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Research Article

Generalizations of Morphic Group Rings

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An element a in a ring R is called left morphic if there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$. R is called left morphic if every element of R is left morphic. An element a in a ring R is called left π-morphic (resp., left G-morphic) if there exists a positive integer n such that a^n (resp., a^n with $a^n \neq 0$) is left morphic. R is called left π-morphic (resp., left G-morphic) if every element of R is left π-morphic (resp., left G-morphic). In this paper, the G-morphic problem and π-morphic problem of group rings are studied.

1. Introduction

An element a in a ring R is said to be left morphic if $R/Ra \cong \mathbf{l}_R(a)$, which is equivalent to that there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$, where $\mathbf{l}_R(a)$ denotes the left annihilator of a in R. R is called left morphic if every element of R is left morphic. Right morphic elements and rings are defined analogously. Nicholson and Sánchez Campos introduced and investigated left morphic rings in [1] (see also [2–4] for more detailed discussion).

Left morphic rings are generalized to left π-morphic rings and left G-morphic rings by Huang and Chen [5]. An element $a \in R$ is called left π-morphic (resp., left G-morphic) if there exists a positive integer n such that a^n (resp., a^n with $a^n \neq 0$) is left morphic. R is called left π-morphic (resp., left G-morphic) if every element of R is left π-morphic (resp., left G-morphic). R is called π-morphic (resp., G-morphic) if it is left and right π-morphic (resp., left and right G-morphic). Moreover, they find examples which show that left π-morphic rings are proper generalizations of left morphic rings, and left G-morphic elements need not be left morphic.

Example 1.1 [5, Example 2.13]. Let $R = F[x,\sigma]/(x^2) = \{a + xb \mid a, b \in F\}$, where F is a field with an isomorphism σ from F to a subfield $\overline{F} \neq F$ and $cx = x\sigma(c)$ for all $c \in F$.

$S = R \oplus R$, then $\lambda = (1, xb) \in S$ (where $b \in F$, but $b \notin \overline{F}$) is left G-morphic, but not left morphic.

The question of when a group ring is morphic was studied by Chen et al. [6]. In this paper, we investigate when a group ring is π-morphic (resp., G-morphic). In Section 2, several general results about π-morphic and G-morphic group rings are obtained. In Section 3, necessary and sufficient conditions for RG to be left G-morphic are also given, where $R = \mathbb{Z}_n$, G is a finite Abelian group. In particular, we prove that if G is a finite Abelian group or a finite p-group, $r \geqslant 1$, then $\mathbb{Z}_{p^r}G$ is π-morphic.

All rings in this paper are associative rings with identity. Let R be a ring and let G be a group. We denote by RG the group ring of G over R. The following concepts in group rings play very important roles in our discussion and will be used frequently later. For any element $u = \Sigma a_i g_i \in RG$, where $a_i \in R$, $g_i \in G$, the augmentation of u, denoted by $\epsilon(u)$, is defined by $\epsilon(u) = \Sigma a_i$. The augmentation ideal of RG, denoted by $\Delta(G)$, is defined by $\Delta(G) = \{u \in RG \mid \epsilon(u) = 0\}$. If G is a cyclic group generated by g, then $\Delta(G) = RG(1-g)$. For any finite subgroup H of G, $\widehat{H}$ is defined to be $\widehat{H} = \sum_{\forall h \in H} h$. When H is a normal subgroup, $\widehat{H}$ is a central element in RG. For any group element $g \in G$ of finite order, define $\widehat{g}$ by $\widehat{g} = 1 + g + \cdots + g^{o(g)-1}$, where $o(g)$ is the order of g. It is not hard to verify that if $o(g) < \infty$, then $\mathbf{l}_{RG}(1-g) = RG\widehat{g}$, and if $|G| < \infty$, then $\mathbf{l}_{RG}(\widehat{G}) = \Delta(G)$. So if G is a finite cyclic group, then $\widehat{G}$ is always left morphic in RG. For more background knowledge about group rings, we refer readers to [7, 8].

2. General results

In this section, several general results about π-morphic and G-morphic group rings are given.

Theorem 2.1. *Let R be a ring and let G be a locally finite group. If RG is left π-morphic (resp., left G-morphic), then R is left π-morphic (resp., left G-morphic).*

Proof. For any $a \in R$, since a is left π-morphic (resp., left G-morphic) in RG, there exist a positive integer n (resp., $a^n \neq 0$) and $u \in RG$ such that $\mathbf{l}_{RG}(a^n) = RGu$ and $\mathbf{l}_{RG}(u) = RGa^n$. Let $u = \sum_{i=1}^{n} a_i g_i$ and $H = \langle g_1, \ldots, g_n \rangle$. Since G is a locally finite group, H is a finite group. Since $a^n u = u a^n = 0$, we have $a^n \epsilon(u) = \epsilon(a^n u) = 0$ and $\epsilon(u) a^n = \epsilon(u a^n) = 0$, where $\epsilon(u)$ is the augmentation of u. Thus $Rb \subseteq \mathbf{l}_R(a^n)$ and $Ra^n \subseteq \mathbf{l}_R(b)$, where $b = \epsilon(u)$. Next we show that in fact, $Rb = \mathbf{l}_R(a^n)$ and $Ra^n = \mathbf{l}_R(b)$. So a is left π-morphic (resp., left G-morphic) in R, and thus R is left π-morphic (resp., left G-morphic).

Let $x \in \mathbf{l}_R(a^n)$. Then $x \in \mathbf{l}_{RG}(a^n) = RGu$, so $x = vu$, $v \in RG$. Taking the augmentation on both sides, we obtain $x = \epsilon(x) = \epsilon(vu) = \epsilon(v)\epsilon(u) = \epsilon(v) b \in Rb$. Therefore, $\mathbf{l}_R(a^n) \subseteq Rb$, and thus $\mathbf{l}_R(a^n) = Rb$. Next, let $y \in \mathbf{l}_R(b)$. Then $yb = 0$. Let $\widehat{H} = \sum_{h \in H} h$. Since $u \in RH$, we have $\widehat{H} u = \epsilon(u)\widehat{H} = b\widehat{H}$. Thus $y\widehat{H}u = yb\widehat{H} = 0$, so $y\widehat{H} \in \mathbf{l}_{RG}(u) = RGa^n$. Hence $y\widehat{H} = \Sigma a_g g a^n$. Comparing the coefficients of the identity on both sides, we obtain that $y = a_e a^n \in Ra^n$, and so $\mathbf{l}_R(b) \subseteq Ra^n$. This implies that $\mathbf{l}_R(b) = Ra^n$. Therefore, a is left π-morphic (resp., left G-morphic) and so is R. □

Corollary 2.2. *If $G = H \times K$ is a locally finite group and RG is left π-morphic (resp., left G-morphic), then RH and RK are both left π-morphic (resp., left G-morphic).*

Proof. Note that $RG = R(H \times K) \cong (RH)K$. By Theorem 2.1, RH is left π-morphic (resp., left G-morphic). Similarly RK is left π-morphic (resp., left G-morphic). $\square$

THEOREM 2.3. *Let G be a locally finite group. If RH is left π-morphic (resp., left G-morphic) for every finite subgroup H of G, then RG is left π-morphic (resp., left G-morphic).*

Proof. Let $u = \sum_{i=1}^{n} a_i g_i$. Now we show that u is left π-morphic (resp., left G-morphic) in RG. Denote $H = \langle g_1, \ldots, g_n \rangle$. Since G is locally finite, H is a finite group. By the assumption, RH is left π-morphic (resp., left G-morphic). Since $u \in RH$, there exist a positive integer n (resp., $u^n \neq 0$) and $c \in RH$ such that $\mathbf{l}_{RH}(u^n) = RHc$ and $\mathbf{l}_{RH}(c) = RHu^n$. Since $u^n c = cu^n = 0$, we have $RGc \subseteq \mathbf{l}_{RG}(u^n)$ and $RGu^n \subseteq \mathbf{l}_{RG}(c)$. We next show that the other inclusions also hold.

Let $v \in \mathbf{l}_{RG}(u^n)$ and let $\{1, g_1', g_2', \ldots\}$ be a left coset representative of H in G. That is, $G = H \cup g_1'H \cup g_2'H \cup \cdots$. Now v can be written as $v = \sum g_i' b_i$, where $b_i \in RH$. Since $0 = vu^n = \sum g_i'(b_i u^n)$ and $b_i u^n \in RH$, we obtain that $b_i u^n = 0$ for all i. So $b_i \in \mathbf{l}_{RH}(u^n) = RHc$, and thus $b_i = c_i c$ for some $c_i \in RH$. It follows that $v = \sum g_i' b_i = \sum (g_i' c_i) c \in RGc$, so $\mathbf{l}_{RG}(u^n) \subseteq RGc$, and thus $\mathbf{l}_{RG}(u^n) = RGc$. Similarly, we can prove that $\mathbf{l}_{RG}(c) = RGu^n$. This shows that u is left π-morphic (resp., left G-morphic) in RG, and therefore RG is left π-morphic (resp., left G-morphic). $\square$

Recall that a group G is called a semidirect product of H by K, denoted by $G = H \rtimes K$, if H, K are subgroups of G such that (1) $H \trianglelefteq G$; (2) $HK = G$; (3) $H \cap K = 1$.

THEOREM 2.4. *Let $G = H \rtimes K$, $|H| < \infty$. If RG is left π-morphic (resp., left G-morphic), then RK is also left π-morphic (resp., left G-morphic).*

Proof. We show that for any $a \in RK$, a is left π-morphic (resp., left G-morphic) in RK. Since a is left π-morphic (resp., left G-morphic) in RG, there exist a positive integer n (resp., $a^n \neq 0$) and $u \in RG$ such that $\mathbf{l}_{RG}(a^n) = RGu$ and $\mathbf{l}_{RG}(u) = RGa^n$. Let $u = \sum u_i k_i$, where $u_i \in RH$, $k_i \in K$ (since $G = H \rtimes K$, the expression of u is unique) and $a^n = \sum a_j k_j$ where $a_j \in R$. Denote $b = \sum \epsilon(u_i) k_i$, so $b \in RK$. We will show that $\mathbf{l}_{RK}(a^n) = RKb$ and $\mathbf{l}_{RK}(b) = RKa^n$. So a is left π-morphic (resp., left G-morphic) in RK, and thus RK is left π-morphic (resp., left G-morphic).

Let $\omega : G \to G/H$ be the natural group homomorphism. We extend ω to a ring homomorphism (still denote it by ω). That is, $\omega : RG \to R(G/H)$ defined by $\omega(\sum a_i g_i) = \sum a_i \omega(g_i)$. Clearly, $\ker(\omega) \cap RK = \{0\}$ and $\omega(v) = \epsilon(v)$ for all $v \in RH$. Since $0 = a^n u$, we have $0 = \omega(a^n)\omega(u) = \omega(a^n)\omega(\sum u_i k_i) = \omega(a^n) \sum \epsilon(u_i)\omega(k_i) = \omega(a^n \sum \epsilon(u_i) k_i) = \omega(a^n b)$. Since $a^n b \in RK$, we conclude that $a^n b = 0$. Similarly, $ba^n = 0$. This shows that $RKb \subseteq \mathbf{l}_{RK}(a^n)$ and $RKa^n \subseteq \mathbf{l}_{RK}(b)$. We next show that the other inclusions also hold.

Let $x \in \mathbf{l}_{RK}(a^n)$. Then $x \in \mathbf{l}_{RG}(a^n) = RGu$. So $x = vu$. Let $v = \sum v_j k_j$ and $c = \sum \epsilon(v_j) k_j$, where $v_j \in RH$, $k_j \in K$. Then $\omega(x) = \omega(v)\omega(u) = \sum \epsilon(v_j)\omega(k_j) \sum \epsilon(u_i)\omega(k_i) = \omega(cb)$. Thus $x - cb \in \ker \omega \cap RK = \{0\}$. Therefore $x = cb \in RKb$. This shows that $\mathbf{l}_{RK}(a^n) \subseteq RKb$, and thus $\mathbf{l}_{RK}(a^n) = RKb$.

Let $y \in \mathbf{l}_{RK}(b)$. Then $yb = 0$. Since $H \trianglelefteq G$, $\widehat{H} = \sum_{h \in H} h$ is central in RG. Now we have $y\widehat{H}u = y\widehat{H} \sum u_i k_i = y \sum \widehat{H}\epsilon(u_i) k_i = y\widehat{H}b = yb\widehat{H} = 0$. So $y\widehat{H} \in \mathbf{l}_{RG}(u) = RGa^n$. Thus $\widehat{H}y = y\widehat{H} = wa^n$, where $w = \sum h_j u_j$, $h_j \in H$, $u_j \in RK$. Hence

$$\sum h_j y = \hat{H} y = wa^n = \sum h_j (u_j a^n). \tag{2.1}$$

Since $H \cap K = \{1\}$, the expression of wa^n is unique. Comparing the coefficients of the identity $h_0 = e$ in (2.1), we obtain $y = u_0 a^n \in RKa^n$. Thus $\mathbf{l}_{RK}(b) \subseteq RKa^n$, and therefore $\mathbf{l}_{RK}(b) = RKa^n$. □

From now on, we always assume that G is a finite group.

Proposition 2.5. *Assume that p is a prime number and $r > 1$. If $\mathbb{Z}_{p^r}G$ is left G-morphic, then p does not divide $|G|$.*

Proof. Assume that $p \mid |G|$. Then there exists $g \in G$ such that $o(g) = p$. Let $u = p^{r-1}\hat{G}$, where $\hat{G} = \sum_{g \in G} g$. Since u is left G-morphic in $\mathbb{Z}_{p^r}G$, there exists a positive integer n such that u^n is left morphic in $\mathbb{Z}_{p^r}G$. Since $u^2 = 0$, u is left morphic in $\mathbb{Z}_{p^r}G$. By Chen et al. [6, Theorem 2.7], this is impossible. So $p \nmid |G|$. □

Theorem 2.6. *Assume that p is a prime number and G is a finite p-group. $\mathbb{Z}_{p^r}G$ is left G-morphic if and only if G is a cyclic group and $r = 1$.*

Proof. "⇒" It follows from Proposition 2.5 that $r = 1$. Since $R = \mathbb{Z}_p$ is a field and G is a finite p-group, RG is a local ring by Nicholson theorem [9]. Because RG is left Artinian, the Jacobson radical $J(RG)$ is nilpotent. Since RG is left G-morphic, RG is left special by Huang and Chen [5, Theorem 2.8]. So it is left morphic. According to Chen et al. [6, Theorem 2.9], G is a cyclic group.

"⇐" If $G = \langle g \rangle$, clearly $\mathbb{Z}_pG$ is a special ring. Therefore it is left G-morphic. □

Theorem 2.7. *Assume that p is a prime number and G is a finite p-group, $r \geqslant 1$, then $\mathbb{Z}_{p^r}G$ is π-morphic.*

Proof. Since $R = \mathbb{Z}_{p^r}$ is local and G is a finite p-group, RG is a local ring by Nicholson'S theorem [9]. Because R is Artinian and G is a finite group, RG is Artinian by Connell [10, Theorem 1], and so the Jacobson radical $J(RG)$ is nilpotent. According to Huang and Chen [5, Lemma 2.10], every element of RG is either nilpotent or invertible. So RG is π-morphic. □

Remark 2.8. By Theorem 2.6, when $r > 1$ and G is a finite p-group, $\mathbb{Z}_{p^r}G$ is not left G-morphic, but by the above theorem, it is π-morphic.

3. Abelian group rings

In this section, we discuss when an Abelian group ring RG is left π-morphic (resp., left G-morphic).

Lemma 3.1 [6, Lemma 3.1]. $(R_1 \oplus R_2 \oplus \cdots \oplus R_s)G \cong \oplus_{i=1}^{s} R_iG$.

Lemma 3.2. *If $R = R_1 \oplus R_2 \oplus \cdots \oplus R_s$ is left π-morphic (resp., left G-morphic), then each R_i is left π-morphic (resp., left G-morphic).*

Proof. For any $r_i \in R_i$, $r = (0,\dots,0,r_i,0,\dots,0) \in R$. Since R is left π-morphic (resp., left G-morphic), there exist $u = (u_1,\dots,u_{i-1},u_i,\dots,u_s) \in R$, where $u_k \in R_k$, $k = 1,\dots,s$, and a positive integer n (resp., $r^n \neq 0$) such that $\mathbf{l}_R(u) = Rr^n$ and $\mathbf{l}_R(r^n) = Ru$, so we have $\mathbf{l}_{R_i}(u_i) = R_i r_i^n$ and $\mathbf{l}_{R_i}(r_i^n) = R_i u_i$. Then r_i is left π-morphic (resp., left G-morphic) in R_i, and thus R_i is left π-morphic (resp., left G-morphic). □

Lemma 3.3. *Let D be a division ring and $s \geqslant 2$. The the following statements are equivalent:*

(1) *$D(C_{m_1} \times \cdots \times C_{m_s})$ is left G-morphic;*

(2) *$D(C_{m_i} \times C_{m_j})$ is left G-morphic for any $1 \leqslant i \neq j \leqslant s$;*

(3) *at most one of $m_1, m_2, \ldots, m_s$ is not invertible in D.*

Proof. We will prove (3)⇒(1)⇒(2)⇒(3).

"(3)⇒(1)" We may assume that $m_1, \ldots, m_{s-1}$ are invertible in D. So $|C_{m_1} \times \cdots \times C_{m_{s-1}}| = m_1 \times \cdots \times m_{s-1}$ is invertible in D. By Maschke's theorem, $D(C_{m_1} \times \cdots \times C_{m_{s-1}})$ is semisimple. It follows from [6, Lemma 3.5] that $D(C_{m_1} \times \cdots \times C_{m_{s-1}} \times C_{m_s})$ is strongly morphic, so it is G-morphic and (2.1) holds.

"(1)⇒(2)" Note that $D(C_{m_1} \times \cdots \times C_{m_s}) \cong D(C_{m_i} \times C_{m_j})(\prod_{k \neq i,j} C_{m_k})$ for any $1 \leqslant i \neq j \leqslant s$. It follows from Theorem 2.1 that $D(C_{m_i} \times C_{m_j})$ is left G-morphic.

"(2)⇒(3)" We prove it by contradiction. We may assume that m_1, m_2 are not invertible in D. Let $\operatorname{char}(D) = p > 0$. By assumption, p divides both m_1 and m_2. So we have $m_i = p^{r_i} t_i$, where $(t_i, p) = 1$, $r_i \geqslant 1$, $i = 1, 2$.

Note that $C_{m_1} \times C_{m_2} \cong (C_{p^{r_1}} \times C_{p^{r_2}}) \times (C_{t_1} \times C_{t_2})$, so $D(C_{m_1} \times C_{m_2}) \cong D(C_{p^{r_1}} \times C_{p^{r_2}}) \times (C_{t_1} \times C_{t_2})$. Since $D(C_{m_1} \times C_{m_2})$ is left G-morphic, $D(C_{p^{r_1}} \times C_{p^{r_2}})$ is left G-morphic by Theorem 2.1. Because $C_{p^{r_1}} \times C_{p^{r_2}}$ is a finite p-group, $D(C_{p^{r_1}} \times C_{p^{r_2}})$ is a local Artinian ring, so the Jacobson radical of this group ring is nilpotent. This ring is a left special ring, and then it is left morphic by Huang and chen [5, Theorem 2.8]. Thus $C_{p^{r_1}} \times C_{p^{r_2}}$ must be cyclic, a contradiction. □

Proposition 3.4. *Let G be a finite Abelian group and $r > 1$. Then $\mathbb{Z}_{p^r} G$ is G-morphic if and only if $(p, |G|) = 1$.*

Proof. "⇐" By Chen et al. [6, Corollary 3.13], if $(p, |G|) = 1$, $\mathbb{Z}_{p^r} G$ is morphic, so it is G-morphic.

"⇒" By Proposition 2.5, if $r > 1$ and $\mathbb{Z}_{p^r} G$ is G-morphic, then $p \nmid |G|$, that is, $(p, |G|) = 1$. □

Theorem 3.5. *Let G be a finite Abelian group. $\mathbb{Z}_n G$ is G-morphic if and only if for each prime number p if $p \mid (n, |G|)$, then $p^2 \nmid n$ and the Sylow p-subgroup G_p of G is cyclic.*

Proof. Let $G = C_{q_1^{t_1}} \times \cdots \times C_{q_m^{t_m}}$, $t_i \geqslant 1$ be a finite Abelian group and let $\alpha = q_1 \cdots q_m$. Suppose that $\mathbb{Z}_n G$ is G-morphic. Let $(n, |G|) = p_1^{r_1} \cdots p_s^{r_s}$. If $r_i > 1$ for some i (i.e., $p_i^2 \mid n$), then $n = p_i^{s_i} n_1$, where $s_i \geqslant r_i > 1$ and $(n_1, p_i) = 1$. Thus $\mathbb{Z}_n G \cong \mathbb{Z}_{p_i^{s_i}} G \oplus \mathbb{Z}_{n_1} G$. Since $\mathbb{Z}_n G$ is G-morphic, $\mathbb{Z}_{p_i^{s_i}} G$ is also G-morphic by Lemma 3.2. By Proposition 3.4, $(p_i, |G|) = 1$. However, $p_i \mid (n, |G|)$. This leads to a contradiction. Thus $r_i \leqslant 1$ for all i. Next we show that $p_i^2 \nmid \alpha$. Otherwise, assume that $p_i^2 \mid \alpha$. There exists $k \neq l$ such that $q_k = q_l = p_i$. Hence $G \cong C_{q_k^{t_k}} \times C_{q_l^{t_l}} \times H$. Since $p_i \mid n$ and $p_i^2 \nmid n$, we have $n = p_i n_1$ with $(p_i, n_1) = 1$. So $\mathbb{Z}_n G \cong$

$\mathbb{Z}_{p_i}G \oplus \mathbb{Z}_{n_1}G$. By Lemma 3.2, $\mathbb{Z}_{p_i}G$ is G-morphic. Since $\mathbb{Z}_{p_i}G \cong \mathbb{Z}_{p_i}(C_{q_k^{t_k}} \times C_{q_l^{t_l}})H$, we conclude that $\mathbb{Z}_{p_i}(C_{q_k^{t_k}} \times C_{q_l^{t_l}}) = \mathbb{Z}_{p_i}(C_{p_i^{t_k}} \times C_{p_i^{t_l}})$ is G-morphic. This contradicts the result of Theorem 2.6. Therefore, $p_i^2 \nmid \alpha$, and thus G_{p_i} is cyclic. $\square$

Remark 3.6. According to Proposition 3.4 and Theorem 3.5, the following group rings are not G-morphic:

$$\mathbb{Z}_4C_2, \qquad \mathbb{Z}_4C_4, \qquad \mathbb{Z}_4(C_2 \times C_2), \qquad \mathbb{Z}_2(C_2 \times C_2), \qquad \mathbb{Z}_2(C_2 \times C_4). \tag{3.1}$$

But by Theorem 2.7, the above group rings are all π-morphic.

Lemma 3.7. *Let R be a ring and let G be a group. If $a \in R$ is left morphic in R, then a is left morphic in RG.*

Proof. If $a \in R$ is left morphic, there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$. Since $ba = ab = 0$, we have $RGb \subseteq \mathbf{l}_{RG}(a)$ and $RGa \subseteq \mathbf{l}_{RG}(b)$. We next show that the other inclusions also hold.

Let $x \in \mathbf{l}_{RG}(a)$, $x = \sum r_j g_j$, where $r_j \in R$, $g_j \in G$. Then $\sum r_j g_j a = 0$ or $\sum (r_j a) g_j = 0$, so all $r_j a = 0$. Thus $r_j \in Rb$ and $r_j = r_j' b$, $r_j' \in R$. Therefore, $x = \sum (r_j' b) g_j = \sum r_j' g_j b \in RGb$. This shows that $\mathbf{l}_{RG}(a) \subseteq RGb$, and thus $\mathbf{l}_{RG}(a) = RGb$.

Using a similar proof, we can show that $\mathbf{l}_{RG}(b) \subseteq RGa$, and thus $\mathbf{l}_{RG}(b) = RGa$. So a is left morphic in RG. $\square$

Recall that if $n = p^u n_1$, $(n_1, p) = 1$, we denote that $p^u \| n$.

Lemma 3.8. *Let p be a prime number, $r \geqslant 1$, $p^r \| m$, and $1 \leqslant n \leqslant m$.*
 (1) *If $(p,n) = 1$, then $p^r \mid C_m^n$.*
 (2) *If $p^t \| n$, $r \geqslant t$, then $p^{r-t} \mid C_m^n$.*

Proof. Let $m = m_1 p^r$, $(m_1, p) = 1$. Then

$$C_m^n = \frac{m(m-1)\cdots(m-(n-1))}{1\cdots(n-1)n} = \frac{m}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n} C_{m-1}^{n-1}. \tag{3.2}$$

 (1) If $(p,n) = 1$, then $(p^r, n) = 1$, so $p^r \mid C_m^n$.
 (2) If $p^t \| n$, $t \leqslant r$, then $n = n_1 p^t$, where $(p, n_1) = 1$, so

$$C_m^n = \frac{m_1 p^r}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n_1 p^t} C_{m-1}^{n-1} = \frac{m_1 p^{r-t}}{n_1} C_{m-1}^{n-1}. \tag{3.3}$$

We have $p^{r-t} \mid C_m^n n_1$. Since $(p, n_1) = 1$, $(p^{r-t}, n_1) = 1$, so $p^{r-t} \mid C_m^n$. $\square$

Proposition 3.9. *Let p be a prime number and let G be a finite Abelian group. If for some r, $t \geqslant 1$, $x \in \mathbb{Z}_{p^r}(C_{p^t} \times G) = \mathbb{Z}_{p^r}(\langle g \rangle \times G)$, then $x^{p^r} \in \mathbb{Z}_{p^r}(C_{p^{t-1}} \times G) = \mathbb{Z}_{p^r}(\langle g^p \rangle \times G)$.*

Proof. For $x \in \mathbb{Z}_{p^r}(C_{p^t} \times G) = (\mathbb{Z}_{p^r}G)C_{p^t} = (\mathbb{Z}_{p^r}G)\langle g\rangle$, $x = r_0 + r_1 g + \cdots + r_{p^t-1}g^{p^t-1}$, where $r_i \in \mathbb{Z}_{p^r}G$. Since

$$\begin{aligned}
&(x_1 + x_2 + \cdots + x_s)^k \\
&\quad= \sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{s-1}=0}^{k_{s-2}} C_k^{k_1}C_{k_1}^{k_2}\cdots C_{k_{s-2}}^{k_{s-1}}x_1^{k-k_1}x_2^{k_1-k_2}\cdots x_{s-1}^{k_{s-2}-k_{s-1}}x_s^{k_{s-1}}, \\
&x^{p^r} = \left(r_0 + r_1 g + \cdots + r_{p^t-1}g^{p^t-1}\right)^{p^r} \\
&\quad= \sum_{n_1=0}^{p^r}\sum_{n_2=0}^{n_1}\cdots\sum_{n_{p^t-1}=0}^{n_{p^t-2}} C_{p^r}^{n_1}C_{n_1}^{n_2}\cdots C_{n_{p^t-2}}^{n_{p^t-1}} r_0^{p^r-n_1}(r_1 g)^{n_1-n_2}\cdots\left(r_{p^t-1}g^{p^t-1}\right)^{n_{p^t-1}}.
\end{aligned} \tag{3.4}$$

□

Claim 3.10. Let n_i be the first number in $n_1,\ldots,n_{p^t-1}$ such that n_i is not divisible by p. Then $p^r \mid C_{p^r}^{n_1}C_{n_1}^{n_2}\cdots C_{n_{i-1}}^{n_i}$.

Proof. If $i = 1$, then $(n_1, p) = 1$, and by Lemma 3.8, $p^r \mid C_{p^r}^{n_1}$.

Now we set $i > 1$. Let $n_k = n_k' p^{u_k}$, $1 \leqslant k \leqslant i-1$, where $(n_k', p) = 1$. Since $C_{n_{k-1}}^{n_k} = C_{n_{k-1}}^{n_{k-1}-n_k}$, we can assume that $u_k \leqslant u_{k-1}$. By Lemma 3.8, we have $p^{u_{k-1}-u_k} \mid C_{n_{k-1}}^{n_k}$, $1 \leqslant k \leqslant i-1$, and $p^{u_{i-1}} \mid C_{n_{i-1}}^{n_i}$ because $(p, n_i) = 1$. So

$$p^{(r-u_1)+(u_1-u_2)+\cdots+(u_{i-2}-u_{i-1})+u_{i-1}} \mid C_{p^r}^{n_1}C_{n_1}^{n_2}\cdots C_{n_{i-1}}^{n_i}. \tag{3.5}$$

Hence, $p^r \mid C_{p^r}^{n_1}C_{n_1}^{n_2}\cdots C_{n_{i-1}}^{n_i}$.

By the above claim, if there exists n_i such that $p \nmid n_i$, then $C_{p^r}^{n_1}C_{n_1}^{n_2}\cdots C_{n_{i-1}}^{n_i} = 0$ in $\mathbb{Z}_{p^r}$. So assume that $p \mid n_j$, $j = 1,\ldots,p^t-1$, and then we have

$$\begin{aligned}
x^{p^r} &= \sum_{p\mid n_1, 0\leqslant n_1\leqslant p^r}\ \sum_{p\mid n_2, 0\leqslant n_2\leqslant n_1}\cdots\sum_{p\mid n_{p^t-1}, 0\leqslant n_{p^t-1}\leqslant n_{p^t-2}} \\
&\qquad\times C_{p^r}^{n_1}C_{n_1}^{n_2}\cdots C_{n_{p^t-2}}^{n_{p^t-1}} r_0^{p^r-n_1}(r_1 g)^{n_1-n_2}\cdots\left(r_{p^t-1}g^{p^t-1}\right)^{n_{p^t-1}} \\
&= \sum c_i\left(g^p\right)^i \in (\mathbb{Z}_{p^r}G)\langle g^p\rangle = (\mathbb{Z}_{p^r G})C_{p^{t-1}}.
\end{aligned} \tag{3.6}$$

□

THEOREM 3.11. *If p is a prime number, $r \geqslant 1$, and G is a finite Abelian group, then $\mathbb{Z}_{p^r}G$ is π-morphic.*

Proof

Case 1. If $(p, |G|) = 1$, then $(p^r, |G|) = 1$. By Chen et al. [6, Corollary 3.13], $\mathbb{Z}_{p^r}G$ is morphic, so $\mathbb{Z}_{p^r}G$ is π-morphic.

Case 2. If $p \mid |G|$, then $G = C_{p^{t_1}} \times \cdots \times C_{p^{t_s}} \times H$, where $(p, |H|) = 1$. Now if $x \in \mathbb{Z}_{p^r}G = \mathbb{Z}_{p^r}(C_{p^{t_2}} \times \cdots \times C_{p^{t_s}} \times H)C_{p^{t_1}}$, then $x^{p^r} \in \mathbb{Z}_{p^r}(C_{p^{t_2}} \times \cdots \times C_{p^{t_s}} \times H)C_{p^{t_1-1}}$ by Proposition 3.9. So we have $x^{k_1} \in \mathbb{Z}_{p^r}(C_{p^{t_2}} \times \cdots \times C_{p^{t_s}} \times H)$ for some k_1. Continuing the process, we get $x^n \in \mathbb{Z}_{p^r}H$ for some n. By Chen et al. [6, Corollary 3.13], $\mathbb{Z}_{p^r}H$ is morphic. So x^n is morphic in $\mathbb{Z}_{p^r}H$. Thus x^n is morphic in $\mathbb{Z}_{p^r}G$ by Lemma 3.7. Hence x is π-morphic in $\mathbb{Z}_{p^r}G$. □

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