# ON $n$-FLAT MODULES AND $n$-VON NEUMANN REGULAR RINGS

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We show that each $R$-module is $n$-flat (resp., weakly $n$-flat) if and only if $R$ is an $(n, n-1)$-ring (resp., a weakly $(n, n-1)$-ring). We also give a new characterization of $n$-von Neumann regular rings and a characterization of weak $n$-von Neumann regular rings for (CH)-rings and for local rings. Finally, we show that in a class of principal rings and a class of local Gaussian rings, a weak $n$-von Neumann regular ring is a (CH)-ring.



## 1. Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. For a nonnegative integer $n$, an $R$-module $E$ is $n$-presented if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to E \to 0$, in which each $F_i$ is a finitely generated free $R$-module. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Also, $\mathrm{pd}_R E$ will denote the projective dimension of $E$ as an $R$-module. Costa [2] introduced a doubly filtered set of classes of rings throwing a brighter light on the structures of non-Noetherian rings. Namely, for nonnegative integers $n$ and $d$, we say that a ring $R$ is an $(n,d)$-ring if $\mathrm{pd}_R(E) \leq d$ for each $n$-presented $R$-module $E$ (as usual, pd denotes projective dimension); and that $R$ is a weak $(n,d)$-ring if $\mathrm{pd}_R(E) \leq d$ for each $n$-presented cyclic $R$-module $E$. The Noetherianness deflates the $(n,d)$-property to the notion of regular ring. However, outside Noetherian settings, the richness of this classification resides in its ability to unify classic concepts such as von Neumann regular, hereditary/Dedekind, and semi-hereditary/Prüfer rings. For instance, see [2–5, 8–10].

We say that $R$ is $n$-von Neumann regular ring (resp., weak $n$-von Neumann regular ring) if it is $(n,0)$-ring (resp., weak $(n,0)$-ring). Hence, the 1-von Neumann regular rings and the weak 1-von Neumann regular rings are exactly the von Neumann regular ring (see [10, Theorem 2.1] for a characterization of $n$-von Neumann regular rings).

According to [1], an $R$-module $E$ is said to be $n$-flat if $\mathrm{Tor}_R^n(E,G) = 0$ for each $n$-presented $R$-module $G$. Similarly, an $R$-module $E$ is said to be weakly $n$-flat if $\mathrm{Tor}_R^n(E,G) = 0$ for each $n$-presented cyclic $R$-module $G$. Consequently, the 1-flat, weakly 1-flat, and flat

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properties are the same. Therefore, each $R$-module is 1-flat or weakly 1-flat if and only if $R$ is a von Neumann regular ring.

In Section 2, we show that each $R$-module is $n$-flat (resp., weakly $n$-flat) if and only if $R$ is an $(n,n-1)$-ring (resp., a weakly $(n,n-1)$-ring). Then we give a wide class of non weakly $(n,d)$-rings for each pair of positive integers $n$ and $d$. In Section 3, we give a new characterization of $n$-von Neumann regular rings. Also, for (CH)-rings and local rings, a characterization of weak $n$-von Neumann regular rings is given. Finally, if $R$ is a principal ring or a local Gaussian ring, we show that $R$ is a weak $n$-von Neumann regular ring which implies that $R$ is a (CH)-ring.

## 2. Rings such that each $R$-module is $n$-flat

Recall that an $R$-module $E$ is said to be $n$-flat (resp., weakly $n$-flat) if $\mathrm{Tor}_R^n(E,G)=0$ for each $n$-presented $R$-module $G$ (resp., $n$-presented cyclic $R$-module $G$). It is clear but important to see that "all $R$-modules are $n$-flat" condition is equivalent to "every $n$-presented module has flat dimension at most $n-1$."

The following result gives us a characterization of those rings modules are $n$-flat (resp., weakly $n$-flat).

Theorem 2.1. *Let $R$ be a commutative ring and let $n\geq 1$ be an integer. Then*

(1) *each $R$-module is $n$-flat if and only if $R$ is an $(n,n-1)$-ring;*

(2) *each $R$-module is weakly $n$-flat if and only if $R$ is a weak $(n,n-1)$-ring.*

*Proof.* (1) For $n=1$, the result is well known. For $n\geq 2$, let $R$ be an $(n,n-1)$-ring and $N$ be an $R$-module. We claim that $N$ is $n$-flat.

Indeed, if $E$ is an $n$-presented $R$-module, then $\mathrm{pd}_R(E)\leq n-1$ since $R$ is an $(n,n-1)$-ring. Hence, $fd_R(E)\leq n-1$ and so $\mathrm{Tor}_R^n(E,N)=0$. Therefore, $N$ is $n$-flat.

Conversely, assume that all $R$-modules are $n$-flat. Prove that $R$ is an $(n,n-1)$-ring. Let $E$ be an $n$-presented $R$-module and consider the exact sequence of $R$-modules

$$0\longrightarrow Q\longrightarrow F_{n-2}\longrightarrow\cdots\longrightarrow F_0\longrightarrow E\longrightarrow 0, \tag{2.1}$$

where $F_i$ is a finitely generated free $R$-module for each $i$ and $Q$ an $R$-module. It follows that $Q$ is a finitely presented $R$-module since $E$ is an $n$-presented $R$-module. On the other hand, $Q$ is a flat $R$-module since $fd_R(E)\leq n-1$ (since all $R$-modules are $n$-flat and $E$ is $n$-presented). Therefore, $Q$ is a projective $R$-module and so $\mathrm{pd}_R(E)\leq n-1$ which implies that $R$ is an $(n,n-1)$-ring.

(2) Mimic the proof of (1), when $E$ is a cyclic $n$-presented replace, $E$ is an $n$-presented. □

Note that, even if all $R$-modules are 2-flat, there may exist an $R$-module which is not flat. An illustration of this situation is shown in the following example.

*Example 2.2.* Let $R$ be a Prüfer domain which is not a field. Then all $R$-modules are 2-flat by [10, Corollary 2.2] since each Prüfer domain is a $(2,1)$-domain. But, there exists an $R$-module which is not flat since $R$ is not a von Neumann regular ring (since $R$ is a domain which is not a field).

Let $A$ be a ring, let $E$ be an $A$-module, and $R = A \propto E$ be the set of pairs $(a,e)$ with pairwise addition and multiplication defined by

$$(a,e)(a',e') = (aa', ae' + a'e). \tag{2.2}$$

$R$ is called the trivial ring extension of $A$ by $E$. For instance, see [7, 9, 11].

It is clear that every Noetherian nonregular ring is an example of a ring which is not a weak $(n,d)$-ring for any $n$, $d$. Now, we give a wide class of rings which are not a weak $(n,d)$-ring (and so not an $(n,d)$-ring) for each pair of positive integers $n$ and $d$.

Proposition 2.3. *Let $A$ be a commutative ring and let $R = A \propto A$ be the trivial ring extension of $A$ by $A$. Then, for each pair of positive integers $n$ and $d$, $R$ is not a weak $(n,d)$-ring. In particular, it is not an $(n,d)$-ring.*

*Proof.* Let $I := R(0,1)(= 0 \propto A)$. Consider the exact sequence of $R$-modules

$$0 \longrightarrow \operatorname{Ker}(u) \longrightarrow R \xrightarrow{u} I \longrightarrow 0, \tag{2.3}$$

where $u(a,e) = (a,e)(0,1) = (0,a)$. Clearly, $\operatorname{Ker}(u) = 0 \propto A = R(0,1) = I$. Therefore, $I$ is $m$-presented for each positive integer $m$ by the above exact sequence. It remains to show that $\operatorname{pd}_R(I) = \infty$.

We claim that $I$ is not projective. Deny. Then the above exact sequence splits. Hence, $I$ is generated by an idempotent element $(0,a)$, where $a \in A$. Then $(0,a) = (0,a)(0,a) = (0,0)$. So, $a = 0$ and $I = 0$, the desired contradiction (since $I \neq 0$). It follows from the above exact sequence that $\operatorname{pd}_R(I) = 1 + \operatorname{pd}_R(I)$ since $\operatorname{Ker}(u) = I$. Therefore, $\operatorname{pd}_R(I) = \infty$ and then $R$ is not a weak $(n,d)$-ring for each pair of positive integers $n$ and $d$. □

*Remark 2.4.* Let $A$ be a commutative ring and let $R = A \propto A$ be the trivial ring extension of $A$ by $A$. Then, for each positive integer $n$, there exists an $R$-module which is not a weakly $n$-flat, in particular it is not $n$-flat, by Theorem 2.1 and Proposition 2.3.

## 3. Characterization of (weak) $n$-von Neumann regular rings

In [10, Theorem 2.1], the author gives a characterization of $n$-von Neumann regular rings ($(n,0)$-rings). In the sequel, we give a new characterization of $n$-von Neumann regular rings. Recall first that $R$ is a (CH)-ring if each finitely generated proper ideal has a nonzero annihilator.

Theorem 3.1. *Let $R$ be a commutative ring. Then $R$ is an $n$-von Neumann regular ring if and only if $R$ is a (CH)-ring and all $R$-modules are $n$-flat.*

*Proof.* Assume that $R$ is $n$-von Neumann regular. Then $R$ is a (CH)-ring by [10, Theorem 2.1]. On the other hand, $R$ is obviously an $(n,n-1)$-ring since it is an $(n,0)$-ring. So, all $R$-modules are $n$-flat by Theorem 2.1.

Conversely, suppose that $R$ is a (CH)-ring and all $R$-modules are $n$-flat. Then, $R$ is an $(n,n-1)$-ring by Theorem 2.1 and hence $R$ is an $n$-von Neumann regular ring by [10, Corollary 2.3] since $R$ is a (CH)-ring. □

The "(CH)" and "all modules are $n$-flat" properties in Theorem 3.1 are not comparable via inclusion as the following two examples show.

*Example 3.2.* Let $R$ be a Prüfer domain which is not a field. Then

(1) all $R$-modules are $n$-flat for each integer $n \geq 2$ by Theorem 2.1 since each Prüfer domain is an $(n, n-1)$-domain;

(2) $R$ is not a (CH)-ring since $R$ is a domain which is not a field.

*Example 3.3.* Let $A$ be a (CH)-ring and let $R = A \propto A$ be the trivial ring extension of $A$ by $A$. Then

(1) $R$ is a (CH)-ring by [11, Lemma 2.6(1)] since $A$ is a (CH)-ring;

(2) $R$ is not an $(n,d)$-ring for each pair of positive integers $n$ and $d$ by Proposition 2.3. In particular, $R$ does not satisfy the property that "all $R$-modules are $n$-flat" by Theorem 2.1.

Now, we give two characterizations of weak $n$-von Neumann regular rings under some hypothesis.

Theorem 3.4. *Let $R$ be a commutative ring and let $n$ be a positive integer.*

(1) *If $R$ is a (CH)-ring, then $R$ is a weak $n$-von Neumann regular ring if and only if all $R$-modules are weakly $n$-flat.*

(2) *If $R$ is a local ring, then $R$ is a weak $n$-von Neumann regular ring if and only if each nonzero proper ideal of $R$ is not $(n-1)$-presented.*

*Proof.* (1) Let $R$ be a (CH)-ring. If $R$ is a weak $(n,0)$-ring, then $R$ is obviously a weak $(n, n-1)$-ring and so each $R$-module is a weakly $n$-flat by Theorem 2.1(2). Conversely, assume that each $R$-module is a weakly $n$-flat. Then, $R$ is a weak $(n, n-1)$-ring by Theorem 2.1(2). Our purpose is to show that $R$ is a weak $(n,0)$-ring. Let $E$ be a cyclic $n$-presented $R$-module and consider the exact sequence of $R$-module

$$0 \longrightarrow Q \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0, \tag{3.1}$$

where $F_i$ is a finitely generated free $R$-module for each $i$ and $Q$ an $R$-module. Hence, $Q$ is a finitely generated projective $R$-module by the same proof of Theorem 2.1(1). Therefore, $E$ is $m$-presented for each positive integer $m$ and so $E$ is a projective $R$-module by mimicking the end of the proof of [10, Theorem 2.1] since $R$ is a (CH)-ring.

(2) If each proper ideal of $R$ is not $(n-1)$-presented, then $R$ is obviously a weak $(n,0)$-ring. Conversely, assume that $R$ is a local weak $(n,0)$-ring. We must show that each proper ideal is not $(n-1)$-presented. Assume to the contrary that $I$ is a proper $(n-1)$-presented ideal of $R$. Then, $R/I$ is an $n$-presented cyclic $R$-module, so $R/I$ is a projective $R$-module since $R$ is a weak $(n,0)$-ring. Hence, the exact sequence of $R$-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \tag{3.2}$$

splits. So, $I$ is generated by an idempotent, that is, there exists $e \in R$ such that $I = Re$ and $e(e-1) = 0$. But $R$ is a local ring, so $I$ is a free $R$-module (since $I$ is a finitely generated projective $R$-module) and then $e(e-1) = 0$ implies that $e - 1 = 0$. So, $I = Re = R$ and

then $I$ is not a proper ideal, a desired contradiction. Hence, each proper ideal of $R$ is not $(n-1)$-presented. □

*Remark 3.5.* In Theorem 3.4(2), the condition $R$ local is necessary. In fact, let $R$ be a von Neumann regular ring (i.e., $(1,0)$-ring) which is not a field. Then, $R$ is a weak $(1,0)$-ring and there exist many finitely generated proper ideals of $R$.

If $R$ is an $(n,0)$-ring, then $R$ is a (CH)-ring by [10, Theorem 2.1]. The $(1,0)$-ring is a (CH)-ring. So we are led to ask the following question.

*Question 1.* If $R$ is a weak $(n,0)$-ring for a positive integer $n \geq 2$, does this imply that $R$ is a (CH)-ring?

If $R$ is a principal ring (i.e., each finitely generated ideal of $R$ is principal) or a local Gaussian ring, we give an affirmative answer to this question.

For a polynomial $f \in R[X]$, denote by $C(f)$—the content of $f$—the ideal of $R$ generated by the coefficients of $f$. For two polynomials $f$ and $g$ in $R[X]$, $C(fg) \subseteq C(f)C(g)$. A polynomial $f$ is called a Gaussian polynomial if this containment becomes equality for every polynomial $g$ in $R[X]$. A ring $R$ is called a Gaussian ring if every polynomial with coefficients in $R$ is a Gaussian polynomial. For instance, see [6].

Proposition 3.6. *Let $R$ be a weak $(n,0)$-ring for a positive integer $n \geq 2$. Then*

(1) *$R$ is a total ring;*

(2) *if $R$ is a principal ring, then $R$ is a (CH)-ring;*

(3) *if $R$ is a local Gaussian ring, then $R$ is a (CH)-ring.*

*Proof.* (1) Let $a(\neq 0)$ be a regular element of $R$. Our aim is to show that $a$ is unit. The ideal $Ra$ is $n$-presented for each positive integer $n$ since $Ra \cong R$ (since $a$ is regular), so $R/Ra$ is $n$-presented for each positive integer $n$ by the exact sequence of $R$-modules $0 \to Ra \to R \to R/Ra \to 0$. Hence, $R/Ra$ is a projective $R$-module (since $R$ is a weak $(n,0)$-ring) and so the above exact sequence splits. Then $Ra$ is generated by an idempotent, that is, there exists $e \in R$ such that $Ra = Re$ and $e(e-1) = 0$. But $e$ is regular since so is $a$ (since $Ra = Re$). Hence, $e(e-1) = 0$ implies that $e - 1 = 0$ and so $Ra = R$, that is, $a$ is unit.

(2) Argue by (1) and since $R$ is principal.

(3) Let $(R,M)$ be a local Gaussian weak $(n,0)$-ring. By the proof (case 1) of [6, Theorem 3.2], it suffices to show that each $a \in M$ is zero divisor. But $R$ is a total ring by (1). Therefore, each $a \in M$ is a zero divisor and this completes the proof of Proposition 3.6. □

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