WEYL TRANSFORMS ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

N. B. HAMADI AND L. T. RACHDI

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For the Riemann-Liouville transform $\mathscr{R}_\alpha$, $\alpha \in \mathbb{R}_+$, associated with singular partial differential operators, we define and study the Weyl transforms W_σ connected with $\mathscr{R}_\alpha$, where σ is a symbol in S^m, $m \in \mathbb{R}$. We give criteria in terms of σ for boundedness and compactness of the transform W_σ.

1. Introduction

In his book [14], Wong studies the properties of pseudodifferential operators arising in quantum mechanics, first envisaged by Weyl [13], as bounded linear operators on $L^2(\mathbb{R}^n)$ (the space of square integrable functions on $\mathbb{R}^n$ with respect to the Lebesgue measure). For this reason, M. W. Wong calls the operators treated in his book Weyl transforms.

Here, we consider the singular partial differential operators

$$\Delta_1 = \frac{\partial}{\partial x},$$
$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x) \in]0,+\infty[\times\mathbb{R},\ \alpha \geqslant 0. \tag{1.1}$$

We associate to Δ_1 and Δ_2 the Riemann-Liouville transform $\mathscr{R}_\alpha$ defined on $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\iint_{-1}^{1} f\left(rs\sqrt{1-t^2}, x+rt\right)\left(1-t^2\right)^{\alpha-1/2}\left(1-s^2\right)^{\alpha-1} dt\,ds & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2}, x+rt\right)\dfrac{dt}{\sqrt{1-t^2}} & \text{if } \alpha = 0. \end{cases} \tag{1.2}$$

For more general integral transforms, we can see [2].

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The transform $\mathscr{R}_\alpha$ generalizes the mean operator defined by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)\,d\theta. \tag{1.3}$$

The mean operator $\mathscr{R}_0$ and its dual play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [5, 6], or in the linearized inverse scattering problem in acoustics [3].

In [1], we have defined a convolution product and a Fourier transform $\mathscr{F}_\alpha$ associated with $\mathscr{R}_\alpha$, and, we have established many harmonic analysis results (inversion formula, Paley-Wiener, and Plancherel theorems, etc.).

Using these results, we define and study, in this paper the Weyl transforms associated with $\mathscr{R}_\alpha$, we give criteria in terms of symbols to prove the boundedness and compactness of these transforms. To obtain these results, we have first defined the Fourier-Wigner transform associated with the operator $\mathscr{R}_\alpha$, and we have established for it an inversion formula.

More precisely, in Section 2, we recall some properties of harmonic analysis for the operator $\mathscr{R}_\alpha$. In Section 3, we define the Fourier-Wigner transform associated with $\mathscr{R}_\alpha$, study some of its properties, and prove an inversion formula.

In Section 4, we introduce the Weyl transform W_σ associated with $\mathscr{R}_\alpha$, with σ a symbol in class S^m, for $m \in \mathbb{R}$, and we give its connection with the Fourier-Wigner transform. We prove that for σ sufficiently smooth, W_σ is a compact operator from $L^2(d\nu)$, the space of square integrable functions on $[0,+\infty[\times\mathbb{R}$, with respect to the measure

$$d\nu(r,x) = \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}} r^{2\alpha+1} dr \otimes dx, \tag{1.4}$$

into itself.

In Section 5, we define W_σ for σ in a certain space $L^p(d\nu\otimes d\gamma)$, with $p \in [1,2]$, and we establish that W_σ is again a compact operator.

In Section 6, we define W_σ for σ in another function space, and use this to prove in Section 7 that for $p > 2$, there exists a function $\sigma \in L^p(d\nu\otimes d\gamma)$, with the property that the Weyl transform W_σ is not bounded on $L^2(d\nu)$.

For more Weyl transforms, we can see [8, 15].

2. Riemann-Liouville transform associated with the operators Δ_1 and Δ_2

In this section, we recall some properties of the Riemann-Liouville transform that we use in the next sections. For more details, see [1].

For all $(\mu,\lambda) \in \mathbb{C}\times\mathbb{C}$, the system

$$\begin{gathered} \Delta_1 u(r,x) = -i\lambda u(r,x), \\ \Delta_2 u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1, \qquad \frac{\partial u}{\partial r}(0,x) = 0, \quad \forall x \in \mathbb{R}, \end{gathered} \tag{2.1}$$

admits a unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda x), \tag{2.2}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(s) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k}, \tag{2.3}$$

and J_α is the Bessel function of first kind and index α (see [7, 12]).

Moreover, we have

$$\sup_{(r,x)\in\mathbb{R}^2}\left|\varphi_{\mu,\lambda}(r,x)\right| = 1 \quad \text{iff } (\mu,\lambda)\in\Gamma, \tag{2.4}$$

where Γ is the set defined by

$$\Gamma = \mathbb{R}^2\cup\left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ |\mu|\leqslant|\lambda|\right\}. \tag{2.5}$$

Proposition 2.1. *The eigenfunction $\varphi_{\mu,\lambda}$ given by (2.2) has the following Mehler integral representation:*

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\iint_{-1}^{1}\cos\left(\mu rs\sqrt{1-t^2}\right)e^{-i\lambda(x+rt)}\left(1-t^2\right)^{\alpha-1/2}\left(1-s^2\right)^{\alpha-1}dt\,ds & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\left(r\mu\sqrt{1-t^2}\right)e^{-i\lambda(x+rt)}\frac{dt}{\sqrt{1-t^2}} & \text{if } \alpha=0. \end{cases} \tag{2.6}$$

This result shows that

$$\varphi_{\mu,\lambda}(r,x) = \mathcal{R}_\alpha\left(\cos(\mu.)\exp(-i\lambda.)\right)(r,x), \tag{2.7}$$

where $\mathcal{R}_\alpha$ is the Riemann-Liouville transform associated with the operators Δ_1 and Δ_2, given in the introduction.

We denote by

(i) $\mathscr{C}_{*,c}(\mathbb{R}^2)$ the subspace of $\mathscr{C}_*(\mathbb{R}^2)$ consisting of functions with compact support;

(ii) $d\nu(r,x)$ the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu(r,x) = c_\alpha r^{2\alpha+1}dr\otimes dx, \tag{2.8}$$

with $c_\alpha = 1/\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)$;

(iii) $L^p(d\nu)$ the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$, satisfying

$$\begin{aligned} \|f\|_{p,\nu} &= \left(\int_{\mathbb{R}}\int_0^{+\infty}\left|f(r,x)\right|^p d\nu(r,x)\right)^{1/p} < +\infty \quad \text{if } p\in[1,+\infty[, \\ \|f\|_{\infty,\nu} &= \operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}}\left|f(r,x)\right| < +\infty \quad \text{if } p=+\infty; \end{aligned} \tag{2.9}$$

(iv) $d\gamma(\mu,\lambda)$ the measure defined on Γ by

$$\iint_\Gamma f(\mu,\lambda)d\gamma(\mu,\lambda) = c_\alpha\left\{\int_{\mathbb{R}}\int_0^{+\infty} f(\mu,\lambda)\left(\mu^2+\lambda^2\right)^\alpha \mu\,d\mu\,d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)\left(\lambda^2-\mu^2\right)^\alpha \mu\,d\mu\,d\lambda\right\}; \tag{2.10}$$

(v) $L^p(d\gamma), p\in[1,+\infty]$, the space of measurable functions on Γ satisfying

$$\|f\|_{p,\gamma} = \left(\iint_\Gamma |f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\right)^{1/p} < +\infty \quad \text{if } p\in[1,+\infty[,$$
$$\|f\|_{\infty,\gamma} = \operatorname*{ess\,sup}_{(\mu,\lambda)\in\Gamma} |f(\mu,\lambda)| < +\infty \quad \text{if } p=+\infty. \tag{2.11}$$

Defintion 2.2. (i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(d\nu)$, for all $(r,x),(s,y)\in[0,+\infty[\times\mathbb{R}$, by

$$\mathcal{T}_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta\,d\theta. \tag{2.12}$$

(ii) The convolution product associated with the Riemann-Liouville transform of $f,g\in L^1(d\nu)$ is defined by

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R},\quad f*g(r,x) = \int_{\mathbb{R}}\int_0^{+\infty}\mathcal{T}_{(r,-x)}\check{f}(s,y)g(s,y)d\nu(s,y), \tag{2.13}$$

where $\check{f}(s,y)=f(s,-y)$.

We have the following properties.

(i) We have the following product formula:

$$\mathcal{T}_{(r,x)}\varphi_{\mu,\lambda}(s,y) = \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y). \tag{2.14}$$

(ii) Let f be in $L^1(d\nu)$. Then, for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, we have

$$\int_{\mathbb{R}}\int_0^\infty \mathcal{T}_{(s,y)}f(r,x)d\nu(r,x) = \int_{\mathbb{R}}\int_0^\infty f(r,x)d\nu(r,x). \tag{2.15}$$

(iii) If $f\in L^p(d\nu)$, $1\leqslant p\leqslant+\infty$, then for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(s,y)}f$ belongs to $L^p(d\nu)$, and we have

$$\|\mathcal{T}_{(s,y)}f\|_{p,\nu} \leqslant \|f\|_{p,\nu}. \tag{2.16}$$

(iv) For $f,g\in L^1(d\nu)$, $f*g$ belongs to $L^1(d\nu)$, and the convolution product is commutative and associative.

(v) For $f\in L^1(d\nu)$, $g\in L^p(d\nu)$, $1<p\leqslant+\infty$, the function $f*g\in L^p(d\nu)$ and

$$\|f*g\|_{p,\nu} \leqslant \|f\|_{1,\nu}\|g\|_{p,\nu}. \tag{2.17}$$

(vi) For $f, g \in \mathscr{C}_{*,c}(\mathbb{R}^2)$, such that $\operatorname{supp} f \subset [-a_1,a_1]\times[-a_2,a_2]$ and $\operatorname{supp} g \subset [-b_1, b_1]\times[-b_2,b_2]$, the function $f * g$ belongs to $\mathscr{C}_{*,c}(\mathbb{R}^2)$ and

$$\operatorname{supp}(f*g) \subset [-(a_1+b_1), a_1+b_1]\times[-(a_2+b_2), a_2+b_2]. \tag{2.18}$$

Defintion 2.3. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu)$, by

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\varphi_{\mu,\lambda}(r,x)\,d\nu(r,x), \tag{2.19}$$

where Γ is the set defined by the relation (2.5).

We have the following properties.

(i) Let f be in $L^1(d\nu)$. For all $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathscr{F}_\alpha(\mathscr{T}_{(r,-x)}f)(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x)\mathscr{F}_\alpha(f)(\mu,\lambda). \tag{2.20}$$

(ii) For $f,g\in L^1(d\nu)$, we have

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathscr{F}_\alpha(f*g)(\mu,\lambda) = \mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(g)(\mu,\lambda). \tag{2.21}$$

(iii) For $f\in L^1(d\nu)$, we have

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathscr{F}_\alpha(f)(\mu,\lambda) = \mathrm{B}\circ\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda), \tag{2.22}$$

where, for every $(\mu,\lambda)\in\mathbb{R}^2$,

$$\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)j_\alpha(r\mu)\exp(-i\lambda x)\,d\nu(r,x), \tag{2.23}$$

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathrm{B}f(\mu,\lambda) = f\left(\sqrt{\mu^2+\lambda^2},\lambda\right). \tag{2.24}$$

(iv) For $f\in L^1(d\nu)$ such that $\mathscr{F}_\alpha(f)\in L^1(d\gamma)$, we have the inversion formula for $\mathscr{F}_\alpha$, for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}$,

$$f(r,x) = \iint_\Gamma \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi}_{\mu,\lambda}(r,x)\,d\gamma(\mu,\lambda). \tag{2.25}$$

Proposition 2.4. *Let f be in $L^p(d\nu)$, with $p\in[1,2]$. Then, $\mathscr{F}_\alpha(f)$ belongs to $L^{p'}(d\gamma)$, with $1/p+1/p'=1$, and $\|\mathscr{F}_\alpha(f)\|_{p',\gamma}\leqslant\|f\|_{p,\nu}$.*

Proof. The mapping $\widetilde{\mathscr{F}}_\alpha$ given by the relation (2.23) is an isometric isomorphism from $L^2(d\nu)$ onto itself, then $\|\widetilde{\mathscr{F}}_\alpha(f)\|_{2,\nu} = \|f\|_{2,\nu}$.

On the other hand, we have $\|\widetilde{\mathscr{F}}_\alpha(f)\|_{\infty,\nu}\leqslant\|f\|_{1,\nu}$.

Thus, from these relations and the Riesz-Thorin theorem [10, 11], we deduce that for all $f\in L^p(d\nu)$, with $p\in[1,2]$, the function $\widetilde{\mathscr{F}}_\alpha(f)$ belongs to $L^{p'}(d\nu)$, with $p'=p/(p-1)$, and we have

$$\|\widetilde{\mathscr{F}}_\alpha(f)\|_{p',\nu}\leqslant\|f\|_{p,\nu}. \tag{2.26}$$

We complete the proof by using the fact that

$$||\mathscr{F}_\alpha(f)||_{p',\gamma} = ||\widetilde{\mathscr{F}}_\alpha(f)||_{p',\nu}, \tag{2.27}$$

which is a consequence of the relation (2.22). □

We denote by (see [1, 9])

(i) $\mathscr{S}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ rapidly decreasing together with all their derivatives, even with respect to the first variable;

(ii) $\mathscr{S}_*(\Gamma)$ the space of functions $f:\Gamma \to \mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that is, for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\Gamma} (1+|\mu|^2+|\lambda|^2)^{k_1} \left| \left(\frac{\partial}{\partial\mu}\right)^{k_2} \left(\frac{\partial}{\partial\lambda}\right)^{k_3} f(\mu,\lambda)\right| < +\infty, \tag{2.28}$$

where

$$\frac{\partial f}{\partial \mu}(\mu,\lambda) = \begin{cases} \dfrac{\partial}{\partial r}(f(r,\lambda)) & \text{if } \mu = r \in \mathbb{R}, \\ \dfrac{1}{i}\dfrac{\partial}{\partial t}(f(it,\lambda)) & \text{if } \mu = it,\ |t| \leqslant |\lambda|. \end{cases} \tag{2.29}$$

Each of these spaces is equipped with its usual topology.

Remark 2.5. From [1], the Fourier transform $\mathscr{F}_\alpha$ is an isomorphism from $\mathscr{S}_*(\mathbb{R}^2)$ onto $\mathscr{S}_*(\Gamma)$. The inverse mapping is given by

$$\forall (r,x) \in \mathbb{R}^2, \quad \mathscr{F}_\alpha^{-1}(f)(r,x) = \iint_\Gamma f(\mu,\lambda)\overline{\varphi}_{\mu,\lambda}(r,x)d\gamma(\mu,\lambda). \tag{2.30}$$

3. Fourier-Wigner transform associated with Riemann-Liouville operator

Defintion 3.1. The Fourier-Wigner transform associated with the Riemann-Liouville operator is the mapping V defined on $\mathscr{S}_*(\mathbb{R}^2)\times\mathscr{S}_*(\mathbb{R}^2)$, for all $((r,x),(\mu,\lambda)) \in \mathbb{R}^2\times\Gamma$, by

$$V(f,g)((r,x),(\mu,\lambda)) = \int_{\mathbb{R}}\int_0^\infty f(s,y)\varphi_{\mu,\lambda}(s,y)\mathscr{T}_{(r,x)}g(s,y)d\nu(s,y). \tag{3.1}$$

Remark 3.2. The transform V can also be written in the forms

(i) $V(f,g)((r,x),(\mu,\lambda)) = \mathscr{F}_\alpha(f\mathscr{T}_{(r,x)}g)(\mu,\lambda)$;

(ii) $V(f,g)((r,x),(\mu,\lambda)) = \check{g} * (\varphi_{\mu,\lambda} f)(r,-x)$,

where $\check{g}(s,y) = g(s,-y)$ and $*$ is the convolution product given in Definition 2.2.

We denote by

(i) $\mathscr{S}_*(\mathbb{R}^2\times\mathbb{R}^2)$ the space of infinitely differentiable functions $f((r,x),(s,y))$ on $\mathbb{R}^2\times\mathbb{R}^2$, even with respect to the variables r and s, and rapidly decreasing together with all their derivatives;

(ii) $\mathcal{S}_*(\mathbb{R}^2\times\Gamma)$ the space of infinitely differentiable functions $f((r,x),(\mu,\lambda))$ on $\mathbb{R}^2\times\Gamma$, even with respect to the variables r and μ, and rapidly decreasing together with all their derivatives;

(iii) $L^p(d\nu\otimes d\nu)$, $1\leqslant p\leqslant+\infty$, the space of measurable functions on $([0,+\infty[\times\mathbb{R})\times([0,+\infty[\times\mathbb{R})$, verifying for $p\in[1,+\infty[$;

$$\|f\|_{p,\nu\otimes\nu}=\left(\iint_{\mathbb{R}}\iint_0^{+\infty}|f((r,x),(s,y))|^p d\nu(r,x)d\nu(s,y)\right)^{1/p}<+\infty, \tag{3.2}$$

for $p=+\infty$,

$$\|f\|_{\infty,\nu\otimes\nu}=\operatorname*{ess\,sup}_{(r,x),(s,y)\in[0,+\infty[\times\mathbb{R}}|f((r,x),(s,y))|<+\infty; \tag{3.3}$$

(iv) $L^p(d\nu\otimes d\gamma)$, $1\leqslant p\leqslant+\infty$, the space similarly defined (with $d\nu(r,x)d\gamma(\mu,\lambda)$ in the integrand).

Proposition 3.3. (i) *The Fourier-Wigner transform V is a bilinear, continuous mapping from $\mathcal{S}_*(\mathbb{R}^2)\times\mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\mathbb{R}^2\times\Gamma)$.*

(ii) *For $p\in]1,2]$,*

$$||V(f,g)||_{p',\nu\otimes\gamma}\leqslant\|f\|_{p,\nu}\|g\|_{p',\nu}. \tag{3.4}$$

The transform V can be extended to a continuous bilinear operator, denoted also by V, from $L^p(d\nu)\times L^{p'}(d\nu)$ into $L^{p'}(d\nu\otimes d\gamma)$, where $p'=p/(p-1)$ is the conjugate exponent of p.

Proof. (i) Let $f,g\in\mathcal{S}_*(\mathbb{R}^2)$, and let F be the function defined on $\mathbb{R}^2\times\mathbb{R}^2$ by

$$F((r,x),(s,y))=f(s,y)\mathcal{T}_{(r,x)}g(s,y). \tag{3.5}$$

Then, we have for all $(s,y),(\mu,\lambda)\in\mathbb{R}^2$,

$$\widetilde{\mathcal{F}}_\alpha\otimes I(F)((\mu,\lambda),(s,y))=j_\alpha(s\mu)\exp(i\lambda y)f(s,y)\widetilde{\mathcal{F}}_\alpha(g)(\mu,\lambda), \tag{3.6}$$

where I is the identity operator. Since $\widetilde{\mathcal{F}}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto itself, we deduce that the function $\widetilde{\mathcal{F}}_\alpha\otimes I(F)$ belongs to the space $\mathcal{S}_*(\mathbb{R}^2\times\mathbb{R}^2)$ and consequently, $F\in\mathcal{S}_*(\mathbb{R}^2\times\mathbb{R}^2)$. Then, (i) follows from the relation

$$V(f,g)((r,x),(\mu,\lambda))=I\otimes\mathcal{F}_\alpha(F)((r,x),(\mu,\lambda)), \tag{3.7}$$

and the fact that $\mathcal{F}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\Gamma)$.

(ii) We get the result from Remark 3.2(i), Proposition 2.4, Minkowski's inequality for integrals (see [4, page 186]), and from the relation (2.16). □

Theorem 3.4. *For all $f,g\in\mathcal{S}_*(\mathbb{R}^2)$, $(\mu,\lambda)\in\Gamma$ and $(r,x)\in\mathbb{R}^2$,*

$$\mathcal{F}_\alpha\otimes\mathcal{F}_\alpha^{-1}(V(f,g))((\mu,\lambda),(r,x))=\overline{\varphi}_{\mu,\lambda}(r,x)f(r,x)\mathcal{F}_\alpha(g)(\mu,\lambda). \tag{3.8}$$

Proof. This theorem follows from the relations (2.20) and (3.7). □

Using the previous theorem and the relation (2.25), we get the following result.

Corollary 3.5. *For* $f,g \in \mathcal{S}_*(\mathbb{R}^2)$,
(i) *for all* $(\mu,\lambda) \in \Gamma$,

$$\int_{\mathbb{R}}\int_0^{\infty} \mathcal{F}_\alpha \otimes \mathcal{F}_\alpha^{-1}\big(V(f,g)\big)\big((\mu,\lambda),(r,x)\big)d\nu(r,x) = \check{\mathcal{F}}_\alpha(f)(\mu,\lambda)\mathcal{F}_\alpha(g)(\mu,\lambda); \tag{3.9}$$

(ii) *for all* $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$\iint_{\Gamma} \mathcal{F}_\alpha \otimes \mathcal{F}_\alpha^{-1}\big(V(f,g)\big)\big((\mu,\lambda),(r,x)\big)d\gamma(\mu,\lambda) = f(r,x)g(r,x). \tag{3.10}$$

Theorem 3.6. *Let* $f,g \in L^1(d\nu)\cap L^2(d\nu)$*, such that* $c = \int_{\mathbb{R}}\int_0^{\infty} g(r,x)d\nu(r,x) \neq 0$*. Then,*

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathcal{F}_\alpha(f)(\mu,\lambda) = \frac{1}{c}\int_{\mathbb{R}}\int_0^{\infty} V(f,g)\big((r,x),(\mu,\lambda)\big)d\nu(r,x). \tag{3.11}$$

Proof. From the relation (3.1), we have for all $(\mu,\lambda)\in\Gamma$,

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{\infty} V(f,g)\big((r,x),(\mu,\lambda)\big)d\nu(r,x)\\ &\quad= \int_{\mathbb{R}}\int_0^{\infty}\Big(\int_{\mathbb{R}}\int_0^{\infty} f(s,y)\varphi_{\mu,\lambda}(s,y)\mathcal{T}_{(r,x)}g(s,y)d\nu(s,y)\Big)d\nu(r,x).\end{aligned} \tag{3.12}$$

Then, the result follows from the relation (2.15), Definition 2.3, the fact that

$$\forall(r,x)\in[0,+\infty[\times\mathbb{R}, \forall(\mu,\lambda)\in\Gamma, \quad \big|\varphi_{\mu,\lambda}(r,x)\big| \leqslant 1, \tag{3.13}$$

and Fubini's theorem. □

Corollary 3.7. *With the hypothesis of Theorem 3.6, if* $\mathcal{F}_\alpha(f)\in L^1(d\gamma)$*, the following inversion formula for the Fourier-Wigner transform* V *holds:*

$$f(r,x) = \frac{1}{c}\iint_{\Gamma}\overline{\varphi}_{\mu,\lambda}(r,x)\Big[\int_{\mathbb{R}}\int_0^{\infty} V(f,g)\big((s,y),(\mu,\lambda)\big)d\nu(s,y)\Big]d\gamma(\mu,\lambda), \tag{3.14}$$

for almost every $(r,x)\in\mathbb{R}^2$.

4. Weyl transform associated with Riemann-Liouville operator

In this section, we introduce and study the Weyl transform and give its connection with the Fourier-Wigner transform. To do this, we must define the class of pseudodifferential operators [14].

Defintion 4.1. Let $m\in\mathbb{R}$. Define S^m to be the set of symbols, consisting of all infinitely differentiable functions $\sigma((r,x),(\mu,\lambda))$ on $\mathbb{R}^2\times\Gamma$, even with respect to the variables r and μ, such that for all $k_1,k_2,k_3,k_4\in\mathbb{N}$, there exists a positive constant $C = C(k_1,k_2,k_3,k_4,m)$

satisfying

$$\left|\left(\frac{\partial}{\partial r}\right)^{k_1}\left(\frac{\partial}{\partial x}\right)^{k_2}\left(\frac{\partial}{\partial \mu}\right)^{k_3}\left(\frac{\partial}{\partial \lambda}\right)^{k_4}\sigma\big((r,x),(\mu,\lambda)\big)\right| \leqslant C\big(1+\mu^2+2\lambda^2\big)^{m-(k_3+k_4)}. \tag{4.1}$$

Defintion 4.2. For $\sigma \in S^m$, $m \in \mathbb{R}$, define the operator H_σ on $\mathcal{S}_*(\mathbb{R}^2) \times \mathcal{S}_*(\mathbb{R}^2)$, for all $(r,x) \in \mathbb{R}^2$,

$$\begin{aligned} H_\sigma(f,g)(r,x) = \iint_\Gamma \bigg\{ \int_{\mathbb{R}} \int_0^\infty &\sigma\big((s,y),(\mu,\lambda)\big)\varphi_{\mu,\lambda}(r,x) \\ &\times V(f,g)\big((s,y),(\mu,\lambda)\big)d\nu(s,y)\bigg\} d\gamma(\mu,\lambda), \end{aligned} \tag{4.2}$$

$$\mathbb{H}_\sigma(f,g) = H_\sigma(f,g)(0,0). \tag{4.3}$$

Proposition 4.3. *Let σ be the symbol given by*

$$\forall (r,x) \in \mathbb{R}^2,\ \forall (\mu,\lambda) \in \Gamma, \quad \sigma\big((r,x),(\mu,\lambda)\big) = -\big(\mu^2+\lambda^2\big). \tag{4.4}$$

Then for $f,g \in \mathcal{S}_*(\mathbb{R}^2)$,

$$\forall (r,x) \in \mathbb{R}^2, \quad \mathrm{H}_\sigma(f,g)(r,x) = c\ell_\alpha f(r,-x), \tag{4.5}$$

where

$$c = \int_{\mathbb{R}} \int_0^\infty g(r,x)d\nu(r,x), \qquad \ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}. \tag{4.6}$$

Proof. From relations (3.1), (4.2) and Fubini's theorem we get, for all $(r,x) \in \mathbb{R}^2$,

$$\begin{aligned} \mathrm{H}_\sigma(f,g)(r,x) = \iint_\Gamma -\big(\mu^2+\lambda^2\big)\varphi_{\mu,\lambda}(r,x)\bigg\{\int_{\mathbb{R}}\int_0^\infty &f(t,z)\varphi_{\mu,\lambda}(t,z) \\ &\times\left[\int_{\mathbb{R}}\int_0^\infty \mathcal{T}_{(t,z)}g(s,y)d\nu(s,y)\right]d\nu(t,z)\bigg\}d\gamma(\mu,\lambda). \end{aligned} \tag{4.7}$$

Now, by relation (2.15), it follows that

$$\mathrm{H}_\sigma(f,g)(r,x) = c\iint_\Gamma -\big(\mu^2+\lambda^2\big)\mathcal{F}_\alpha(f)(\mu,\lambda)\varphi_{\mu,\lambda}(r,x)d\gamma(\mu,\lambda). \tag{4.8}$$

The result follows from relation (2.25) and the fact that

$$\forall (\mu,\lambda) \in \Gamma, \quad -\big(\mu^2+\lambda^2\big)\mathcal{F}_\alpha(f)(\mu,\lambda) = \mathcal{F}_\alpha\big(\ell_\alpha f\big)(\mu,\lambda). \tag{4.9}$$

□

Defintion 4.4. Let $\sigma \in S^m$, $m < -(\alpha+3/2)$. The Weyl transform associated with the Riemann-Liouville operator is the mapping W_σ defined on $\mathcal{S}_*(\mathbb{R}^2)$, for all $(r,x) \in \mathbb{R}^2$, by

$$W_\sigma(f)(r,x) = \iint_\Gamma \left[\int_{\mathbb{R}}\int_0^\infty \varphi_{\mu,\lambda}(r,x)\sigma\big((s,y),(\mu,\lambda)\big)\mathcal{T}_{(r,x)}f(s,y)d\nu(s,y)\right]d\gamma(\mu,\lambda). \tag{4.10}$$

Theorem 4.5. *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$. The Weyl transform W_σ is a continuous mapping from $\mathcal{S}_*(\mathbb{R}^2)$ into itself.*

Proof. Let $f \in \mathcal{S}_*(\mathbb{R}^2)$, since $\widetilde{\mathcal{F}}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto itself, and

$$\forall (\mu,\lambda) \in \mathbb{R}^2, \quad \widetilde{\mathcal{F}}_\alpha(\mathcal{T}_{(r,x)}f)(\mu,\lambda) = j_\alpha(r\mu)\exp(i\lambda x)\widetilde{\mathcal{F}}_\alpha(f)(\mu,\lambda), \tag{4.11}$$

we deduce that for all $(r,x) \in [0,+\infty[\times\mathbb{R}$, the function $(s,y) \mapsto \mathcal{T}_{(r,x)}f(s,y)$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$. Then, by the inversion formula for $\widetilde{\mathcal{F}}_\alpha$, we get, for all $(s,y) \in \mathbb{R}^2$;

$$\mathcal{T}_{(r,x)}f(s,y) = \int_{\mathbb{R}}\int_0^{+\infty} j_\alpha(r\mu)\exp(i\lambda x)\widetilde{\mathcal{F}}_\alpha(f)(\mu,\lambda)j_\alpha(s\mu)\exp(i\lambda y)d\nu(\mu,\lambda). \tag{4.12}$$

By Definition 4.4 and Fubini's theorem, we obtain, for all $(r,x) \in \mathbb{R}^2$,

$$\begin{aligned} &W_\sigma(f)(r,x) \\ &\quad= \iint_\Gamma \varphi_{\mu,\lambda}(r,x)\bigg[\int_{\mathbb{R}}\int_0^\infty \widetilde{\mathcal{F}}_\alpha(f)(t,z)j_\alpha(rt)\exp(ixz) \\ &\qquad\qquad \times\bigg\{\int_{\mathbb{R}}\int_0^\infty \sigma((s,y),(\mu,\lambda))j_\alpha(st)\exp(iyz)d\nu(s,y)\bigg\}d\nu(t,z)\bigg]d\gamma(\mu,\lambda) \\ &\quad= \iint_\Gamma \varphi_{\mu,\lambda}(r,x)\bigg[\int_{\mathbb{R}}\int_0^\infty \widetilde{\mathcal{F}}_\alpha(f)(t,z)j_\alpha(rt)\exp(ixz) \\ &\qquad\qquad \times\widetilde{\mathcal{F}}_\alpha^{-1}(\sigma((\cdot,\cdot),(\mu,\lambda)))(t,z)d\nu(t,z)\bigg]d\gamma(\mu,\lambda). \end{aligned} \tag{4.13}$$

Now, the function

$$((t,z),(\mu,\lambda)) \longmapsto \widetilde{\mathcal{F}}_\alpha^{-1}(\sigma((\cdot,\cdot),(\mu,\lambda)))(t,z) \tag{4.14}$$

belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$.

On the other hand, the mapping $f \mapsto G_f$, given for all $((t,z),(\mu,\lambda)) \in \mathbb{R}^2 \times \Gamma$ by

$$G_f((t,z),(\mu,\lambda)) = \widetilde{\mathcal{F}}_\alpha(f)(t,z)\widetilde{\mathcal{F}}_\alpha^{-1}(\sigma((\cdot,\cdot),(\mu,\lambda)))(t,z), \tag{4.15}$$

is continuous from $\mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$, and for all $(r,x) \in \mathbb{R}^2$, we have

$$\begin{aligned} W_\sigma(f)(r,x) &= \iint_\Gamma \bigg(\int_{\mathbb{R}}\int_0^\infty G_f((t,z),(\mu,\lambda))j_\alpha(rt)\exp(ixz)\overline{\varphi}_{\mu,\lambda}(r,-x)d\nu(t,z)\bigg)d\gamma(\mu,\lambda) \\ &= \widetilde{\mathcal{F}}_\alpha^{-1} \otimes \mathcal{F}_\alpha^{-1}(G_f)((r,x),(r,-x)). \end{aligned} \tag{4.16}$$

Since $\mathcal{F}_\alpha^{-1}$ is an isomorphism from $\mathcal{S}_*(\Gamma)$ onto $\mathcal{S}_*(\mathbb{R}^2)$, we deduce that $\widetilde{\mathcal{F}}_\alpha^{-1} \otimes \mathcal{F}_\alpha^{-1}$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$ onto $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$. □

Lemma 4.6. *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$. Then, the function k defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by*

$$k((r,x),(s,y)) = \iint_\Gamma \varphi_{\mu,\lambda}(r,x)\mathcal{T}_{(r,-x)}(\sigma((\cdot,\cdot),(\mu,\lambda)))(s,y)d\gamma(\mu,\lambda) \tag{4.17}$$

belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. The function k can be written in the form

$$k((r,x),(s,y)) = \mathcal{T}_{(r,-x)}(I \otimes \mathcal{F}_\alpha^{-1}(\sigma)((\cdot,\cdot),(r,-x)))(s,y). \tag{4.18}$$

Since the Fourier transform $\mathcal{F}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto $\mathcal{S}_*(\Gamma)$, we deduce that the function $I \otimes \mathcal{F}_\alpha^{-1}(\sigma)$ belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$.

Then, the lemma follows from the fact that for all $g \in \mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$, the function

$$((r,x),(s,y)) \longmapsto \mathcal{T}_{(r,-x)}(g((\cdot,\cdot),(r,-x)))(s,y) \tag{4.19}$$

belongs to $\mathcal{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$. □

Theorem 4.7. *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$.*

(i) *For all $f \in \mathcal{S}_*(\mathbb{R}^2)$,*

$$\forall (r,x) \in \mathbb{R}^2, \quad W_\sigma(f)(r,x) = \int_\mathbb{R}\int_0^\infty k((r,x),(s,y))f(s,y)d\nu(s,y). \tag{4.20}$$

(ii) *For $f \in \mathcal{S}_*(\mathbb{R}^2)$ and $p, p' \in [1,+\infty]$ such that $1/p + 1/p' = 1$,*

$$||W_\sigma(f)||_{p',\nu} \leqslant \|k\|_{p',\nu\otimes\nu}\|f\|_{p,\nu}. \tag{4.21}$$

(iii) *For $p \in [1,+\infty[$, the operator W_σ can be extended to a bounded operator from $L^p(d\nu)$ into $L^{p'}(d\nu)$.*

In particular

$$W_\sigma : L^2(d\nu) \longmapsto L^2(d\nu) \tag{4.22}$$

is a Hilbert-Schmidt operator, and consequently it is compact.

Proof. (i) Let f be in $\mathcal{S}_*(\mathbb{R}^2)$. From Definition 4.4, for all $(\mu,\lambda) \in \mathbb{R}^2$, we have

$$\begin{aligned} W_\sigma(f)(r,x) &= \iint_\Gamma \left(\int_\mathbb{R}\int_0^\infty \varphi_{\mu,\lambda}(r,x)\sigma((s,y),(\mu,\lambda))\mathcal{T}_{(r,x)}f(s,y)d\nu(s,y)\right)d\gamma(\mu,\lambda) \\ &= \iint_\Gamma \varphi_{\mu,\lambda}(r,x)\left(\int_\mathbb{R}\int_0^\infty \sigma((s,y),(\mu,\lambda))\mathcal{T}_{(r,x)}f(s,y)d\nu(s,y)\right)d\gamma(\mu,\lambda). \end{aligned} \tag{4.23}$$

Using Fubini's theorem, and the equality

$$\begin{aligned} &\int_\mathbb{R}\int_0^\infty \sigma((s,y),(\mu,\lambda))\mathcal{T}_{(r,x)}f(s,y)d\nu(s,y) \\ &\quad = \int_\mathbb{R}\int_0^\infty f(s,y)\mathcal{T}_{(r,-x)}(\sigma((\cdot,\cdot),(\mu,\lambda)))(s,y)d\nu(s,y), \end{aligned} \tag{4.24}$$

we get

$$\begin{aligned} W_\sigma(f)(r,x) &= \int_{\mathbb{R}}\int_0^{\infty} f(s,y)\Big\{\iint_\Gamma \varphi_{\mu,\lambda}(r,x)\mathcal{T}_{(r,-x)}\big(\sigma((\cdot,\cdot),(\mu,\lambda))\big)(s,y)d\gamma(\mu,\lambda)\Big\}d\nu(s,y) \\ &= \int_{\mathbb{R}}\int_0^{\infty} f(s,y)k((r,x),(s,y))d\nu(s,y). \end{aligned} \tag{4.25}$$

(ii) follows from (i), Hölder's inequality, and Lemma 4.6.

(iii) From (ii) and the fact that the space $\mathcal{S}_*(\mathbb{R}^2)$ is dense in $L^p(d\nu)$, $p \in [1,+\infty[$, we deduce that W_σ can be extended to a continuous mapping from $L^p(d\nu)$ into $L^{p'}(d\nu)$.

By Lemma 4.6, the kernel k belongs to $L^2(d\nu \otimes d\nu)$, hence W_σ is a Hilbert-Schmidt operator. In particular, it is compact. □

Theorem 4.8. *Let $\sigma \in S^m$, $m < -(\alpha+3/2)$. For all $f,g \in \mathcal{S}_*(\mathbb{R}^2)$, we have*

$$\mathbb{H}_\sigma(f,g) = \left\langle \frac{W_\sigma(g)}{\overline{f}} \right\rangle, \tag{4.26}$$

where $\langle \cdot/\cdot \rangle$ is the inner product of $L^2(d\nu)$.

Proof. From Definition (3.1) and relations (4.2), (4.3), we get

$$\begin{aligned} \mathbb{H}_\sigma(f,g) = \iint_\Gamma \Big\{\int_{\mathbb{R}}\int_0^{\infty} \sigma((r,x),(\mu,\lambda))\Big(\int_{\mathbb{R}}\int_0^{\infty} f(s,y)\varphi_{\mu,\lambda}(s,y) \\ \times \mathcal{T}_{(r,x)}g(s,y)d\nu(s,y)\Big)d\nu(r,x)\Big\}d\gamma(\mu,\lambda). \end{aligned} \tag{4.27}$$

Using Fubini's theorem, we obtain

$$\begin{aligned} \mathbb{H}_\sigma(f,g) = \int_{\mathbb{R}}\int_0^{\infty} f(s,y)\Big\{\iint_\Gamma \varphi_{(\mu,\lambda)}(s,y)\Big(\int_{\mathbb{R}}\int_0^{\infty} \sigma((r,x),(\mu,\lambda)) \\ \times \mathcal{T}_{(r,x)}g(s,y)d\nu(r,x)\Big)d\gamma(\mu,\lambda)\Big\}d\nu(s,y). \end{aligned} \tag{4.28}$$

The theorem follows from Definition 4.4 and the fact that for all $((r,x),(s,y)) \in [0,+\infty[\times\mathbb{R}$,

$$\mathcal{T}_{(r,x)}g(s,y) = \mathcal{T}_{(s,y)}g(r,x). \tag{4.29}$$

□

5. Weyl transform associated with symbol in $L^p(d\nu \otimes d\gamma)$, $1 \leqslant p \leqslant 2$

In this section, we will see that relation (4.26) allows us to prove that the Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$, $1 \leqslant p \leqslant 2$, is a compact operator.

We denote by $\mathcal{B}(L^2(d\nu))$ the $\mathbb{C}^*$-algebra of bounded operators ψ from $L^2(d\nu)$ into itself, equipped with the norm

$$\|\psi\|_* = \sup_{\|f\|_{2,\nu}=1} \big\|\psi(f)\big\|_{2,\nu}. \tag{5.1}$$

Theorem 5.1. *For $p \in [1,2]$, there exists a unique bounded operator Q from $L^p(d\nu \otimes d\gamma)$ into $\mathcal{B}(L^2(d\nu)) : \sigma \mapsto Q_\sigma$, such that for all $f,g \in \mathcal{S}_*(\mathbb{R}^2)$,*

$$\begin{gathered}\left\langle \frac{Q_\sigma(g)}{\overline{f}} \right\rangle = \iint_\Gamma \left(\int_{\mathbb{R}} \int_0^\infty \sigma((r,x),(\mu,\lambda))\, V(f,g)((r,x),(\mu,\lambda))\, d\nu(r,x) \right) d\gamma(\mu,\lambda), \\ \big\|Q_\sigma\big\|_* \leqslant \|\sigma\|_{p,\nu\otimes\gamma}.\end{gathered} \tag{5.2}$$

Proof. (i) The case $p = 2$.

Let $\sigma \in \mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$. For $g \in \mathcal{S}_*(\mathbb{R}^2)$, we put $Q_\sigma(g) = W_\sigma(g)$.

From Theorem 4.8, we obtain

$$\begin{aligned}\left\langle \frac{Q_\sigma(g)}{\overline{f}} \right\rangle &= \left\langle \frac{W_\sigma(g)}{\overline{f}} \right\rangle = \mathbb{H}_\sigma(f,g) \\ &= \iint_\Gamma \left(\int_{\mathbb{R}} \int_0^\infty \sigma((r,x),(\mu,\lambda))\, V(f,g)((r,x),(\mu,\lambda))\, d\nu(r,x) \right) d\gamma(\mu,\lambda).\end{aligned} \tag{5.3}$$

On the other hand, from Proposition 3.3(ii) and Cauchy-Shwartz inequality, we have

$$\left| \left\langle \frac{Q_\sigma(g)}{\overline{f}} \right\rangle \right| \leqslant \|\sigma\|_{2,\nu\otimes\gamma} \|f\|_{2,\nu} \|g\|_{2,\nu}. \tag{5.4}$$

This implies that $Q_\sigma \in \mathcal{B}(L^2(d\nu))$ and

$$\big\|Q_\sigma\big\|_* \leqslant \|\sigma\|_{2,\nu\otimes\gamma}. \tag{5.5}$$

We complete the proof by using the fact that the space $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$ is dense in $L^2(d\nu \otimes d\gamma)$.

(ii) The case $p = 1$ can be obtained by the same way.

(iii) Using the cases $p = 1$, $p = 2$, and the Riesz-Thorin theorem [10, 11], we complete the proof for all $p \in [1,2]$. □

Remark 5.2. In the following, the operator Q_σ will be denoted by W_σ.

Theorem 5.3. *For $\sigma \in L^p(d\nu \otimes d\gamma)$, $1 \leqslant p \leqslant 2$, the operator W_σ from $L^2(d\nu)$ into itself is a compact operator.*

Proof. Let $\sigma \in L^p(d\nu \otimes d\gamma)$, $1 \leqslant p \leqslant 2$, and let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$, such that

$$\big\|\sigma_k - \sigma\big\|_{p,\nu\otimes\gamma} \xrightarrow[k\to+\infty]{} 0. \tag{5.6}$$

From relation (5.5), we have $\|W_{\sigma_k} - W_\sigma\|_* \leqslant \|\sigma_k - \sigma\|_{p,\nu\otimes\gamma}$. This implies that

$$W_{\sigma_k} \xrightarrow[k\to+\infty]{} W_\sigma, \quad \text{in } \mathcal{B}(L^2(d\nu)). \tag{5.7}$$

But from Theorem 4.7, we know that for all $k \in \mathbb{N}$, the operator W_{σ_k} is compact, then the result of the theorem follows from the fact that the subspace $\mathcal{K}(L^2(d\nu))$ of $\mathcal{B}(L^2(d\nu))$ consisting of compact operators is a closed ideal of $\mathcal{B}(L^2(d\nu))$. □

6. Weyl transform with symbol in $S'_*(\mathbb{R}^2 \times \Gamma)$

We denote by

(i) $\mathcal{S}'_*(\mathbb{R}^2)$ the space of tempered distributions on $\mathbb{R}^2$, even with respect to the first variable. It is the topological dual of $\mathcal{S}_*(\mathbb{R}^2)$;

(ii) $\mathcal{S}'_*(\mathbb{R}^2 \times \Gamma)$ the space of tempered distributions on $\mathbb{R}^2 \times \Gamma$, even with respect to the first variables of $\mathbb{R}^2$ and Γ. It is the topological dual of $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$.

Defintion 6.1. For $\sigma \in \mathcal{S}'_*(\mathbb{R}^2 \times \Gamma)$ and $g \in \mathcal{S}_*(\mathbb{R}^2)$, define the operator $W_\sigma(g)$ on $\mathcal{S}_*(\mathbb{R}^2)$, by

$$[W_\sigma(g)](f) = \sigma(V(f,g)), \quad f \in \mathcal{S}_*(\mathbb{R}^2), \tag{6.1}$$

where V is the mapping given by (3.1).

Remark 6.2. From Proposition 3.3, it is clear that $W_\sigma(g)$ given by (6.1) belongs to $S'_*(\mathbb{R}^2)$.

For a slowly increasing function h on $\mathbb{R}^2 \times \Gamma$, we denote by σ_h the element of $S'_*(\mathbb{R}^2 \times \Gamma)$ defined by

$$\sigma_h(F) = \iint_\Gamma \int_{\mathbb{R}} \int_0^\infty F((r,x),(\mu,\lambda))h((r,x),(\mu,\lambda))\,d\nu(r,x)\,d\gamma(\mu,\lambda). \tag{6.2}$$

Then, we have the following.

Proposition 6.3. *Let $\sigma_1 \in S'_*(\mathbb{R}^2 \times \Gamma)$, given by the function equal to 1. One has*

$$W_{\sigma_1}(g) = c\delta, \tag{6.3}$$

where $c = \int_{\mathbb{R}} \int_0^\infty g(r,x)\,d\nu(r,x)$ and δ is the Dirac distribution at $(0,0)$.

Proof. By relation (6.1), we have for all f in $\mathcal{S}_*(\mathbb{R}^2)$,

$$\begin{aligned} [W_{\sigma_1}(g)](f) &= \sigma_1(V(f,g)), \\ &= \iint_\Gamma \left(\int_{\mathbb{R}} \int_0^\infty V(f,g)((r,x)(\mu,\lambda))\,d\nu(r,x) \right) d\gamma(\mu,\lambda), \end{aligned} \tag{6.4}$$

and by Theorem 3.6

$$[W_{\sigma_1}(g)](f) = c \iint_\Gamma \mathcal{F}_\alpha(f)(\mu,\lambda)\,d\gamma(\mu,\lambda). \tag{6.5}$$

We complete the proof by using relation (2.25). □

Remark 6.4. From Proposition 6.3, we deduce that there exists $\sigma \in \mathcal{S}'_*(\mathbb{R}^2 \times \Gamma)$ given by a function in $L^\infty(\mathbb{R}^2 \times \Gamma)$, such that for all $g \in \mathcal{S}_*(\mathbb{R}^2)$ satisfying

$$c = \int_{\mathbb{R}} \int_0^\infty g(r,x)\,d\nu(r,x) \neq 0, \tag{6.6}$$

the distribution $W_\sigma(g)$ is not given by a function of $L^2(d\nu)$.

7. Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$, $2 < p < \infty$

Theorem 7.1. *Let $p \in]2,+\infty[$. There exists a function $\sigma \in L^p(d\nu \otimes d\gamma)$, such that the Weyl transform W_σ defined by (6.1) is not a bounded linear operator on $L^2(d\nu)$.*

We break down the proof into two lemmas, of which the theorem is an immediate consequence.

Lemma 7.2. *Let $2 < p < \infty$. Suppose that for all $\sigma \in L^p(d\nu \otimes d\gamma)$, the Weyl transform W_σ given by relation (6.1) is a bounded linear operator on $L^2(d\nu)$. Then, there exists a positive constant M such that*

$$||W_\sigma||_* \leqslant M\|\sigma\|_{p,\nu\otimes\gamma}, \quad \forall \sigma \in L^p(d\nu \otimes d\gamma). \tag{7.1}$$

Proof. Under the assumption of the lemma, there exists for each $\sigma \in L^p(d\nu \otimes d\gamma)$ a positive constant C_σ such that

$$||W_\sigma(g)||_{2,\nu} \leqslant C_\sigma\|g\|_{2,\nu}, \quad \text{for } g \in L^2(d\nu). \tag{7.2}$$

Let $f,g \in \mathcal{S}_*(\mathbb{R}^2)$ such that $\|f\|_{2,\nu} = \|g\|_{2,\nu} = 1$, and let us define the operator

$$Q_{f,g} : L^p(d\nu \otimes d\gamma) \longrightarrow \mathbb{C} \tag{7.3}$$

by

$$Q_{f,g}(\sigma) = \left\langle \frac{W_\sigma(g)}{\overline{f}} \right\rangle. \tag{7.4}$$

Then,

$$\sup_{\|f\|_{2,\nu}=\|g\|_{2,\nu}=1} |Q_{f,g}(\sigma)| \leqslant C_\sigma. \tag{7.5}$$

By the Banach-Steinhauss theorem, the operator $Q_{f,g}$ is bounded on $L^p(d\nu \otimes d\gamma)$, then there exists a positive constant M such that

$$||Q_{f,g}||_* = \sup_{\|\sigma\|_{p,\nu\otimes\gamma}=1} |Q_{f,g}(\sigma)| \leqslant M. \tag{7.6}$$

From this, we deduce that for all $f,g \in \mathcal{S}_*(\mathbb{R}^2)$, and $\sigma \in L^p(d\nu \otimes d\gamma)$, we have

$$\left|\left\langle \frac{W_\sigma(g)}{\overline{f}} \right\rangle\right| \leqslant M\|\sigma\|_{p,\nu\otimes\gamma}\|f\|_{2,\nu}\|g\|_{2,\nu}, \tag{7.7}$$

which implies (7.1). □

Lemma 7.3. *For $2 < p < \infty$, there is no positive constant M satisfying (7.1).*

Proof. Suppose that there exists $M > 0$ such that relation (7.1) holds.

Let p' be such that $1/p + 1/p' = 1$, then $p' \in]1,2[$.

We consider for $f,g \in \mathcal{S}_*(\mathbb{R}^2)$, the function $V(f,g)$ given by the relation (3.1). We have

$$||V(f,g)||_{p',\nu\otimes\gamma} = \sup_{\|\sigma\|_{p,\nu\otimes\gamma}=1} \left| \left\langle \frac{W_\sigma(g)}{\overline{f}} \right\rangle \right| \leqslant \sup_{\|\sigma\|_{p,\nu\otimes\gamma}=1} ||W_\sigma(g)||_{2,\nu} \|f\|_{2,\nu}, \tag{7.8}$$

and consequently

$$||V(f,g)||_{p',\nu\otimes\gamma} \leqslant M\|f\|_{2,\nu}\|g\|_{2,\nu}. \tag{7.9}$$

Now, let $f,g \in L^2(d\nu)$, we choose sequences $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ in $\mathcal{S}_*(\mathbb{R}^2)$, approximating f and g in the $\|\cdot\|_{2,\nu}$-norm.

From (7.9), we get

$$||V(f_k,g_k)||_{p',\nu\otimes\gamma} \leqslant M||f_k||_{2,\nu}||g_k||_{2,\nu}, \tag{7.10}$$

which implies that $(V(f_k,g_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^{p'}(d\nu\otimes d\gamma)$. Then, it converges to some function F in $L^{p'}(d\nu\otimes d\gamma)$.

Now, using Proposition 3.3, we deduce that $F = V(f,g)$, and

$$\forall f,g \in L^2(d\nu), \quad ||V(f,g)||_{p',\nu\otimes\gamma} \leqslant M\|f\|_{2,\nu}\|g\|_{2,\nu}. \tag{7.11}$$

We will exhibit an example where the relation (7.11) leads to a contradiction. Let f be defined on $\mathbb{R}^2$, even with respect to the first variable, and supported in $[-1,1]\times[-1,1]$. Then, for all $((r,x),(\mu,\lambda)) \in \mathbb{R}^2\times\Gamma$,

$$|V(f,f)((r,x),(\mu,\lambda))| \leqslant |f| * |\check{f}|(r,-x), \tag{7.12}$$

where $*$ is the convolution product given by Definition 2.2. From (2.18), we deduce that for all $(\mu,\lambda)\in\Gamma$, the function $(r,x)\mapsto V(f,f)((r,x),(\mu,\lambda))$ is supported in $[-2,2]\times[-2,2]$.

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}
&\left(\iint_\Gamma \left| \int_{-2}^{2}\int_0^2 V(f,f)((r,x),(\mu,\lambda))\,d\nu(r,x)\right|^{p'} d\gamma(\mu,\lambda)\right)^{1/p'} \\
&\quad\leqslant \left(\int_{-2}^{2}\int_0^2 d\nu(r,x)\right)^{1/p} \left(\iint_\Gamma \int_{-2}^{2}\int_0^{+\infty} |V(f,f)((r,x),(\mu,\lambda))|^{p'} d\nu(r,x)\,d\gamma(\mu,\lambda)\right)^{1/p'} \\
&\quad= \left(\int_{-2}^{2}\int_0^2 d\nu(r,x)\right)^{1/p} ||V(f,f)||_{p',\nu\otimes\gamma} \leqslant M\left(\int_{-2}^{2}\int_0^2 d\nu(r,x)\right)^{1/p} \|f\|_{2,\nu}^2.
\end{aligned} \tag{7.13}$$

The last inequality follows from (7.9). Now, Theorem 3.6 implies that the function

$$(\mu,\lambda) \longmapsto \int_{\mathbb{R}}\int_0^{+\infty} V(f,f)((r,x),(\mu,\lambda))\,d\nu(r,x) = c\mathscr{F}_\alpha(f)(\mu,\lambda) \tag{7.14}$$

belongs to $L^{p'}(d\gamma)$, here $c = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\,d\nu(r,x)$.

If we pick $c = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\,d\nu(r,x) \neq 0$, and the last inequality, we deduce that the function $\mathscr{F}_\alpha(f)$ belongs to $L^{p'}(d\gamma)$, and

$$\|\mathscr{F}_\alpha(f)\|_{p',\gamma} \leqslant \frac{M}{|c|}\left(\int_{-2}^{2}\int_0^{2} d\nu(r,x)\right)^{1/p} \|f\|_{2,\nu}^2. \tag{7.15}$$

In the following, we consider the particular function f given by

$$f(r,x) = |r|^{\beta}\mathbf{1}_{[-1,1]}(r)\mathbf{1}_{[-1,1]}(x), \tag{7.16}$$

where $\mathbf{1}_{[-1,1]}$ is the characteristic function of the interval $[-1,1]$.

This function belongs to $L^1(d\nu)\cap L^2(d\nu)$, for $\beta > -(\alpha+1)$, and we have

$$\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)\sqrt{2\pi}}\frac{\sin\lambda}{\lambda}\int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)\,dr, \tag{7.17}$$

so

$$\|\widetilde{\mathscr{F}}_\alpha(f)\|_{p',\nu}^{p'} = \frac{2^{p'}}{\left(2^{\alpha}\Gamma(\alpha+1)\sqrt{2\pi}\right)^{p'+1}}\int_{\mathbb{R}}\left|\frac{\sin\lambda}{\lambda}\right|^{p'} d\lambda \times \int_0^{+\infty}\left|\int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)\,dr\right|^{p'}\mu^{2\alpha+1}\,d\mu. \tag{7.18}$$

However

$$\int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)\,dr = \frac{1}{\mu^{\beta+2\alpha+2}}\int_0^{\mu} r^{\beta+2\alpha+1} j_\alpha(r)\,dr. \tag{7.19}$$

Using the asymptotic expansion of j_α (see [7, 12]), given by

$$j_\alpha(r) = \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}r^{\alpha+1/2}}\left[\cos\left(r-\alpha\frac{\pi}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{r}\right)\right], \quad \text{as } (r \longrightarrow +\infty), \tag{7.20}$$

we deduce that for $-(\alpha+1) < \beta < -(\alpha+1/2)$, the integral

$$a = \int_0^{+\infty} r^{\beta+2\alpha+1} j_\alpha(r)\,dr \tag{7.21}$$

exists and is finite. This involves that

$$\int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)dr \sim \frac{a}{\mu^{\beta+2\alpha+2}}, \quad \text{as } (\mu \longrightarrow +\infty). \tag{7.22}$$

Then, there exist $A, B > 0$ such that for

$$\mu > A, \quad \left| \int_0^1 r^{\beta+2\alpha+1} j_\alpha(r\mu)dr \right| \geqslant \frac{B}{\mu^{\beta+2\alpha+2}}. \tag{7.23}$$

Replacing in relation (7.18), we get

$$\|\widetilde{\mathscr{F}}_\alpha(f)\|_{p',\gamma}^{p'} \geqslant \frac{(2B)^{p'}}{(2^\alpha\Gamma(\alpha+1)\sqrt{2\pi})^{p'+1}} \int_{\mathbb{R}} \left|\frac{\sin\lambda}{\lambda}\right|^{p'} d\lambda \int_A^{+\infty} \frac{d\mu}{\mu^{p'(2\alpha+\beta+2)-2\alpha-1}}. \tag{7.24}$$

Thus, for $\beta < -(2\alpha+2) + (2\alpha+2/p')$,

$$\|\mathscr{F}_\alpha(f)\|_{p',\gamma}^{p'} = \|\widetilde{\mathscr{F}}_\alpha(f)\|_{p',\nu}^{p'} = +\infty. \tag{7.25}$$

This shows that relation (7.15) is false if we pick

$$\beta \in \left] -(\alpha+1), \min\left(-\left(\alpha+\frac{1}{2}\right), -(2\alpha+2) + \frac{2\alpha+2}{p'}\right)\right[. \tag{7.26}$$

□

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N. B. Hamadi: Department of Mathematics, Faculty of Sciences of Tunis, University Tunis, El Manar 2092, Tunis, Tunisia
E-mail address: nadia.zouari@edunet.tn

L. T. Rachdi: Department of Mathematics, Faculty of Sciences of Tunis, University Tunis, El Manar 2092, Tunis, Tunisia
E-mail address: lakhdartannech.rachdi@fst.rnu.tn