# AUXILIARY PRINCIPLE FOR GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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We introduce and study a new class of generalized nonlinear variational-like inequalities and prove an existence theorem of solutions for this kind of generalized nonlinear variational-like inequalities. By using the auxiliary principle technique, we construct a new iterative scheme for solving the class of the generalized nonlinear variational-like inequalities. The convergence of the sequence generated by the iterative algorithm is also discussed. Our results extend and unify the corresponding results due to Ding, Liu, Ume, Kang, Yao, and others.



## 1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in many diverse fields of pure and applied sciences. It is well known that one of the most important problems in variational inequality theory is the development of efficient and implementable iterative algorithms for solving various classes of variational inequalities and variational inclusions. In [3–38] there are a lot of iterative algorithms for finding the approximate solutions of various variational inequalities. Glowinski et al. [8] had developed the auxiliary principle technique. By using the auxiliary principle technique, Ding [3, 4], Ding and Tan [5], and Ding and Yao [6], Liu et al. [21, 28], Zeng et al. [37], Zeng et al. [38], and others suggested several iterative algorithms to compute approximate solutions for some classes of general nonlinear mixed variational inequalities and variational-like inequalities in reflexive Banach spaces.

Motivated and inspired by the research work in [3–38], in this paper, we introduce and study a new class of generalized nonlinear variational-like inequalities and prove an existence theorem of solutions for this kind of generalized nonlinear variational-like inequalities. By applying the result due to Chang [1, 2] and the auxiliary principle technique, we suggest a new iterative scheme for solving the class of generalized nonlinear variational-like inequalities. The convergence of the sequence generated by the iterative

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algorithm is also discussed. Our results extend and unify the corresponding results due to Ding [3], Liu et al. [21], Yao [36], and others.

## 2. Preliminaries

Throughout this paper, we assume that $H$ is a real Hilbert space with dual space $H^*$ and that $\langle u,v\rangle$ is the dual pairing between $u \in H$ and $v \in H^*$. Let $K$ be a nonempty closed convex subset of $H$, and let $A,B,C : K \to H$, $N : H \times H \times H \to H$, and $\eta : K \times K \to H^*$ be mappings. Suppose that $a : K \times K \to (-\infty,\infty)$ is a coercive continuous bilinear form, that is, there exist positive constants $c,d > 0$ such that

(C1) $a(v,v) \geq c\|v\|^2$ for all $v \in K$;

(C2) $|a(u,v)| \leq d\|u\| \cdot \|v\|$ for all $u,v \in K$. It follows from (C1) and (C2) that $c \leq d$. Let $b : K \times K \to (-\infty,+\infty)$ be nondifferentiable and satisfy the following conditions:

(C3) $b$ is linear in the first argument;

(C4) $b$ is convex in the second argument;

(C5) $b$ is bounded, that is, there exists a constant $l > 0$ satisfying

$$\left|b(u,v)\right| \leq l\|u\| \cdot \|v\| \quad \forall u,v \in K; \tag{2.1}$$

(C6) $b(u,v) - b(u,w) \leq b(u,v-w)$ for all $u,v,w \in K$.

Now we consider the following generalized nonlinear variational-like inequality.

For given $f \in H$, find $u \in K$ such that

$$\langle N(Au,Bu,Cu) - f, \eta(v,u)\rangle + a(u,v-u) \geq b(u,u) - b(u,v) \quad \forall v \in K. \tag{2.2}$$

*Special cases.* If $N(Au,Bu,Cu) = Au - Bu$, $f = 0$, and $b(u,v) = \varphi(v)$ for all $u,v \in K$, where $\varphi : H \to (-\infty,+\infty)$ is a functional, then the generalized nonlinear variational-like inequality (2.2) is equivalent to finding $u \in K$ such that

$$\langle Au - Bu, \eta(v,u)\rangle \geq \varphi(u) - \varphi(v) \quad \forall v \in K, \tag{2.3}$$

which was introduced and studied by Ding [3].

If $N(Au,Bu,Cu) = Au - Bu$, $f = 0$, $b(u,v) = \varphi(v)$, and $\eta(u,v) = gu - gv$ for all $u,v \in K$, where $g : K \to H^*$ is a mapping, then the generalized nonlinear variational-like inequality (2.2) is equivalent to finding $u \in K$ such that

$$\langle Au - Bu, gv - gu\rangle \geq \varphi(u) - \varphi(v) \quad \forall v \in K, \tag{2.4}$$

which was studied by Yao [36].

*Definition 2.1.* Let $A,B : K \to H$, $N : H \times H \times H \to H$, and $\eta : K \times K \to H^*$ be mappings.

(1) $A$ is said to be *Lipschitz continuous* with constant $r$ if there exists a constant $r > 0$ such that

$$\|Ax - Ay\| \leq r\|x - y\| \quad \forall x,y \in K. \tag{2.5}$$

(2) $N$ is said to be *$\eta$-strongly monotone* with constant $s$ with respect to $A$ in the first argument if there exists a constant $s > 0$ such that

$$\langle N(Ax,u,v) - N(Ay,u,v), \eta(x,y)\rangle \geq s\|x-y\|^2 \quad \forall x,y \in K,\ \forall u,v \in H. \tag{2.6}$$

(3) $N$ is said to be *$\eta$-monotone* with respect to $A$ in the second argument if

$$\langle N(u,Ax,v) - N(v,Ay,v), \eta(x,y)\rangle \geq 0 \quad \forall x,y \in K,\ \forall u,v \in H. \tag{2.7}$$

(4) $N$ is said to be *Lipschitz continuous* with constant $t$ in the third argument if there exists a constant $t > 0$ such that

$$\|N(u,v,x) - N(u,v,y)\| \leq t\|x-y\| \quad \forall x,y,u,v \in H. \tag{2.8}$$

(5) $N$ is said to be *$\eta$-hemicontinuous* with respect to $A$ and $B$ in the first and second arguments if for any $x,y,z \in K$, the mapping $g : [0,1] \to (-\infty,\infty)$ defined by $g(t) = \langle N(A(tx+(1-t)y), B(tx+(1-t)y), z), \eta(x,y)\rangle$ is continuous at $0^+$.

(6) $\eta$ is said to be *Lipschitz continuous* with constant $s$ if there exists a constant $s > 0$ such that

$$\|\eta(x,y)\| \leq s\|x-y\| \quad \forall x,y \in K. \tag{2.9}$$

(7) $\eta$ is said to be *strongly monotone* with constant $t$ if there exists a constant $t > 0$ such that

$$\langle x-y, \eta(x,y)\rangle \geq t\|x-y\|^2 \quad \forall x,y \in K. \tag{2.10}$$

Lemma 2.2 [1, 2]. *Let $X$ be a nonempty closed convex subset of a Hausdorff linear topological space $E$, and let $\phi,\psi : X \times X \to R$ be mappings satisfying the following conditions:*

(a) *$\psi(x,y) \leq \phi(x,y)$ for all $x,y \in X$, and $\psi(x,x) \geq 0$ for all $x \in X$;*

(b) *for each $x \in X$, $\phi(x,\cdot)$ is upper semicontinuous on $X$;*

(c) *for each $y \in X$, the set $\{x \in X : \psi(x,y) < 0\}$ is a convex set;*

(d) *there exists a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0,y) < 0$ for all $y \in X \setminus K$.*

*Then there exists $\hat{y} \in K$ such that $\phi(x,\hat{y}) \geq 0$ for all $x \in X$.*

## 3. Auxiliary problem and algorithm

Now we consider the following auxiliary problem with respect to the generalized nonlinear variational-like inequality (2.2). For any given $u \in K$, find $\hat{w} \in K$ such that

$$\begin{aligned} \langle \hat{w}, \eta(v,\hat{w})\rangle \geq{}& \langle u, \eta(v,\hat{w})\rangle - \rho\langle N(A\hat{w}, B\hat{w}, Cu) - f, \eta(v,\hat{w})\rangle \\ &- \rho a(\hat{w}, v-\hat{w}) - \rho b(u,v) + \rho b(u,\hat{w}) \quad \forall v \in K, \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant.

Theorem 3.1. *Let $K$ be a nonempty closed convex subset of $H$ and $f \in H$. Suppose that $a : K \times K \to (-\infty,\infty)$ satisfies* (C1) *and* (C2)*, $b : K \times K \to (-\infty,\infty)$ satisfies* (C3)–(C6)*, and $A,B,C : K \to H$ and $N : H \times H \times H \to H$ are mappings such that $N$ is $\eta$-hemicontinuous with respect to $A$ and $B$ in the first and second arguments. Let $\eta : K \times K \to H^*$ be Lipschitz continuous with constant $\delta$ and strongly monotone with constant $\tau$, for each $y \in K$, let $\eta(\cdot,y)$ be continuous and $\eta(y,x) = -\eta(x,y)$ for all $x,y \in K$. Assume that $N$ is $\eta$-strongly monotone with constant $\alpha$ with respect to $A$ in the first argument and $\eta$-monotone with respect to $B$ in the second argument. If for given $x,y,z \in H$ and $v \in K$, the mapping $\langle N(x,y,z),\eta(v,\cdot)\rangle$ is concave and upper semicontinuous, then the auxiliary problem (3.1) has a unique solution in $K$.*

*Proof.* Let $u$ be in $K$. Define the functionals $\phi$ and $\psi : K \times K \to R$ by

$$\begin{aligned}\phi(v,w) &= \langle v,\eta(v,w)\rangle - \langle u,\eta(v,w)\rangle + \rho\langle N(Av,Bv,Cu) - f,\eta(v,w)\rangle \\ &\quad + \rho a(v,v-w) - \rho b(u,w) + \rho b(u,v), \\ \psi(v,w) &= \langle w,\eta(v,w)\rangle - \langle u,\eta(v,w)\rangle + \rho\langle N(Aw,Bw,Cu) - f,\eta(v,w)\rangle \\ &\quad + \rho a(w,v-w) - \rho b(u,w) + \rho b(u,v)\end{aligned} \tag{3.2}$$

for all $v,w \in K$.

We check that the functionals $\phi$ and $\psi$ satisfy all the conditions of Lemma 2.2 in the weak topology. It is easy to see for all $v,w \in K$,

$$\begin{aligned}&\phi(v,w) - \psi(v,w) \\ &\quad = \langle v-w,\eta(v,w)\rangle + \rho\langle N(Av,Bv,Cu) - N(Aw,Bv,Cu),\eta(v,w)\rangle \\ &\qquad + \rho\langle N(Aw,Bv,Cu) - N(Aw,Bw,Cu),\eta(v,w)\rangle + \rho a(v-w,v-w) \\ &\quad \geq \left[\tau + \rho(\alpha + c)\right]\|v-w\|^2 \geq 0,\end{aligned} \tag{3.3}$$

which implies that $\phi$ and $\psi$ satisfy the condition (1) of Lemma 2.2. Since $a$ is a coercive continuous bilinear form, it follows that $a(v,v-w)$ is weakly upper semicontinuous with respect to $w$. Note that $b$ is convex and lower semicontinuous in the second argument and for given $x,y,z \in H$, $v \in K$, the mapping $\langle N(x,y,z),\eta(v,\cdot)\rangle$ is concave and upper semicontinuous. Therefore $\phi(v,\cdot)$ is weakly upper semicontinuous in the second argument and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. That is, the conditions (2) and (3) of Lemma 2.2 hold. Let $\bar{v} \in K$. Put

$$\begin{aligned}L &= \left[\tau + \rho(\alpha + c)\right]^{-1}\left[\delta\|u - \bar{v}\| + \rho d\|\bar{v}\| + \rho\delta\|N(A\bar{v},B\bar{v},Cu) - f\| + \rho l\|u\|\right], \\ M &= \{w \in K : \|w - \bar{v}\| \leq L\}.\end{aligned} \tag{3.4}$$

Clearly, $M$ is a weakly compact subset of $K$ and for any $w \in K \setminus M$,

$$\begin{aligned}
\psi(\bar{v},w) = {} & \langle w,\eta(\bar{v},w)\rangle - \langle u,\eta(\bar{v},w)\rangle \\
& + \rho\langle N(Aw,Bw,Cu) - f,\eta(\bar{v},w)\rangle \\
& + \rho a(w,\bar{v}-w) - \rho b(u,w) + \rho b(u,\bar{v}) \\
\leq {} & -\langle w-\bar{v},\eta(w,\bar{v})\rangle + \langle u-\bar{v},\eta(w,\bar{v})\rangle \\
& - \rho\langle N(Aw,Bw,Cu) - N(A\bar{v},Bw,Cu),\eta(w,\bar{v})\rangle \\
& - \rho\langle N(A\bar{v},Bw,Cu) - N(A\bar{v},B\bar{v},Cu),\eta(w,\bar{v})\rangle \\
& - \rho\langle N(A\bar{v},B\bar{v},Cu) - f,\eta(w,\bar{v})\rangle \\
& - \rho a(w-\bar{v},w-\bar{v}) - \rho a(\bar{v},w-\bar{v}) + \rho b(u,\bar{v}-w) \\
\leq {} & -\|w-\bar{v}\|\Big\{[\tau+\rho(\alpha+c)]\|w-\bar{v}\| - \delta\|u-\bar{v}\| \\
& \qquad - \rho d\|\bar{v}\| - \rho\delta\|N(A\bar{v},B\bar{v},Cu) - f\| - \rho l\|u\|\Big\} < 0,
\end{aligned} \tag{3.5}$$

which means that the condition (4) of Lemma 2.2 holds. Thus Lemma 2.2 ensures that there exists $\hat{w} \in K$ such that $\phi(v,\hat{w}) \geq 0$ for all $v \in K$, that is,

$$\begin{aligned}
\langle v,\eta(v,\hat{w})\rangle \geq {} & \langle u,\eta(v,\hat{w})\rangle - \rho\langle N(Av,Bv,Cu) - f,\eta(v,\hat{w})\rangle \\
& - \rho a(v,v-\hat{w}) - \rho b(u,v) + \rho b(u,\hat{w}) \quad \forall v \in K.
\end{aligned} \tag{3.6}$$

Let $t$ be in $(0,1]$ and let $v$ be in $K$. Replacing $v$ by $v_t = tv + (1-t)\hat{w}$ in (3.6), we see that

$$\begin{aligned}
\langle v_t,\eta(v_t,\hat{w})\rangle \geq {} & \langle u,\eta(v_t,\hat{w})\rangle - \rho\langle N(Av_t,Bv_t,Cu) - f,\eta(v_t,\hat{w})\rangle \\
& - \rho a(v_t,v_t-\hat{w}) - \rho b(u,v_t) + \rho b(u,\hat{w}) \quad \forall v \in K.
\end{aligned} \tag{3.7}$$

Notice that $b$ is convex in the second argument and $\langle N(x,y,z),\eta(v,\cdot)\rangle$ is concave and upper semicontinuous. From (C6) and (3.7) we infer that

$$\begin{aligned}
t[\langle v_t,\eta(v,\hat{w})\rangle] \geq t\Big[ & \langle u,\eta(v,\hat{w})\rangle - \rho\langle N(Av_t,Bv_t,Cu) - f,\eta(v,\hat{w})\rangle \\
& - \rho a(v_t,v-\hat{w}) - \rho b(u,v) + \rho b(u,\hat{w})\Big] \quad \forall v \in K,
\end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned}
\langle v_t,\eta(v,\hat{w})\rangle \geq {} & \langle u,\eta(v,\hat{w})\rangle - \rho\langle N(Av_t,Bv_t,Cu) - f,\eta(v,\hat{w})\rangle \\
& - \rho a(v_t,v-\hat{w}) - \rho b(u,v) + \rho b(u,\hat{w}) \quad \forall v \in K.
\end{aligned} \tag{3.9}$$

Letting $t \to 0^+$ in the above inequality, we conclude that

$$\begin{aligned}\langle \widehat{w}, \eta(v,\widehat{w})\rangle \geq{}& \langle u, \eta(v,\widehat{w})\rangle - \rho\langle N(A\widehat{w}, B\widehat{w}, Cu) - f, \eta(v,\widehat{w})\rangle \\ & - \rho a(\widehat{w}, v-\widehat{w}) - \rho b(u,v) + \rho b(u,\widehat{w}) \quad \forall v \in K.\end{aligned} \tag{3.10}$$

That is, $\widehat{w}$ is a solution of (3.1). Now we prove the uniqueness. For any two solutions $w_1, w_2 \in K$ of (3.1) with respect to $u$, we know that

$$\begin{aligned}\langle w_1, \eta(v,w_1)\rangle \geq{}& \langle u, \eta(v,w_1)\rangle - \rho\langle N(Aw_1, Bw_1, Cu) - f, \eta(v,w_1)\rangle \\ & - \rho a(w_1, v-w_1) - \rho b(u,v) + \rho b(u,w_1),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\langle w_2, \eta(v,w_2)\rangle \geq{}& \langle u, \eta(v,w_2)\rangle - \rho\langle N(Aw_2, Bw_2, Cu) - f, \eta(v,w_2)\rangle \\ & - \rho a(w_2, v-w_2) - \rho b(u,v) + \rho b(u,w_2)\end{aligned} \tag{3.12}$$

for all $v \in K$. Taking $v = w_2$ in (3.11) and $v = w_1$ in (3.12), we get that

$$\begin{aligned}\langle w_1, \eta(w_2,w_1)\rangle \geq{}& \langle u, \eta(w_2,w_1)\rangle - \rho\langle N(Aw_1, Bw_1, Cu) - f, \eta(w_2,w_1)\rangle \\ & - \rho a(w_1, w_2-w_1) - \rho b(u,w_2) + \rho b(u,w_1), \\ \langle w_2, \eta(w_1,w_2)\rangle \geq{}& \langle u, \eta(w_1,w_2)\rangle - \rho\langle N(Aw_2, Bw_2, Cu) - f, \eta(w_1,w_2)\rangle \\ & - \rho a(w_2, w_1-w_2) - \rho b(u,w_1) + \rho b(u,w_2).\end{aligned} \tag{3.13}$$

Adding these inequalities, we deduce that

$$\begin{aligned}\tau\|w_1 - w_2\|^2 \leq{}& -\rho\langle N(Aw_1, Bw_1, Cu) - N(Aw_2, Bw_1, Cu), \eta(w_1,w_2)\rangle \\ & - \rho\langle N(Aw_2, Bw_1, Cu) - N(Aw_2, Bw_2, Cu), \eta(w_1,w_2)\rangle \\ & - \rho a(w_1 - w_2, w_1 - w_2) \\ \leq{}& -\rho(\alpha + c)\|w_1 - w_2\|^2,\end{aligned} \tag{3.14}$$

which yields $w_1 = w_2$. That is, $\widehat{w}$ is the unique solution of (3.1). This completes the proof. □

By Theorem 3.1, we suggest the following algorithms for solving the generalized nonlinear variational-like inequality (2.2).

Algorithm 3.2. *Suppose that* $a : K \times K \to (-\infty,\infty)$ *satisfies* (C1), (C2), $b : K \times K \to (-\infty,\infty)$ *satisfies* (C3)–(C6), *and* $A, B, C : K \to H$, $N : H \times H \times H \to H$ *and* $\eta : K \times K \to H^*$ *are mappings. For given* $f \in H$ *and* $u_0 \in K$, *compute the sequence* $\{u_n\}_{n\geq 0} \subset K$ *by the following iterative scheme:*

$$\begin{aligned}\langle u_{n+1}, \eta(v,u_{n+1})\rangle \geq{}& \langle u_n, \eta(v,u_{n+1})\rangle - \rho\langle N(Au_{n+1}, Bu_{n+1}, Cu_n) - f, \eta(v,u_{n+1})\rangle \\ & - \rho a(u_{n+1}, v-u_{n+1}) - \rho b(u_n,v) + \rho b(u_n,u_{n+1}) + \langle e_n, \eta(v,u_{n+1})\rangle\end{aligned} \tag{3.15}$$

*for all* $v \in K$ *and* $n \geq 0$, *where* $\{e_n\}_{n\geq 0} \subset H$ *and* $\rho > 0$ *is a constant.*

## 4. Existence and convergence

In this section, we prove the existence of solution for the generalized nonlinear variational-like inequality (2.2) and discuss the convergence of the sequence generated by Algorithm 3.2.

THEOREM 4.1. *Let $a$, $b$, $A$, $B$, $N$, $\eta$ be as in Theorem 3.1. Let $C : K \to H$ be Lipschitz continuous with constant $\xi$. Assume that $N$ is Lipschitz continuous with constant $\sigma$ in the third argument and strongly monotone with constant $\beta$ with respect to $C$ in the third argument and*

$$\sigma\xi \geq \beta, \qquad k = \frac{l-\alpha-c}{\delta}, \qquad p = \frac{\tau}{\delta}, \qquad \lim_{n\to\infty} \|e_n\| = 0. \tag{4.1}$$

*If there exist a constant $\rho$ satisfying*

$$0 < \rho < \frac{\tau}{l-\alpha-c}, \tag{4.2}$$

*and one of the following conditions:*

$$\begin{gathered} \left|\rho - \frac{\beta - pk}{\sigma^2\xi^2 - k^2}\right| < \frac{\sqrt{(\beta-pk)^2 - (\sigma^2\xi^2 - k^2)(1-p^2)}}{\sigma^2\xi^2 - k^2}, \\ \sigma\xi > k, \quad |\beta - pk| > \sqrt{(\sigma^2\xi^2 - k^2)(1-p^2)}, \\ \left|\rho - \frac{pk-\beta}{k^2 - \sigma^2\xi^2}\right| > \frac{\sqrt{(\beta-pk)^2 + (k^2 - \sigma^2\xi^2)(1-p^2)}}{k^2 - \sigma^2\xi^2}, \quad \sigma\xi < k, \end{gathered} \tag{4.3}$$

*then the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.2 converges strongly to some $u \in K$ and $u$ is a solution of the generalized nonlinear variational-like inequality (2.2).*

*Proof.* It follows from the proof of Theorem 3.1 that there exists a mapping $G : K \to K$ satisfying $G(u) = w$, where $w$ is the unique solution of (3.1) for each $u \in K$. Next we show that $G$ is a contraction mapping. Let $u_1$ and $u_2$ be arbitrary elements in $K$. Using (3.1), we see that

$$\begin{aligned} \langle Gu_1, \eta(v, Gu_1)\rangle \geq{}& \langle u_1, \eta(v, Gu_1)\rangle - \rho\langle N(A(Gu_1), B(Gu_1), Cu_1) - f, \eta(v, Gu_1)\rangle \\ & - \rho a(Gu_1, v - Gu_1) - \rho b(u_1, v) + \rho b(u_1, Gu_1), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \langle Gu_2, \eta(v, Gu_2)\rangle \geq{}& \langle u_2, \eta(v, Gu_2)\rangle - \rho\langle N(A(Gu_2), B(Gu_2), Cu_2) - f, \eta(v, Gu_2)\rangle \\ & - \rho a(Gu_2, v - Gu_2) - \rho b(u_2, v) + \rho b(u_2, Gu_2) \end{aligned} \tag{4.5}$$

for all $v \in K$. Letting $v = Gu_2$ in (4.4) and $v = Gu_1$ in (4.5), and adding these inequalities, we arrive at

$$\begin{aligned}
\tau\|Gu_1 - Gu_2\|^2 &\le \langle Gu_1 - Gu_2, \eta(Gu_1, Gu_2)\rangle \\
&\le \langle u_1 - u_2 - \rho(N(A(Gu_2), B(Gu_2), Cu_1) \\
&\qquad\qquad - N(A(Gu_2), B(Gu_2), Cu_2)), \eta(Gu_1, Gu_2)\rangle \\
&\quad - \rho\langle N(A(Gu_1), B(Gu_1), Cu_1) \\
&\qquad - N(A(Gu_2), B(Gu_1), Cu_1), \eta(Gu_1, Gu_2)\rangle \\
&\quad - \rho\langle N(A(Gu_2), B(Gu_1), Cu_1) \\
&\qquad - N(A(Gu_2), B(Gu_2), Cu_1), \eta(Gu_1, Gu_2)\rangle \\
&\quad - \rho a(Gu_1 - Gu_2, Gu_1 - Gu_2) + \rho b(u_1 - u_2, Gu_2 - Gu_1) \\
&\le \left[\delta\sqrt{1 - 2\rho\beta + (\rho\sigma\xi)^2} + \rho l\right]\|u_1 - u_2\|\|Gu_1 - Gu_2\| \\
&\quad - \rho(\alpha + c)\|Gu_1 - Gu_2\|^2,
\end{aligned} \tag{4.6}$$

that is,

$$\|Gu_1 - Gu_2\| \le \theta\|u_1 - u_2\|, \tag{4.7}$$

where

$$\theta = \frac{\delta\sqrt{1 - 2\rho\beta + (\rho\sigma\xi)^2} + \rho l}{\tau + \rho(\alpha + c)} < 1, \tag{4.8}$$

by (4.2) and one of (4.3). Therefore, $G : K \to K$ is a contraction mapping and has a unique fixed point $u \in K$. It follows from (3.1) that

$$\begin{aligned}
\langle u, \eta(v, u)\rangle \ge \langle u, \eta(v, u)\rangle &- \rho\langle N(Au, Bu, Cu) - f, \eta(v, u)\rangle \\
&- \rho a(u, v - u) - \rho b(u, v) + \rho b(u, u) \quad \forall v \in K,
\end{aligned} \tag{4.9}$$

which implies that

$$\langle N(Au, Bu, Cu) - f, \eta(v, u)\rangle + a(u, v - u) \ge b(u, u) - b(u, v) \quad \forall v \in K, \tag{4.10}$$

that is, $u$ is a solution of the generalized nonlinear variational-like inequality (2.2).

Next, we consider the convergence of the iterative sequence generated by Algorithm 3.2. Taking $v = u_{n+1}$ in (4.9) and $v = u$ in (3.15), and adding these inequalities, we have

$$\begin{aligned}
\tau\|u_{n+1}-u\|^2 &\le \langle u_{n+1}-u,\eta(u_{n+1},u)\rangle \\
&\le \langle u_n-u-\rho(N(Au,Bu,Cu_n)-N(Au,Bu,Cu)),\eta(u_{n+1},u)\rangle \\
&\quad -\rho\langle N(Au_{n+1},Bu_{n+1},Cu_n)-N(Au,Bu_{n+1},Cu),\eta(u_{n+1},u)\rangle \\
&\quad -\rho\langle N(Au,Bu_{n+1},Cu_n)-N(Au,Bu,Cu_n),\eta(u_{n+1},u)\rangle \\
&\quad -\rho a(u_{n+1}-u,u_{n+1}-u)+\rho b(u_n-u,u-u_{n+1})+\langle e_n,\eta(u_{n+1},u)\rangle \\
&\le \left[\delta\sqrt{1-2\rho\beta+(\rho\sigma\xi)^2}+\rho l\right]\|u_n-u\|\|u_{n+1}-u\| \\
&\quad -\rho(\alpha+c)\|u_{n+1}-u\|^2+\|e_n\|\|u_{n+1}-u\|
\end{aligned} \tag{4.11}$$

for all $n \ge 1$. That is,

$$\|u_{n+1}-u\| \le \theta\|u_n-u\|+\|e_n\| \to 0 \quad \text{as } n \longrightarrow \infty, \tag{4.12}$$

where $\theta$ is defined by (4.8). It follows from (4.1) and (4.12) that the iterative sequence $\{u_n\}_{n\ge0}$ generated by Algorithm 3.2 converges strongly to $u$. This completes the proof. □

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## References

[1] S. S. Chang, *Variational Inequalitity and Complementarity Theory with Applications*, Shanghai Scientific Technology, Shanghai, 1991.
[2] ______, *On the existence of solutions for a class of quasi-bilinear variational inequalities*, Journal of Systems Science and Mathematical Sciences **16** (1996), no. 2, 136–140 (Chinese).
[3] X. P. Ding, *Algorithm of solutions for mixed-nonlinear variational-like inequalities in reflexive Banach space*, Applied Mathematics and Mechanics **19** (1998), no. 6, 521–529.
[4] ______, *Existence and algorithm of solutions for nonlinear mixed variational-like inequalities in Banach spaces*, Journal of Computational and Applied Mathematics **157** (2003), no. 2, 419–434.
[5] X. P. Ding and K.-K. Tan, *A minimax inequality with applications to existence of equilibrium point and fixed point theorems*, Colloquium Mathematicum **63** (1992), no. 2, 233–247.
[6] X. P. Ding and J.-C. Yao, *Existence and algorithm of solutions for mixed quasi-variational-like inclusions in Banach spaces*, Computers & Mathematics with Applications **49** (2005), no. 5-6, 857–869.
[7] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
[8] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, Studies in Mathematics and Its Applications, vol. 8, North-Holland, Amsterdam, 1981.

[9] N. J. Huang and C.-X. Deng, *Auxiliary principle and iterative algorithms for generalized set-valued strongly nonlinear mixed variational-like inequalities*, Journal of Mathematical Analysis and Applications **256** (2001), no. 2, 345–359.

[10] Z. Liu, L. Debnath, S. M. Kang, and J. S. Ume, *Completely generalized multivalued nonlinear quasi-variational inclusions*, International Journal of Mathematics and Mathematical Sciences **30** (2002), no. 10, 593–604.

[11] ———, *On the generalized nonlinear quasi-variational inclusions*, Acta Mathematica et Informatica Universitatis Ostraviensis **11** (2003), 81–90.

[12] ———, *Sensitivity analysis for parametric completely generalized nonlinear implicit quasivariational inclusions*, Journal of Mathematical Analysis and Applications **277** (2003), no. 1, 142–154.

[13] ———, *Generalized mixed quasivariational inclusions and generalized mixed resolvent equations for fuzzy mappings*, Applied Mathematics and Computation **149** (2004), no. 3, 879–891.

[14] Z. Liu and S. M. Kang, *Comments on the papers involving variational and quasivariational inequalities for fuzzy mappings*, Mathematical Sciences Research Journal **7** (2003), no. 10, 394–339.

[15] ———, *Generalized multivalued nonlinear quasi-variational inclusions*, Mathematische Nachrichten **253** (2003), no. 1, 45–54.

[16] ———, *Convergence and stability of perturbed three-step iterative algorithm for completely generalized nonlinear quasivariational inequalities*, Applied Mathematics and Computation **149** (2004), no. 1, 245–258.

[17] Z. Liu, S. M. Kang, and J. S. Ume, *On general variational inclusions with noncompact valued mappings*, Advances in Nonlinear Variational Inequalities **5** (2002), no. 2, 11–25.

[18] ———, *Completely generalized multivalued strongly quasivariational inequalities*, Publicationes Mathematicae Debrecen **62** (2003), no. 1-2, 187–204.

[19] ———, *Generalized variational inclusions for fuzzy mappings*, Advances in Nonlinear Variational Inequalities **6** (2003), no. 1, 31–40.

[20] ———, *The solvability of a class of quasivariational inequalities*, Advances in Nonlinear Variational Inequalities **6** (2003), no. 2, 69–78.

[21] Z. Liu, J. S. Ume, and S. M. Kang, *General strongly nonlinear quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings*, Journal of Optimization Theory and Applications **114** (2002), no. 3, 639–656.

[22] ———, *Resolvent equations technique for general variational inclusions*, Proceedings of the Japan Academy. Series A. Mathematical Sciences **78** (2002), no. 10, 188–193.

[23] ———, *Nonlinear variational inequalities on reflexive Banach spaces and topological vector spaces*, International Journal of Mathematics and Mathematical Sciences **2003** (2003), no. 4, 199–207.

[24] ———, *Completely generalized quasivariational inequalities*, Advances in Nonlinear Variational Inequalities **7** (2004), no. 1, 35–46.

[25] ———, *Completely generalized quasivariational inequalities*, International Journal of Pure and Applied Mathematics **16** (2004), no. 1, 101–112.

[26] ———, *General variational inclusions and general resolvent equations*, Bulletin of the Korean Mathematical Society **41** (2004), no. 2, 241–256.

[27] ———, *Stability of Noor iterations with errors for generalized nonlinear complementarity problems*, Acta Mathematica. Academiae Paedagogicae Nyíregyháziensis. New Series **20** (2004), no. 1, 53–61.

[28] ———, *Generalized nonlinear variational-like inequalities in reflexive Banach spaces*, Journal of Optimization Theory and Applications **126** (2005), no. 1, 157–174.

[29] Z. Liu, Li Wang, S. M. Kang, and J. S. Ume, *Sensitivity analysis of solutions for generalized parametric nonlinear variational inequalities*, International Journal of Pure and Applied Mathematics **6** (2003), no. 1, 77–91.

[30] P. D. Panagiotopoulos and G. E. Stavroulakis, *New types of variational principles based on the notion of quasidifferentiability*, Acta Mechanica **94** (1992), no. 3-4, 171–194.

[31] J. Parida and A. Sen, *A variational-like inequality for multifunctions with applications*, Journal of Mathematical Analysis and Applications **124** (1987), no. 1, 73–81.

[32] A. H. Siddiqi and Q. H. Ansari, *Strongly nonlinear quasivariational inequalities*, Journal of Mathematical Analysis and Applications **149** (1990), no. 2, 444–450.

[33] ______, *General strongly nonlinear variational inequalities*, Journal of Mathematical Analysis and Applications **166** (1992), no. 2, 386–392.

[34] G. Q. Tian, *Generalized quasi-variational-like inequality problem*, Mathematics of Operations Research **18** (1993), no. 3, 752–764.

[35] J. C. Yao, *The generalized quasi-variational inequality problem with applications*, Journal of Mathematical Analysis and Applications **158** (1991), no. 1, 139–160.

[36] ______, *Existence of generalized variational inequalities*, Operations Research Letters **15** (1994), no. 1, 35–40.

[37] L.-C. Zeng, S.-M. Guu, and J.-C. Yao, *Iterative algorithm for completely generalized set-valued strongly nonlinear mixed variational-like inequalities*, Computers & Mathematics with Applications **50** (2005), no. 5-6, 935–945.

[38] L. C. Zeng, S. Schaible, and J. C. Yao, *Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities*, Journal of Optimization Theory and Applications **124** (2005), no. 3, 725–738.

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