EXPLICIT INVERSE OF THE PASCAL MATRIX PLUS ONE

SHENG-LIANG YANG AND ZHONG-KUI LIU

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This paper presents a simple approach to invert the matrix $P_n + I_n$ by applying the Euler polynomials and Bernoulli numbers, where P_n is the Pascal matrix.

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1. Introduction

The Pascal matrix has been known since ancient times, and it arises in many different areas of mathematics. However, it has been studied carefully only recently, see [1, 3–5]. For any integer $n > 0$, the $n \times n$ Pascal matrix P_n is defined with the binomial coefficients by

$$P_n(i, j) = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

It is known that the $n \times n$ inverse matrix P_n^{-1} is given by

$$P_n(i, j) = \begin{cases} (-1)^{i-j} \binom{i-1}{j-1} & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.2)$$

The Hadamard product $A \circ B$ of two matrices is the matrix obtained by coordinate-wise multiplication: $(A \circ B)(i, j) = A(i, j)B(i, j)$. Let Γ_n be the $n \times n$ lower triangular matrices defined by

$$\Gamma_n(i, j) = \begin{cases} (-1)^{i-j} & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \quad (1.3)$$

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then the inverse of the Pascal matrix can be represented as the Hadamard product $P_n^{-1} = P_n \circ \Gamma_n$. For example, if $n = 5$, then

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix},$$

$$P_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}. \quad (1.4)$$

Now we consider the sum of the Pascal matrix and the identity matrix $P_n + I_n$, where I_n is the $n \times n$ identity matrix. We call $P_n + I_n$ the Pascal matrix plus one simply. An interesting fact is that the inverse of $P_n + I_n$ is related to P_n closely. For instance,

$$P_6 + I_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 \\ 1 & 4 & 6 & 4 & 2 & 0 \\ 1 & 5 & 10 & 10 & 5 & 2 \end{pmatrix},$$

$$(P_6 + I_6)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{4} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{3}{4} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{4}{8} & 0 & -\frac{4}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{10}{8} & 0 & -\frac{5}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}. \quad (1.5)$$

This suggests that there may exist a sequence of constants $\{a_n\}_{n=0}^{\infty}$ such that $(P_n + I_n)^{-1} = P_n \circ \Delta_n$, where the matrix Δ_n is a lower triangular matrix with generic element $\Delta_n(i, j) = a_{i-j}$ when $i \geq j$. Aggarwala and Lamoureaux [2] have showed that these constants are values of the Dirichlet eta function evaluated at negative integers, or more generally, certain polylogarithm functions evaluated at the number -1 . In this note, we will give a new simple approach to invert the matrix $P_n + I_n$ by applying the Euler polynomials. As a result, we will show that these constants are values of the Euler polynomials evaluated at the number 0.

The Euler polynomials $E_n(x)$ are defined by means of the following generating function (see [7]):

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{tx}}{e^t + 1}, \quad (1.6)$$

since $\sum_{n=0}^{\infty} (E_n(x+1) + E_n(x))(t^n/n!) = \sum_{n=0}^{\infty} E_n(x+1)(t^n/n!) + \sum_{n=0}^{\infty} E_n(x)(t^n/n!) = 2e^{t(x+1)}/(e^t + 1) + 2e^{tx}/(e^t + 1) = 2e^{tx} = \sum_{n=0}^{\infty} 2x^n(t^n/n!)$. Comparing the coefficients of $t^n/n!$ in this equation, we obtain

$$E_n(x+1) + E_n(x) = 2x^n, \quad n \geq 0. \quad (1.7)$$

The following lemmas are well known and can be found in [9], we give a short proof for the sake of completeness.

LEMMA 1.1. For all $n \geq 0$,

$$E_n(x+y) = \sum_{k=0}^n \binom{n}{k} E_k(x) y^{n-k}, \quad (1.8)$$

$$E_n(x+1) = \sum_{k=0}^n \binom{n}{k} E_k(x). \quad (1.9)$$

Proof. $\sum_{n=0}^{\infty} E_n(x+y)(t^n/n!) = 2e^{t(x+y)}/(e^t + 1) = (2e^{tx}/(e^t + 1))e^{ty} = (\sum_{n=0}^{\infty} E_n(x)(t^n/n!))(\sum_{n=0}^{\infty} y^n(t^n/n!)) = \sum_{n=0}^{\infty} (\sum_{k=0}^n \binom{n}{k} E_k(x) y^{n-k})(t^n/n!)$. Comparing the coefficients of $t^n/n!$ in this equation, we obtain (1.8). In particular, when $y = 1$, we get (1.9).

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From (1.7) and (1.9), we obtain

$$\frac{1}{2} \sum_{k=0}^n \binom{n}{k} E_k(x) + \frac{1}{2} E_n(x) = x^n, \quad n \geq 0. \quad (1.10)$$

If we set $x = 0$ in (1.8), we get $E_n(y) = \sum_{k=0}^n \binom{n}{k} E_{n-k}(0) y^k$, that is,

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} E_{n-k}(0) x^k, \quad n \geq 0. \quad (1.11)$$

Let $E(x)$ and $X(x)$ be the $n \times 1$ matrices defined by $E(x) = [E_0(x), E_1(x), \dots, E_{n-1}(x)]^T$, $X(x) = [1, x, \dots, x^{n-1}]^T$, and let $\bar{E}_n$ be $n \times n$ lower triangular matrices defined by

$$\bar{E}_n(i, j) = \begin{cases} \binom{i-1}{j-1} E_{i-j}(0) & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.12)$$

Then (1.10), (1.11) can be represented as matrix equations, respectively,

$$\begin{aligned} \frac{1}{2} (P_n + I_n) E(x) &= X(x), \\ E(x) &= \bar{E}_n X(x). \end{aligned} \quad (1.13)$$

Thus, we have

$$\begin{aligned} & (P_n + I_n)^{-1} \\ &= \frac{1}{2} \bar{E}_n \\ &= \frac{1}{2} \begin{pmatrix} \binom{0}{0} E_0(0) & 0 & 0 & \cdots & 0 \\ \binom{1}{0} E_1(0) & \binom{1}{1} E_0(0) & 0 & \cdots & 0 \\ \binom{2}{0} E_2(0) & \binom{2}{1} E_1(0) & \binom{2}{2} E_0(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0} E_{n-1}(0) & \binom{n-1}{1} E_{n-2}(0) & \binom{n-1}{2} E_{n-3}(0) & \cdots & \binom{n-1}{n-1} E_0(0) \end{pmatrix}. \end{aligned} \quad (1.14)$$

□

The Bernoulli numbers B_n are defined by (see [7])

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \quad (1.15)$$

It is known (see [6, 8]) that the Euler polynomials can be expressed by the Bernoulli numbers as

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k x^{n+1-k}. \quad (1.16)$$

Putting $x = 0$ in (1.16) gives

$$E_n(0) = \frac{2(1 - 2^{n+1})B_{n+1}}{n+1}, \quad (1.17)$$

for all integers $n \geq 0$. Therefore, we obtain an explicit inverse of the Pascal matrix plus one as follows.

THEOREM 1.2. *For $n \geq 1$, the $n \times n$ inverse matrix $Q_n = (P_n + I_n)^{-1}$ is given by*

$$Q_n(i, j) = \begin{cases} \frac{1}{2} \binom{i-1}{j-1} E_{i-j}(0) & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j; \end{cases} \quad (1.18)$$

or

$$Q_n(i, j) = \begin{cases} \binom{i-1}{j-1} \frac{(1 - 2^{i-j+1})B_{i-j+1}}{i-j+1} & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j. \end{cases} \quad (1.19)$$

In view of the Hadamard product, the inverse matrix $(P_n + I_n)^{-1}$ is the Hadamard product of the Pascal matrix P_n and the matrix Δ_n , where Δ_n is the $n \times n$ lower triangular matrices defined by

$$\Delta_n(i, j) = \begin{cases} \frac{1}{2} E_{i-j}(0) & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j; \end{cases} \quad (1.20)$$

or

$$\Delta_n(i, j) = \begin{cases} \frac{(1 - 2^{i-j+1})B_{i-j+1}}{i-j+1} & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j. \end{cases} \quad (1.21)$$

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The two functions, $\text{Euler}(n, x)$ and $\text{Bernoulli}(n)$, in the *combinat* library of the computer algebra system *Maple* are very useful in obtaining the matrix Q_n . For example, for $n = 8$, we get

$$Q_8 = (P_8 + I_8)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{3}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{5}{4} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} & 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ \frac{17}{16} & 0 & -\frac{21}{4} & 0 & \frac{35}{8} & 0 & -\frac{7}{4} & \frac{1}{2} \end{pmatrix}. \quad (1.22)$$

Note that $Q_n(i, j) = 0$ whenever $i < j$ or $i = j + 2, j + 4, j + 6, \dots$.

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References

- [1] L. Aceto and D. Trigiante, *The matrices of Pascal and other greats*, The American Mathematical Monthly **108** (2001), no. 3, 232–245.
- [2] R. Aggarwala and M. P. Lamoureux, *Inverting the Pascal matrix plus one*, The American Mathematical Monthly **109** (2002), no. 4, 371–377.
- [3] M. Bayat and H. Teimoori, *Pascal K-eliminated functional matrix and its property*, Linear Algebra and Its Applications **308** (2000), no. 1–3, 65–75.
- [4] R. Brawer and M. Pirovino, *The linear algebra of the Pascal matrix*, Linear Algebra and Its Applications **174** (1992), 13–23.
- [5] G. S. Call and D. J. Velleman, *Pascal's matrices*, The American Mathematical Monthly **100** (1993), no. 4, 372–376.
- [6] G.-S. Cheon, *A note on the Bernoulli and Euler polynomials*, Applied Mathematics Letters **16** (2003), no. 3, 365–368.
- [7] L. Comtet, *Advanced Combinatorics. The Art of Finite and Infinite Expansions*, D. Reidel, Dordrecht, 1974.

- [8] K. H. Rosen, J. G. Michaels, J. L. Gross, J. W. Grossman, and D. R. Shier (eds.), *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Florida, 2000.
- [9] H. M. Srivastava and Á. Pintér, *Remarks on some relationships between the Bernoulli and Euler polynomials*, Applied Mathematics Letters **17** (2004), no. 4, 375–380.

Sheng-liang Yang: Department of Applied Mathematics, Lanzhou University of Technology,
Lanzhou, Gansu 730050, China
E-mail address: slyang@lut.cn

Zhong-kui Liu: Department of Mathematics, Northwest Normal University, Lanzhou,
Gansu 730070, China
E-mail address: liuzk@nwnu.edu.cn