alpha}(f) = \mathfrak{R}_{\alpha} (\Delta_1 f).$$
(5.2)

Proof. (i) We know that the operators Δ_1 , Δ_2 , $\partial^2/\partial r^2$, and ${}^t\mathfrak{R}_{\alpha}$ are continuous mappings from $\mathscr{G}_*(\mathbb{R}^2)$ into itself. Then, by applying the usual Fourier transform Λ_{α} , we have

$$\Lambda_{\alpha}({}^{t}\mathfrak{R}_{\alpha}(\Delta_{2}f))(\mu,\lambda) = -\mu^{2}\Lambda_{\alpha} \circ {}^{t}\mathfrak{R}_{\alpha}(f)(\mu,\lambda) = \Lambda_{\alpha}\left(\frac{\partial^{2}}{\partial r^{2}}{}^{t}\mathfrak{R}_{\alpha}(f)\right)(\mu,\lambda),$$

$$\Lambda_{\alpha}(\Delta_{1}{}^{t}\mathfrak{R}_{\alpha}f)(\mu,\lambda) = i\lambda\Lambda_{\alpha}({}^{t}\mathfrak{R}_{\alpha}(f))(\mu,\lambda) = \Lambda_{\alpha}({}^{t}\mathfrak{R}_{\alpha}(\Delta_{1}f))(\mu,\lambda).$$
(5.3)

Consequently, (i) follows from the fact that Λ_{α} is an isomorphism from $\mathscr{G}_*(\mathbb{R}^2)$ onto itself.

(ii) We obtain the result from (i), Lemma 2.4, and the fact that for $f \in \mathscr{C}_*(\mathbb{R}^2)$, and $g \in \mathfrak{D}_*(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \Delta_2 f(r,x) g(r,x) d\nu(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r,x) \Delta_2 g(r,x) d\nu(r,x).$$
(5.4)

THEOREM 5.2. (i) The Riemann-Liouville transform \Re_{α} is a transmutation operator of

$$\frac{\partial^2}{\partial r^2}, \Delta_1 \quad into \ \Delta_2, \ \Delta_1$$
 (5.5)

from

$$\mathscr{G}_{*,0}(\mathbb{R}^2)$$
 onto $\mathscr{G}^0_*(\mathbb{R}^2)$. (5.6)

(ii) The dual transform ${}^{t}\mathfrak{R}_{\alpha}$ is a transmutation operator of

$$\Delta_2, \Delta_1 \quad into \ \frac{\partial^2}{\partial r^2}, \Delta_1$$
 (5.7)

from

$$\mathscr{G}^0_*(\mathbb{R}^2) \quad onto \, \mathscr{G}_{*,0}(\mathbb{R}^2).$$
 (5.8)

This theorem follows from Proposition 5.1 and the fact that \mathfrak{R}_{α} is an isomorphism from $\mathscr{G}_{*,0}(\mathbb{R}^2)$ onto $\mathscr{G}_{*}^0(\mathbb{R}^2)$ and ${}^t\mathfrak{R}_{\alpha}$ is an isomorphism from $\mathscr{G}_{*}^0(\mathbb{R}^2)$ onto $\mathscr{G}_{*,0}(\mathbb{R}^2)$.

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C. Baccar: Department of Mathematics, Faculty of Sciences of Tunis, University Tunis El Manar, 2092 Tunis, Tunisia

N. B. Hamadi: Department of Mathematics, Faculty of Sciences of Tunis, University Tunis El Manar, 2092 Tunis, Tunisia

L. T. Rachdi: Department of Mathematics, Faculty of Sciences of Tunis, University Tunis El Manar, 2092 Tunis, Tunisia *E-mail address*: lakhdartannech.rachdi@fst.rnu.tn



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