

Research Article

The k -Zero-Divisor Hypergraph of a Commutative Ring

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The concept of the zero-divisor graph of a commutative ring has been studied by many authors, and the k -zero-divisor hypergraph of a commutative ring is a nice abstraction of this concept. Though some of the proofs in this paper are long and detailed, any reader familiar with zero-divisors will be able to read through the exposition and find many of the results quite interesting. Let R be a commutative ring and k an integer strictly larger than 2. A k -uniform hypergraph $H_k(R)$ with the vertex set $Z(R, k)$, the set of all k -zero-divisors in R , is associated to R , where each k -subset of $Z(R, k)$ that satisfies the k -zero-divisor condition is an edge in $H_k(R)$. It is shown that if R has two prime ideals P_1 and P_2 with zero their only common point, then $H_k(R)$ is a bipartite (2-colorable) hypergraph with partition sets $P_1 - Z'$ and $P_2 - Z'$, where Z' is the set of all zero divisors of R which are not k -zero-divisors in R . If R has a nonzero nilpotent element, then a lower bound for the clique number of $H_3(R)$ is found. Also, we have shown that $H_3(R)$ is connected with diameter at most 4 whenever $x^2 \neq 0$ for all 3-zero-divisors x of R . Finally, it is shown that for any finite nonlocal ring R , the hypergraph $H_3(R)$ is complete if and only if R is isomorphic to $Z_2 \times Z_2 \times Z_2$.

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1. Introduction

The notion of a zero-divisor graph $\Gamma(R)$ of a commutative ring R was first introduced by Beck in [1] and was further investigated in [2], where the authors were interested in colorings of $\Gamma(R)$, though their vertex set included the zero element. In [3–9] the authors, using the set of nonzero zero divisors of R as vertex set of $\Gamma(R)$, were interested in examining the interplay between the ring-theoretic properties of R and the graph-theoretic properties

of $\Gamma(R)$. In this paper, we extend the concept of a zero-divisor of a commutative ring R to that of a k -zero-divisor and investigate the interplay between the ring-theoretic properties of R and the graph-theoretic properties of its associated k -uniform hypergraph $H_k(R)$. In this section, we define and study some examples of k -zero-divisors and recall some definitions from graph theory. In Section 2, we define and study some basic properties of the k -uniform hypergraph $H_k(R)$ and k -zero-divisors of a commutative ring R . Finally, in the last section, we merely concentrate on the properties of 3-zero-divisor hypergraphs.

Definition 1.1. Let R be a commutative ring and $k \geq 2$ a fixed integer. A nonzero nonunit element a_1 in R is said to be a k -zero-divisor in R if there exist $k - 1$ distinct nonunit elements a_2, a_3, \dots, a_k in R different from a_1 such that $a_1 a_2 a_3 \cdots a_k = 0$ and the product of no elements of any proper subset of $A = \{a_1, a_2, \dots, a_k\}$ is zero.

Clearly, a 2-zero-divisor in R is a zero divisor, but the converse is not true in general. For example, 2 is a zero divisor in Z_4 , but it is not a 2-zero-divisor.

Remark 1.2. In the literature, on zero-divisor graphs, the edges are defined to be between the distinct nonzero zero-divisors in order to construct a graph with no loops. Here, we assume distinctness of the elements in Definition 1.1 for k -zero-divisors in order to have a k -uniform hypergraph, for any fixed integer $k \geq 3$. Note that the graph constructed by 2-zero-divisors is exactly the same as the zero-divisor graph of a ring.

Example 1.3. The element 2 in Z_{30} is a 3-zero-divisor since $2 \cdot 3 \cdot 5 = 0$, and the product of no elements of any proper subset of $\{2, 3, 5\}$ is zero.

By $Z(R, k)$ we denote the set of all k -zero-divisors of R . It is not difficult to show that the statement “the product of no elements of any proper subset of A is zero” or the statement “the product of no elements of any $(k - 1)$ -subset of A is zero” can be used in Definition 1.1 equivalently. Clearly, from Definition 1.1, every element of the set $\{a_2, a_3, \dots, a_k\}$ is a k -zero-divisor in R . It is clear that every k -zero-divisor in R is also a zero divisor in R , but, the converse is not true in general. For example, the element 2 is a zero divisor, but not a 3-zero-divisor in Z_{10} .

We review some basic graph-theoretic definitions, and for the necessary definitions and notations of hypergraphs, we refer the reader to standard texts of graph theory such as [10]. A hypergraph is a pair (V, E) of disjoint sets, where the elements of E are nonempty subsets (of any cardinality) of V . The elements of V are the vertices, and the elements of E are the edges of the hypergraph. The hypergraph $H = (V, E)$ is called k -uniform whenever every edge e of H is of size k . A k -uniform hypergraph H is called complete if every k -subset of the vertices is an edge of H . The definition of a clique and the clique number of a k -uniform hypergraph are taken from [11, 12] as follows.

Let H be a k -uniform hypergraph. A subset A of $V(H)$ is called a *clique* of H if every k -subset of A is an edge of H . The *clique number* of H , denoted by $\omega(H)$, is defined to be

$$\omega(H) = \frac{\max\{|A| \mid A \text{ is a clique}\}}{k - 1}. \quad (1.1)$$

An r -coloring of a hypergraph $H = (V, E)$ is a map $c : V \rightarrow \{1, 2, \dots, r\}$ such that for every edge e of H , there exist at least two vertices x and y in e with $c(x) \neq c(y)$. The smallest integer r such that H has an r -coloring is called the chromatic number of H and is denoted by $\chi(H)$. In [11], it is shown that for any k -uniform hypergraph H , $\chi(H) \geq \lceil \omega(H) \rceil$. A path in a hypergraph H is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \dots, v_k$ such that v_i, v_{i+1} is in e_i for all $1 \leq i \leq k - 1$. The number of edges of a path is its length. The distance between two vertices x and y of H , denoted by $d_H(x, y)$, is the length of the shortest path from x to y . If no such path between x and y exists, we set $d_H(x, y) = \infty$. The greatest distance between any two vertices in H is called the diameter of H and is denoted by $\text{diam}(H)$. The hypergraph H is said to be connected whenever $\text{diam}(H) < \infty$. A cycle in a hypergraph H is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_1$ such that v_i, v_{i+1} are in e_i for all $1 \leq i \leq k - 1$ with $v_k, v_1 \in e_k$. The girth of a hypergraph H containing a cycle, denoted by $\text{gr}(H)$, is the smallest size of the length of cycles of H .

2. k -zero-divisor hypergraphs

In this section, we define and study some properties of the k -uniform hypergraph $H_k(R)$, the k -zero-divisors of a commutative ring R , and provide some examples.

Definition 2.1. A ring R is said to be a k -integral domain whenever $Z(R, k)$, the set of all k -zero-divisors of R , is the empty set.

Example 2.2. Let (R, M) be a local ring with maximal ideal $M \neq 0$ such that $M^2 = 0$. Then R is a 3-integral domain which is not an integral domain.

Example 2.3. For any integer $k \geq 3$, we have the following results.

- (1) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime decomposition of n , where $p_i \neq p_j$ whenever $i \neq j$ and $1 \leq \alpha_i$ for all $i, j = 1, 2, \dots, r$. Then Z_n is a k -integral domain whenever $\sum_{i \leq r} \alpha_i \leq k - 1$.
- (2) Let $n_i = p_{1_i}^{\alpha_{1_i}} p_{2_i}^{\alpha_{2_i}} \cdots p_{r_i}^{\alpha_{r_i}}$ be the prime decomposition of n_i for distinct primes p_{j_i} 's and $1 \leq \alpha_{j_i}$ for all $1 \leq i \leq t$ and $j = 1, 2, \dots, r$. Then $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_t}$ is a k -integral domain whenever

$$\sum_{j \leq r_1} \alpha_{j_1} + \sum_{j \leq r_2} \alpha_{j_2} + \cdots + \sum_{j \leq r_t} \alpha_{j_t} \leq k - 1. \tag{2.1}$$

- (3) Let F be a field and let $f(x)$ be a polynomial in $F[x]$ such that $f(x) = P_1(x)^{\alpha_1} P_2(x)^{\alpha_2} \cdots P_r(x)^{\alpha_r}$, where $P_i(x) \in F[x]$ are distinct irreducible polynomials and $1 \leq \alpha_i$ for all $1 \leq i \leq r$. Then $F[x]/(f(x))$ is a k -integral domain whenever $\sum_{i \leq r} \alpha_i \leq k - 1$.
- (4) Let R_i be an integral domain for each $i = 1, 2, \dots, n$. Then $R = R_1 \times R_2 \times \cdots \times R_n$ is a k -integral domain whenever $n \leq k - 1$.

By [13], it is true that a nonintegral domain with a finite number of zero divisors is finite. Similarly, we pose the following question for the rings with a finite number of k -zero-divisors.

Question 1. Does the finiteness of k -zero-divisors in a non- k -integral domain R imply the finiteness of zero-divisors or, equivalently, finiteness of R ?

Definition 2.4. For any fixed integer $k \geq 3$, an ideal P of a ring R is said to be k -prime whenever for any set $A = \{a_1, a_2, \dots, a_k\}$ of nonzero, distinct, and nonunit elements of R , $a_1 a_2 \cdots a_k \in P$ implies that the product of the elements of a proper subset of A is in P .

Note that by this definition, every prime ideal of R is a k -prime ideal of R .

Example 2.5. Let (R_1, M_1) and (R_2, M_2) be two local rings with nonzero maximal ideals M_1 and M_2 , respectively. We show that $M_1 \times M_2$ is a 3-prime ideal in $R = R_1 \times R_2$ which is not a prime ideal in R . Let (a_1, b_1) , (a_2, b_2) , and (a_3, b_3) be arbitrary elements in $R_1 \times R_2$, where for each $1 \leq i \leq 3$, (a_i, b_i) is a nonzero nonunit in R . Clearly, $(a_1, b_1) \cdot (a_2, b_2) \cdot (a_3, b_3) = (a_1 a_2 a_3, b_1 b_2 b_3) \in M_1 \times M_2$ implies that at least one of the elements a_i 's (b_j 's) belongs to M_1 (M_2) for some i (j) in $\{1, 2, 3\}$. In this case, there always exists a proper subset of $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ such that the product of its elements belongs to $M_1 \times M_2$. But since $(1, 0) \cdot (0, 1) \in M_1 \times M_2$ and neither of the elements $(1, 0)$ and $(0, 1)$ is in $M_1 \times M_2$, then $M_1 \times M_2$ is a 3-prime ideal in $R_1 \times R_2$ which is not a prime ideal in R .

The following theorem is similar to the well-known fact on the relationship between prime ideals and integral domains.

THEOREM 2.6. *Let P be an ideal in the ring R . Then R/P is a k -integral domain if P is a k -prime ideal.*

The proof follows directly from the definition, and we leave it to the reader.

The converse of the above theorem is not true in general. For example, the ideal $\langle 8 \rangle$ generated by 8 in Z_{48} is not a 3-prime ideal, but $Z_{48}/\langle 8 \rangle$ is a 3-integral domain.

Next, we extend the concept of zero-divisor graph of a commutative ring R to that of a k -zero-divisor hypergraph.

Definition 2.7. Let R be a commutative ring (with $1 \neq 0$) and let $Z(R, k)$ be the set of all k -zero-divisors in R . Associate a k -uniform hypergraph $H_k(R)$ to R with vertex set $Z(R, k)$, and for distinct elements x_1, x_2, \dots, x_k in $Z(R, k)$, the set $\{x_1, x_2, \dots, x_k\}$ is an edge of $H_k(R)$ if and only if $x_1 x_2 \cdots x_k = 0$ and the product of elements of no $(k - 1)$ -subset of $\{x_1, x_2, \dots, x_k\}$ is zero.

Clearly, from the above definition we can conclude that for any $k \geq 3$, $H_k(R)$ is the empty set if and only if R is a k -integral domain.

THEOREM 2.8. *Let R be a non- k -integral domain. If there exist prime ideals P_1 and P_2 in R such that $P_1 \cap P_2 = \{0\}$, then $\chi(H_k(R)) = 2$.*

Proof. Since $P_1 \cap P_2 = \{0\}$, then $P_1 \cup P_2$ is equal to the set of all zero divisors of R . On the other hand, since each k -zero-divisor is also a zero divisor, each k -zero-divisor must belong to the prime ideals P_1 or P_2 . Consider the function $c : V(H_k(R)) \rightarrow \{1, 2\}$ given by

$$c(x) = \begin{cases} 1, & x \in P_1, \\ 2, & x \in P_2. \end{cases} \tag{2.2}$$

In order to prove that c is a 2-coloring of $H_k(R)$, we need to show that there is no edge e in $H_k(R)$ such that every vertex of e obtains the same color. Without loss of generality, let $e = \{x_1, x_2, \dots, x_k\}$ be an edge of $H_k(R)$ such that $c(x_1) = c(x_2) = \dots = c(x_k) = 1$. Since $x_1 x_2 \dots x_k = 0 \in P_2$ and P_2 is a prime ideal of R , then $x_i \in P_2$ for at least one $1 \leq i \leq k$, which is a contradiction. Therefore, $\chi(H_k(R)) \leq 2$. On the other hand, since R is not a k -integral domain, then $H_k(R)$ has at least one edge, which implies that $\chi(H_k(R)) \geq 2$, and the proof is complete. \square

Remark 2.9. From the above theorem, it is clear that $H_k(R)$ is a bipartite hypergraph with partition sets $V(H_k(R)) \cap P_1$ and $V(H_k(R)) \cap P_2$. Note that in [4], it is shown that for any reduced ring R , the zero-divisor graph $\Gamma(R)$ is bipartite if and only if there exist two distinct prime ideals P_1 and P_2 of R such that $P_1 \cap P_2 = \{0\}$. In addition, if $\Gamma(R)$ is bipartite, then it is a complete bipartite graph.

Remark 2.10. By considering the ring $R = Z_2 \times Z_2 \times Z_2$, we see that $\chi(H_3(R)) = 2$. But there are no prime ideals P_1 and P_2 in R satisfying the condition of Theorem 2.8. Therefore, the converse of Theorem 2.8 is not true in general.

THEOREM 2.11. *Let $R = R_1 \times R_2 \times \dots \times R_n$, where R_i is an integral domain for each $i = 1, 2, \dots, n$.*

- (1) *If $n = k$, then $\chi(H_k(R)) = 2$.*
- (2) *If $n = k + t$, then $\chi(H_k(R)) \leq 2 + t$ for all $t \geq 0$.*

Proof. Let $k = n$. We claim that

$$Z(R, k) = \{(a_1, a_2, \dots, a_k) \mid \text{exactly one of the } a_i\text{'s is zero for } 1 \leq i \leq k\}. \tag{2.3}$$

It is obvious that any k -zero-divisor must have at least one zero component. Let $x_1 = (a_{11}, a_{12}, \dots, a_{1k})$ be a k -zero-divisor with at least two zero components. Without loss of generality, assume that $a_{11} = a_{12} = 0$. Consequently, there exist $x_2, x_3, \dots, x_k \in V(H_k(R))$ such that $\{x_1, x_2, \dots, x_k\} \in E(H_k(R))$, where $x_i = (a_{i1}, a_{i2}, \dots, a_{ik})$ for all $1 \leq i \leq k$. Thus, $\prod_{i \geq 1} a_{ij} = 0$ for each $j \geq 3$. Now since R_j is an integral domain, then for each fixed $j \geq 3$, there exists at least one i_j with $1 \leq i \leq k$ such that $a_{i_j j} = 0$. Let I be the set of all i_j 's such that $a_{i_j j} = 0$ for the smallest i in the set $\{1, 2, \dots, k\}$. Thus, we have $x_1 \prod_{i \in I} x_i = 0$ and since $|I| \leq k - 2$, we have a contradiction. Now let $x_1 = (a_1, a_2, \dots, a_k) \in R$ such that exactly one and only one of the components is zero. Without loss of generality, assume that $a_1 = 0$. Let $x_i = (1, 1, \dots, 1, 0, 1, 1, \dots, 1)$, where the i th component is the only zero component of x_i for $2 \leq i \leq k$. It is obvious that $\{x_1, x_2, \dots, x_k\} \in E(H_k(R))$ and the claim is true. Consider the function $c : V(H_k(R)) \rightarrow \{1, 2\}$ given by

$$c(x) = \begin{cases} 1 & \text{the first component of } x \text{ is zero,} \\ 2 & \text{otherwise.} \end{cases} \tag{2.4}$$

It is easy to see that c is a 2-coloring of $H_k(R)$, and since $H_k(R)$ has at least one edge, $\chi(H_k(R)) = 2$.

For the proof of part 2, assume $n = k + t$ with $t \geq 0$ a fixed integer. The proof is by induction on t . From part 1, the first step of induction for $t = 0$ is true. Now, assume that

$t \geq 1$ and the result is true for $k + t$. Let $c : V(H_k(R_1 \times R_2 \times \cdots \times R_{k+t})) \rightarrow \{1, 2, \dots, t + 2\}$ be a $t + 2$ -coloring of $H_k(R_1 \times R_2 \times \cdots \times R_{k+t})$. Consider the function $c' : V(H_k(R_1 \times R_2 \times \cdots \times R_{k+t+1})) \rightarrow \{1, 2, \dots, t + 3\}$ given by

$$c'(x) = \begin{cases} c(x) & \text{the last component of } x \text{ is zero,} \\ t + 3 & \text{otherwise.} \end{cases} \tag{2.5}$$

From this, it is not difficult to show that c' is a $(t + 3)$ -coloring of $H_k(R_1 \times R_2 \times \cdots \times R_{k+t+1})$, and the proof is complete. \square

As a very special case of the above theorem, it is easy to show that the chromatic number of $H_3(Z_2^4)$ and $H_3(Z_2^5)$ is 3. Note that the chromatic number of $H_3(Z_2^5)$ is strictly less than $2 + (5 - 3)$, and the chromatic number of $H_3(Z_2^4)$ equal to 3 shows that the bound is sharp.

3. 3-zero-divisor hypergraphs

In this section, we only focus on some graph-theoretic properties of $H_3(R)$. We show that $H_3(R)$ is connected with diameter at most 4 provided that $x^2 \neq 0$ for all 3-zero-divisors x in R . We find a necessary and sufficient condition for its completeness, and we also find a lower bound for its clique number.

THEOREM 3.1. *Let $H_3(R)$ be the 3-zero-divisor hypergraph of a ring R such that $x^2 \neq 0$ for every 3-zero-divisor $x \in R$. Then $H_3(R)$ is connected and*

$$\text{diam}(H_3(R)) \leq 4. \tag{3.1}$$

Proof. For the proof of the theorem, it is enough to show that for each two edges $e_1 = \{a_1, a_2, a_3\}$ and $e_2 = \{b_1, b_2, b_3\}$ of $H_3(R)$, there exist edges e_3 and e_4 which satisfy one of the following conditions:

$$e_3 \cap e_1 \neq \emptyset, \quad e_3 \cap e_2 \neq \emptyset, \tag{*1}$$

or

$$e_3 \cap e_1 \neq \emptyset, \quad e_4 \cap e_2 \neq \emptyset, \quad e_4 \cap e_3 \neq \emptyset. \tag{*2}$$

Consequently, for the rest of the proof, we can always assume that $a_i \neq b_j$ and $a_i \neq -b_j$ for all $i, j \in \{1, 2, 3\}$. Let G be the bipartite graph constructed as follows: $V(G) = e_1 \cup e_2$ and $a_i b_j \in E(G)$ if and only if $a_i b_j = 0$ in the ring R .

Suppose G has two isolated vertices, one in e_1 and the other in e_2 . For example, $\deg_G(a_3) = \deg_G(b_3) = 0$. If there exists an element $c \in \{a_1, a_2, b_1, b_2\}$ such that $a_3 b_3 c = 0$, then $e_3 = \{a_3, b_3, c\}$ satisfies $(*1)$. Suppose that this is not the case. If $a_3 b_3 \notin \{a_1, a_2, b_1, b_2\}$, then $e_3 = \{a_1, a_2, a_3 b_3\}$ and $e_4 = \{b_1, b_2, a_3 b_3\}$ satisfy $(*2)$. Otherwise without loss of generality, assume that $a_3 b_3 = a_1$. Then $e_3 = \{a_1, b_1, b_2\}$ satisfies $(*1)$. The rest of our proof depends on the number of edges of G .

Case 1. Suppose $|E(G)| \leq 2$. Then G has two isolated vertices, one in e_1 and the other in e_2 .

Case 2. Suppose $|E(G)| = 3$. We study this case for four different subcases as follows.

Case 2.1. Assume the degree of each vertex of G is one and

$$E(G) = \{a_1b_1, a_2b_2, a_3b_3\}. \quad (3.2)$$

Consider the set $\{a_1, a_2b_3, b_1 + b_2\}$. If $a_1 = a_2b_3$, then $a_1b_2 = 0$ is a contradiction. If $a_1 = b_1 + b_2$, then $b_1a_2a_3 = 0$, and $e_3 = \{b_1, a_2, a_3\}$ satisfies $(*_1)$. If $b_1 + b_2 = a_2b_3$, then $a_1b_2a_3 = 0$, and $e_3 = \{a_1, b_2, a_3\}$ satisfies $(*_1)$. Otherwise, $e_3 = \{a_1, a_2b_3, b_1 + b_2\}$ is an edge. Similarly if we consider the set $\{b_1, a_2b_3, a_1 + a_3\}$, then we find an edge e_3 which satisfies $(*_1)$ or $e_4 = \{b_1, a_2b_3, a_1 + a_3\}$ is an edge with e_3 and e_4 satisfying $(*_2)$.

Case 2.2. Assume that the degree of exactly one of the vertices of G is one. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_3\}. \quad (3.3)$$

Consider the set $\{a_2, a_3b_1, a_1 + b_3\}$. If $a_2 = a_3b_1$, then $a_1a_2 = 0$ implies a contradiction. If $a_2 = a_1 + b_3$, then $a_2b_2b_1 = 0$, and $e_3 = \{a_2, b_2, b_1\}$ satisfies $(*_1)$. If $a_1 + b_3 = a_3b_1$, then $a_3b_1b_2b_1 = 0$. In this case if $a_3 = b_1b_2$, then $a_1a_3 = 0$, also, $b_1 = b_1b_2$ implies that $b_1b_3 = 0$, which in both cases we have a contradiction. Therefore, $e_3 = \{a_3, b_1b_2, b_1\}$ is an edge which satisfies $(*_1)$. If none of the above conditions holds, then the set $e_3 = \{a_2, a_3b_1, a_1 + b_3\}$ is an edge. Now consider the set $\{b_2, a_3b_1, b_3\}$. Similarly, we find an edge e_3 which satisfies $(*_1)$, or $e_4 = \{a_2, a_3b_1, a_1 + b_3\}$ is an edge where e_3 and e_4 satisfy $(*_2)$.

Case 2.3. Let the degree of two vertices of G be two. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2\}. \quad (3.4)$$

In this case, $\deg_G(a_3) = \deg_G(b_3) = 0$, and the proof is complete.

Case 2.4. Assume that the degree of one vertex of G is three. Without loss of generality, suppose

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3\}. \quad (3.5)$$

Suppose that $a_1^2a_2 \neq 0$. Consider the set $\{a_1a_2 - b_1, a_1, a_3\}$. If $a_1a_2 - b_1 = a_1$, then $b_2b_1 = 0$ is a contradiction. If $a_1a_2 - b_1 = a_3$, then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge satisfying $(*_1)$. In the other case, $e_3 = \{a_1a_2 - b_1, a_1, a_3\}$ is an edge. Similarly, if we consider the set $\{a_1a_2 - b_1, b_2, b_3\}$, we will find an edge e_3 that satisfies $(*_1)$, or $e_4 = \{a_1a_2 - b_1, b_2, b_3\}$ is an edge with e_3 and e_4 that satisfy $(*_2)$. Now let $a_1^2a_2 = 0$. Consider the set $\{a_1 - b_1, a_1, a_2\}$. If $a_1 - b_1 = a_2$, then $a_2b_3b_2 = 0$, and therefore $e_3 = \{a_2, b_2, b_3\}$ is an edge satisfying $(*_1)$. In the other case, $e_3 = \{a_1 - b_1, a_1, a_2\}$ is an edge. Similarly, if we consider the set $\{a_1 - b_1, b_2, b_3\}$, we will find a contradiction, or $e_4 = \{a_1a_2 - b_1, b_2, b_3\}$ is an edge with e_3 and e_4 that satisfy $(*_2)$.

Case 3. Suppose $|E(G)| = 4$. We study this case using four different subcases as follows.

Case 3.1. Assume the degree of one vertex of G is three. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_3\}. \quad (3.6)$$

Consider the set $\{a_3b_1, a_2, a_1 + b_3\}$. If $a_3b_1 = a_2$, then $a_3b_3b_1 = 0$, and therefore $e_3 = \{a_3, b_1, b_3\}$ is an edge satisfying (\ast_1) . If $a_3b_1 = a_1 + b_3$, then $a_1^2 = 0$ is a contradiction. If $a_2 = a_1 + b_3$, then $b_3^2 = 0$ is a contradiction. In the other case, $e_3 = \{a_3b_1, a_2, a_1 + b_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_1, b_2, b_3\}$, we will find an edge e_3 that satisfies (\ast_1) , or $e_4 = \{a_3b_1, b_2, b_3\}$ is an edge with e_3 and e_4 that satisfy (\ast_2) .

Case 3.2. Assume that the degree of four vertices of G is two. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}. \quad (3.7)$$

In this case, $\deg_G(a_3) = \deg_G(b_3) = 0$, and the proof is complete.

Case 3.3. Let the degree of three vertices of G be two. Suppose without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2, a_2b_3\}. \quad (3.8)$$

Consider the set $\{a_3b_3, a_1, a_2\}$. If $a_3b_3 = a_1$ or a_2 , then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge that satisfies (\ast_1) . In the other case, $e_3 = \{a_3b_3, a_1, a_2\}$ is an edge. Similarly, if we consider the set $\{a_3b_3, b_1, b_2\}$, we will find an edge e_3 that satisfies (\ast_1) , or $e_4 = \{a_3b_3, b_1, b_2\}$ is an edge with e_3 and e_4 that satisfy (\ast_2) .

Case 3.4. Assume that the degree of two vertices of G is two. In this case, there might be two different nonisomorphic cases. Without loss of generality, for one case we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2, a_3b_3\}, \quad (3.9)$$

and in the other case

$$E(G) = \{a_1b_1, a_1b_2, a_2b_3, a_3b_3\}. \quad (3.10)$$

In the first case, consider the set $\{a_3b_1, a_2, a_1 + b_2\}$. If $a_3b_1 = a_2$, then $a_3b_1b_2 = 0$, and therefore $e_3 = \{a_3, b_1, b_2\}$ is an edge that satisfies (\ast_1) . If $a_3b_1 = a_1 + b_2$, then $a_1^2 = 0$ is a contradiction. Also, $a_2 = a_1 + b_2$ implies that $b_2^2 = 0$, which is a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, b_1a_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_3, b_1, a_1 + b_2\}$, we will find an edge e_3 that satisfies (\ast_1) , or $e_4 = \{a_3b_3, b_1, a_1 + b_2\}$ is an edge with e_3 and e_4 that satisfy (\ast_2) .

Similarly, for the second case, by considering the sets $\{a_1 + b_1, a_2, a_3\}$ and $\{a_1 + b_1, b_2, b_3\}$, we find an edge e_3 that satisfies (\ast_1) , or two edges e_3 and e_4 that satisfy (\ast_2) .

Case 4. Suppose $|E(G)| = 5$. We continue our investigation for five different nonisomorphic subcases as follows.

Case 4.1. Without loss of generality, we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_2b_2\}. \quad (3.11)$$

Consider the set $\{a_3b_3, a_2, a_1 + b_2\}$. If $a_3b_3 = a_2$, then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge that satisfies $(*_1)$. If $a_3b_3 = a_1 + b_2$, then $a_1^2 = 0$, which is a contradiction. If $a_1 + b_2 = a_2$, then $b_1b_2 = 0$ is a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, a_3b_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_3, b_1, b_2\}$, we will find an edge e_3 which satisfies $(*_1)$, or $e_4 = \{a_3b_3, b_1, b_2\}$ is an edge with e_3 and e_4 that satisfy $(*_2)$.

Case 4.2. Assume without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_3b_2\}. \quad (3.12)$$

Consider the set $\{a_1 + b_1, a_2, b_2\}$. If $a_1 + b_1 = a_2$, then $b_1^2 = 0$ is a contradiction. If $a_1 + b_1 = b_2$, then $a_1^2 = 0$ implies a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, a_3b_3\}$ is an edge that satisfies $(*_1)$.

Case 4.3. Assume without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_3b_1\}. \quad (3.13)$$

Consider the set $\{a_1 + b_1, a_2, b_2\}$. If $a_1 + b_1 = a_2$, then $b_1^2 = 0$ is a contradiction. If $a_1 + b_1 = b_2$, then $a_2a_3b_2 = 0$, and $e_3 = \{a_2, b_2, a_3\}$ is an edge which satisfies $(*_1)$. In the other case, $e_3 = \{a_1 + b_1, a_2, b_2\}$ is an edge that satisfies $(*_1)$.

Case 4.4. Without loss of generality, we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_3b_3\}. \quad (3.14)$$

Consider the set $\{a_3 + b_1, a_1, b_3\}$. If $a_3 + b_1 = a_1$ or $a_3 + b_1 = b_3$, then $a_1a_2b_3 = 0$, and $e_3 = \{a_1, a_2, b_3\}$ is an edge that satisfies $(*_1)$. In the other case, $e_3 = \{a_3 + b_1, a_1, b_3\}$ is an edge which satisfies $(*_1)$.

Case 4.5. Assume without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_3\}. \quad (3.15)$$

Consider the set $\{a_1 + b_2, a_2, b_1\}$. If $a_1 + b_2 = a_2$, then $b_2^2 = 0$. If $a_1 + b_2 = b_1$, then $a_1^2 = 0$, which is a contradiction. Therefore $e_3 = \{a_1 + b_2, a_2, b_1\}$ is an edge that satisfies $(*_1)$.

Case 5. Suppose $|E(G)| = 6$. We study three different nonisomorphic subcases as follows.

Case 5.1. Without loss of generality, we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_2b_2, a_3b_1\}. \quad (3.16)$$

Consider the sets $\{a_1 + b_1, a_2, a_3\}$ and $\{a_1 + b_1, b_2, b_3\}$. If $a_1 + b_1 = a_2$, then $b_1 b_2 = 0$. If $a_1 + b_1 = a_3$, then $b_1^2 = 0$. Also, $a_1 + b_1 = b_2$ or $a_1 + b_1 = b_3$ implies that $a_1^2 = 0$, and in either case, we have a contradiction. Therefore, $e_3 = \{a_1 + b_1, a_2, a_3\}$ and $e_4 = \{a_1 + b_1, b_2, b_3\}$ are two edges that satisfy $(*_2)$.

Case 5.2. Without loss of generality, we can assume that

$$E(G) = \{a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2, a_3 b_3\}. \tag{3.17}$$

Consider the set $\{a_1 + b_3, a_3, b_1\}$. If $a_1 + b_3 = a_3$, then $b_3^2 = 0$. Also $a_1^2 = 0$ whenever $a_1 + b_3 = b_1$, which is a contradiction. Therefore, $e_3 = \{a_1 + b_3, a_3, b_1\}$ is an edge that satisfies $(*_1)$.

Case 5.3. Assume without loss of generality that

$$E(G) = \{a_1 b_1, a_1 b_3, a_2 b_1, a_2 b_2, a_3 b_2, a_3 b_3\}. \tag{3.18}$$

In this case, similar to the above subcase, $e_3 = \{a_1 + b_3, a_3, b_1\}$ is an edge which satisfies $(*_1)$.

Case 6. Suppose that $7 \leq |E(G)| \leq 9$. In this case, there always exist two vertices with degree three, one from e_1 and the other from e_2 . Let $d_G(a_1) = d_G(b_1) = 3$. Consider the sets $\{a_1 + b_1, a_2, a_3\}$ and $\{a_1 + b_1, b_2, b_3\}$. If $a_1 + b_1 = a_2$ or a_3 , then $b_1^2 = 0$; and if $a_1 + b_1 = b_2$ or b_3 , then $a_1^2 = 0$, which is a contradiction in all cases. Therefore, $e_3 = \{a_1 + b_1, a_2, a_3\}$ and $e_4 = \{a_1 + b_1, b_2, b_3\}$ are two edges that satisfy $(*_2)$. \square

Remark 3.2. From the above theorem and the fact that

$$\text{gr}(H_3(R)) \leq 2 \text{diam}(H_3(R)) + 1, \tag{3.19}$$

we can conclude that the diameter and girth of any hypergraph $H_3(R)$ containing a cycle and satisfying the conditions in the above theorem are bounded by 4 and 9, respectively. Note that a similar result for a zero-divisor graph $\Gamma(R)$ is studied in [5, 8, 9, 14] as follows.

- (1) $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R)) \leq 3$.
- (2) If $\Gamma(R)$ contains a cycle, then $\text{gr}(\Gamma(R)) \leq 4$.

LEMMA 3.3. Let R be a finite ring with $|R| \geq 4$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or there exist two distinct elements x and y in $R - \{0, 1\}$ such that $xy \neq 0$.

Proof. For the case $|R| = 4$, it is clear that R is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2[x]/\langle x^2 \rangle$ or \mathbb{Z}_4 , which implies the desired result. Next, we study the case for $|R| \geq 5$ by a contrary method. Suppose $R - \{0, 1\} = \{a_1, a_2, \dots, a_m\}$, $m \geq 3$, and $a_i a_j = 0$ for all $1 \leq i \neq j \leq m$. It is clear that $a_2 + 1$ is different from 0 and 1. Otherwise, $a_1 = 0$ or $a_2 = 0$, which is a contradiction to the choice of a_1 and a_2 . If $a_2 + 1 \neq a_1$, then $a_1(a_2 + 1) = 0$, and we have $a_1 = 0$, which is a contradiction. Thus, $a_2 + 1 = a_1$. Similarly, $a_1 a_3 = 0$, and $a_3 + 1 = a_1$ implies that $a_3 = a_2$, which is a contradiction. \square

In the next theorem, we give a necessary and sufficient condition for a hypergraph $H_3(R)$ to be complete. In the process of the following proof, we consider the obvious fact

that $H_3(Z_2 \times Z_2 \times Z_2)$ has only one edge, and necessarily it is a complete hypergraph. Note that for a detailed study of the completeness of a zero-divisor graph $\Gamma(R)$, the reader is referred to [5].

THEOREM 3.4. *Let R be a finite nonlocal ring. Then $H_3(R)$ is complete if and only if $R = Z_2 \times Z_2 \times Z_2$.*

Proof. The sufficient part of the theorem is trivial, because $H_3(R)$ has only one edge, and therefore is complete whenever $R = Z_2 \times Z_2 \times Z_2$. Suppose that $H_3(R)$ is complete. It is a well-known fact that any finite ring R is isomorphic to the product of local rings. Thus, assume that $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring for all $i = 1, 2, \dots, n$. Now, we study the following cases for different values of n .

Case 1. Suppose $n \geq 4$. It is clear that $e_1 = \{x_1, x_2, x_3\}$ and $e_2 = \{y_1, y_2, y_3\}$ with

$$\begin{aligned} x_1 &= (1, 1, 0, 0, \dots, 0), & x_2 &= (1, 0, 1, 0, \dots, 0), & x_3 &= (0, 1, 1, 0, \dots, 0), \\ y_1 &= (1, 0, 0, 1, \dots, 0), & y_2 &= (1, 1, 0, 0, \dots, 0), & y_3 &= (0, 1, 0, 1, \dots, 0) \end{aligned} \quad (3.20)$$

are two edges of $H_3(R)$. Clearly, $H_3(R)$ is not complete since $\{x_1, x_2, y_1\}$ is not an edge of $H_3(R)$.

Case 2. Let $R = R_1 \times R_2 \times R_3$. Without loss of generality, suppose that $|R_1| \geq 3$. Let $x \in R_1 - \{0, 1\}$. Obviously, $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \in E(H_3(R))$ and $\{(x, 1, 0), (1, 0, 1), (0, 1, 1)\} \in E(H_3(R))$. But $\{(x, 1, 0), (1, 0, 1), (1, 1, 0)\} \notin E(H_3(R))$, which implies that $H_3(R)$ is not complete. Hence, we can conclude that $|R_i| \leq 2$ and $R = Z_2 \times Z_2 \times Z_2$.

Case 3. Let $R = R_1 \times R_2$. If $H_3(R)$ does not have any vertices, we do not have anything to prove. Therefore, first we assume that $|R_i| \geq 4$ for each $1 \leq i \leq 2$ and investigate the following subcases.

Case 3.1. The square of one of the components of some 3-zero-divisor of R is zero. Let (a, b) be a 3-zero-divisor in R with $a^2 = 0$ and let $e = \{(a, b), (c, d), (f, g)\}$ be an edge of $H_3(R)$. Since $Z_2 \times Z_2$ is not a local ring, by Lemma 3.3 there exist distinct elements x and y in $R_2 - \{0, 1\}$ such that $xy \neq 0$. Now, from the fact that $\{(a, 1), (a, x), (1, 0)\}$ and $\{(a, 1), (a, y), (1, 0)\}$ are in $E(H_3(R))$ and $\{(a, x), (a, y), (a, 1)\} \notin E(H_3(R))$, we can conclude that $H_3(R)$ is not complete.

Case 3.2. The square of none of the components of any 3-zero-divisor of R is zero. Suppose that $e = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ is an edge of $H_3(R)$. In this case, there always exists $i \in \{1, 2, 3\}$, say $i = 1$, such that $a_1 a_2 \neq 0$ and $a_1 a_3 \neq 0$, or similarly, $b_1 b_2 \neq 0$ and $b_1 b_3 \neq 0$. Otherwise, the product of two elements of e will be zero, which contradicts the definition for e to be an edge in $H_3(R)$. Without loss of generality, we assume that $a_1 a_2 \neq 0$ and $a_1 a_3 \neq 0$. By using Lemma 3.3, similar to Case 3.1, there exist distinct elements x and y in $R_2 - \{0, 1\}$ such that $xy \neq 0$. Since $\{(a_1, 0), (a_2, x), (a_3, 1)\}$ and $\{(a_1, 0), (a_2, y), (a_3, 1)\}$ are the edges of $H_3(R)$, and $\{(a_2, x), (a_2, y), (a_3, 1)\}$ is not an edge of $H_3(R)$, then $H_3(R)$ is not complete.

Next, we assume that the size of one of the rings R_i 's is 2, where $i = 1, 2$. Without loss of generality, assume that $R_2 = Z_2$. It is clear that R does not have any 3-zero-divisors

whenever R_1 is an integral domain. Thus, R_1 has at least four elements. Obviously, the edges of $H_3(R)$ cannot be different from the following forms:

$$\{(a, 0), (b, 0), (c, 0)\}, \quad \{(a, 1), (b, 0), (c, 0)\}, \quad \{(a, 1), (b, 1), (c, 0)\}. \quad (3.21)$$

Case 3.3. Let $H_3(R)$ have an edge of the form $\{(a, 0), (b, 0), (c, 0)\}$.

Then $\{(a, 1), (b, 0), (c, 0)\} \in E(H_3(R))$, $\{(a, 0), (b, 1), (c, 0)\} \in E(H_3(R))$, and $\{(a, 0), (b, 0), (c, 1)\} \in E(H_3(R))$. In this case, the completeness of $H_3(R)$ implies that $\{(a, 1), (b, 1), (c, 1)\} \in E(H_3(R))$, which is a contradiction.

Case 3.4. Suppose $\{(a, 1), (b, 0), (c, 0)\}$ is an edge of $H_3(R)$. Therefore, $b \neq c$, $ab \neq 0$, $ac \neq 0$, and $bc \neq 0$. In this subcase, we study two different cases:

(a) The first components of two elements of $\{(a, 1), (b, 0), (c, 0)\}$ are equal. For example, assume $a = b$. Thus, $\{(a^2, 1), (1, 0), (c, 1)\} \in E(H_3(R))$ whenever $a^2 \neq c$. In this case, $c \neq 1$, and the completeness of $H_3(R)$ implies that $\{(a, 1), (1, 0), (c, 0)\} \in E(H_3(R))$, which contradicts $ac \neq 0$.

On the other hand if $a^2 = c$, we have $a^4 = 0$, which implies $a^3 \neq a$. Therefore, $\{(a, 1), (a^3, 1), (1, 0)\}$ is an edge of $H_3(R)$, which contradicts $ac \neq 0$.

(b) Let $a \neq b$ and $a \neq c$. In this case, $\{(a, 1), (b, 1), (c, 0)\}$ and $\{(a, 1), (b, 0), (c, 1)\}$ are in $E(H_3(R))$. Consequently, the completeness of $H_3(R)$ implies that $\{(a, 1), (b, 1), (c, 1)\} \in E(H_3(R))$, which is a contradiction.

Case 3.5. Let all the edges of $H_3(R)$ be of the form $\{(a, 1), (b, 1), (c, 0)\}$. Assume that $\{(a, 1), (b, 1), (c, 0)\}$ and $\{(a', 1), (b', 1), (c', 0)\}$ are two edges of $H_3(R)$. Therefore, by the completeness of $H_3(R)$, one of the sets

$$\{(a, 1), (b, 1), (a', 1)\}, \quad \{(a, 1), (b, 1), (b', 1)\}, \quad \{(a, 1), (c, 0), (c', 0)\} \quad (3.22)$$

should be an edge of $H_3(R)$. This is a contradiction to the definition of an edge or to Case 3.4. Now, we can conclude that $H_3(R)$ has only one edge of the form $\{(a, 1), (b, 1), (c, 0)\}$, where $ac \neq 0$ and $bc \neq 0$. Furthermore, if $ab \neq 0$, then $\{(a, 1), (b, 0), (c, 0)\}$ is an edge of $H_3(R)$, which is a contradiction. Thus, $ab = 0$. Consequently, $c \neq 1$ implies that $\{(a, 1), (b, 1), (c, 0)\}$, $\{(a, 1), (b, 1), (1, 0)\}$ and $\{(a, 1), (b, 1), (-1, 0)\}$ are edges in $H_3(R)$, which is a contradiction. Hence, we can conclude that $\{(a, 1), (b, 1), (1, 0)\}$ is the only edge of $H_3(R)$ and $1 = -1$ in R_1 . Next, we show that $a^2 = a$ and $b^2 = b$. Since $\{(a, 1), (b, 1), (a + 1, 0)\}$ is not an edge in $H_3(R)$, $ba = 0$, and $b \neq 0$, then $b(a + 1) \neq 0$, and we must have $a(a + 1) = 0$, which implies that $a^2 = a$. By a similar argument, we can conclude that $b^2 = b$. Suppose $x \in R_1 - \{0, 1, a, b\}$. Since $\{(a, 1), (b, 1), (x, 0)\}$ is not an edge of $H_3(R)$, then $ax = 0$ or $bx = 0$. Without loss of generality, suppose that $ax = 0$. Now, since $b + x \neq b$, $\{(a, 1), (b + x, 1), (1, 0)\}$ is not an edge of $H_3(R)$. Therefore, $b + x = 0$ or $b + x = a$. If $b + x = 0$, we have $b = x$, which is a contradiction. Let $b + x = a$. Then $x = b + a$, and therefore $a(b + a) = 0$, which implies that $a = a^2 = 0$, a contradiction. Thus, $\{0, 1, a, b\}$ are the only elements of R_1 . Since R_1 is a local ring with 4 elements, then $R_1 = Z_4$ or $R_1 = Z_2[x]/\langle x^2 \rangle$. In either case, $R = R_1 \times Z_2$ does not have any edges, and $H_3(R)$ is not complete.

Finally, since the proof of the case $R_2 = Z_3$ is similar to the above argument, we leave the rest of the proof to the reader. \square

Remark 3.5. Bounds for $\omega(\Gamma(R))$ are given by using nilpotent elements of R as studied in [6] as follows. Let R be a commutative ring and $0 \neq x \in \text{nil}(R)$, and let n be the least positive integer such that $x^n = 0$.

(1) If $n = 2t$, then $\omega(\Gamma(R)) \geq 2^t - 1$.

(2) If $n = 2t + 1$, then $\omega(\Gamma(R)) \geq 2^t$.

Similarly, in the next theorem, we give a lower bound for the clique number of $H_3(R)$ using the index of nilpotence as studied in [6] for a zero-divisor graph $\Gamma(R)$.

THEOREM 3.6. *Let x be an element of a commutative ring R such that $x^n = 0$ and $x^{(n-1)} \neq 0$. Then*

$$\omega(H_3(R)) \geq \begin{cases} 2^{2t-2} & \text{if } n = 3t, \\ \frac{2^{2t-1} + 1}{2} & \text{otherwise.} \end{cases} \quad (3.23)$$

Proof. For $n = 3t$, the set

$$A = \{x^t(1 + a_1x + a_2x^2 + \cdots + a_{2t-1}x^{2t-1}) \mid a_i \in \{0, 1\}, 1 \leq i \leq 2t - 1\} \quad (3.24)$$

is a clique of size 2^{2t-1} .

Similarly, for $n = 3t + 1$ and $n = 3t + 2$, the set

$$A = \{x^{t+1}(1 + a_1x + a_2x^2 + \cdots + a_{2t-1}x^{2t-1}) \mid a_i \in \{0, 1\}, 1 \leq i \leq 2t - 1\} \cup \{x^t\} \quad (3.25)$$

is a clique of size $2^{2t-1} + 1$. \square

THEOREM 3.7. *For any integer $m \geq 3$, there exists an integer n such that*

$$\omega(H_3(Z_2^n)) \geq \frac{m}{2}, \quad (3.26)$$

where $Z_2^n = Z_2 \times Z_2 \times \cdots \times Z_2$ (n times).

Proof. For $m = 3$, it is clear that the set $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a clique of size 3 in $H_3(Z_2^3)$. Suppose that $\{a_1, a_2, \dots, a_m\}$ is a clique of size m in $H_3(Z_2^{n'})$. Let $n = n' + m$. We define b_i in $H_3(Z_2^n)$ to be the n -tuple whose first n' components are exactly a_i and all the other components are 0, except the $(n' + i)$ th component, which is 1 for all $1 \leq i \leq m$. Let b_{m+1} be the n -tuple whose first n' components are 0 and all the other m components are 1. Now, it is easy to see that $\{b_1, b_2, \dots, b_{m+1}\}$ is a clique of size $m + 1$ in $H_3(Z_2^n)$. Note that n satisfies the recursion relation $x_m = x_{m-1} + m - 1$, where $m \geq 4$ and $x_3 = 3$. \square

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.8. *The chromatic number of $H_3(Z_2^n)$ goes to infinity as n approaches infinity. That is,*

$$\lim_{n \rightarrow \infty} \chi(H_3(Z_2^n)) = \infty. \quad (3.27)$$

We conclude this section by posing a question on the isomorphism of the rings of 3-zero-divisor hypergraphs. In [6], it is shown that for any finite reduced commutative rings A and B which are not fields, then $\Gamma(A) \cong \Gamma(B)$ as graphs if and only if $A \cong B$ as rings. Furthermore, in [3], this result is generalized to the case that if A is a finite reduced ring which is not isomorphic to $Z_2 \times Z_2$ or Z_6 with B a ring such that $\Gamma(A) \cong \Gamma(B)$, then $A \cong B$. Also, in [7], it is shown that A and its total quotient ring $T(A)$ have isomorphic zero-divisor graphs.

Question 2. Let A and B be two commutative rings. Under what condition(s) does the isomorphism of $H_3(A)$ and $H_3(B)$ imply the isomorphism of A and B ?

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