Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 57607, 13 pages
doi:10.1155/2007/57607

Research Article

On the First Curve in the FučiK Spectrum with Weights for a Mixed p-Laplacian

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Received 27 April 2007; Accepted 9 August 2007

Recommended by Peter Basarab-Horwath

We show the existence of a first curve in the Fučik spectrum with weights for the p-Laplacian under mixed boundary conditions. We also study the asymptotic behavior of this first curve.

1. Introduction

We investigate the following asymmetric elliptic problem:

$$\begin{gathered}\Delta_p u = \lambda\left[m(x)\left(u^+\right)^{p-1} - n(x)\left(u^-\right)^{p-1}\right] \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_2,\end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u with $1 < p < \infty$, and λ is a real parameter. Moreover, Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, whose boundary $\partial\Omega$ is made of two disjoint nonempty closed sets Γ_1 and Γ_2 which are smooth manifolds of dimension $N-1$, m and n are weights which may be indefinite and unbounded, $\partial/\partial\nu$ denotes the exterior normal derivative, and $u^{\pm} = \max\{\pm u, 0\}$.

The main motivation for considering problem (1.1) arises from the study of the Fučik spectrum. This spectrum is defined as the set Σ of those $(\alpha,\beta) \in \mathbb{R}^2$ such that the problem

$$\begin{gathered}-\Delta_p u = \alpha m(x)\left(u^+\right)^{p-1} - \beta n(x)\left(u^-\right)^{p-1} \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_2,\end{gathered} \tag{1.2}$$

has a nontrivial solution u. One recovers the spectrum of (1.1) by taking $\alpha = \beta$ in (1.2). Another relation between (1.1) and (1.2) comes from the fact that the line of slope s through the origin of $\mathbb{R}^2$ meets Σ at a point $(\alpha, \beta = s\alpha)$ if and only if α is an eigenvalue for (1.1) for the weights m and sn.

Of special interest for our purposes is the work in [1] where a study of problems such as (1.1) and (1.2) was carried out in the case of the Dirichlet boundary conditions.

In [1], the existence of the first nontrivial curve in the Fučik spectrum was derived and some of its properties were established. In particular, it was shown that if both m and n change signs, then Σ contains a first hyperbolic-like curve in each quadrant of $\mathbb{R}^2$. Moreover, the asymptotic behavior of these first curves was shown to depend on the supports of the weights. The case of the Neumann boundary conditions was considered later in [2] where, contrary to what happened in the Dirichlet case, the asymptotic behavior of the first curve did not depend on the supports of the weights.

Thus, the concern now naturally arises to study the Fučik spectrum under other boundary conditions, particularly the asymptotic behavior of the first curve. Our purpose in this work is to investigate a case which is somehow intermediate between Dirichlet's and Neumann's, that is, the case of the classical "mixed" boundary conditions. While trying to adapt the approach in [1, 2] to the present situation, new difficulties arise in connection with the lack of regularity of the eigenfunctions. It is well known that weak solutions of degenerate elliptic quasilinear equations, more generally the one considered here, under Dirichlet or Neumann boundary conditions are essentially bounded in Ω and at least of class $C^{\alpha}_{\mathrm{loc}}(\Omega)$ (cf. [3–7]). A little more regularity result is gained when the weights are bounded. In fact, the results in [6–8] imply that solutions of (1.1) under Dirichlet or Neumann boundary conditions for m and n bounded are at least of class $C^{1,\alpha}(\overline{\Omega})$.

In the case of mixed boundary conditions considered here, one can easily adapt the above regularity results and derives that any solution of (1.1) (or (1.2)) is also of class $C^{\alpha}_{\mathrm{loc}}$ and is essentially bounded. However, to our knowledge, there is no result stating a $C^{1,\alpha}(\overline{\Omega})$ regularity when the weights are bounded.

As in [1], we will construct a positive nonprincipal eigenvalue for (1.1) by applying a version of the mountain pass theorem to the functional $\int_{\Omega} |\nabla u|^p$ restricted to the C^1 manifold:

$$M_{m,n} := \left\{ u \in E(\Omega), B_{m,n}(u) = \int_{\Omega} \left[m(u^+)^p + n(u^-)^p \right] = 1 \right\}, \tag{1.3}$$

where the space $E(\Omega)$ will be specified later. In Section 4, we show that the eigenvalue constructed in Section 3 is the first eigenvalue of (1.1) which is greater than $\lambda_1(m)$ and $\lambda_1(n)$ (where $\lambda_1(m)$, resp., $\lambda_1(n)$, is the positive principal eigenvalue of p-Laplacian with weight m, resp., n, under the above mixed boundary conditions). Some of the properties of this distinguished eigenvalue are also briefly indicated. In Section 5, we study the Fučik spectrum. We show in particular that if m and n both change signs in Ω, then each of the four quadrants in the (α, β) plane contains a first nontrivial curve of Σ, which is hyperbolic-like and has a variational characterization. We also study the asymptotic behavior of these first curves. For instance, the first curve in $\mathbb{R}^+ \times \mathbb{R}^+$ is asymptotic to the

line $\lambda_1(m)\times\mathbb{R}$ if $N\geq p$ or $N<p$ and the support of n^+ intersects Γ_1, but it is not asymptotic to that line if $N<p$ and the support of n^+ is compact in Ω or does not intersect Γ_1. A similar result holds of course for the support of m^+ with respect to the line $\mathbb{R}\times\lambda_1(n)$ in $\mathbb{R}^+\times\mathbb{R}^+$. These results can be generalized to the other quadrants. Section 6 is devoted to the particular case $m=n\equiv 1$ in one dimension. The Fučik spectrum in this case is described explicitly so that the asymptotic values of the first curve are computed. Those values illustrate as well the result stated in Section 5, in the general case. Section 2 is a preliminary section and contains particularly some results relative to the usual principal eigenvalue.

Let us conclude this introduction with some general definitions related to the (PS) condition. Let E be a real Banach space and let $M:=\{u\in E: g(u)=1\}$, where $g\in C^1(E,\mathbb{R})$ and 1 is a regular value of g. Let $f\in C^1(E,\mathbb{R})$ and consider the restriction $\widetilde{f}$ of f to M. The functional $\widetilde{f}$ is said to satisfy the (PS) condition on M if for any sequence, $u_k\in M$ such that $\widetilde{f}(u_k)$ is bounded and $\|\widetilde{f}'(u_k)\|_*\to 0$. Then, u_k admits a converging subsequence. Here, $\|\widetilde{f}'(u_k)\|_*$ denotes the norm of the restriction of $f'(u)$ to the tangent space $T_uM:=\{v\in E:\langle g'(u),v\rangle=0\}$, where $\langle,\rangle$ is the pairing between E and its dual.

2. Preliminaries

Throughout this paper, Ω will be a smooth bounded domain in $\mathbb{R}^N$, $N\geq 1$, with $\partial\Omega=\Gamma_1\cup\Gamma_2$, where Γ_1 and Γ_2 are two closed disjoint nonempty sets which are smooth manifolds of dimension $N-1$. The weights m,n will be assumed to belong to $L^r(\Omega)$ with $r>N/p$ if $N\geq p$ and $r=1$ if $p>N$. We also assume, unless otherwise stated, that

$$m^+\not\equiv 0,\ n^+\not\equiv 0\quad \text{in } \Omega. \tag{2.1}$$

We will work in the space $E(\Omega)$ which is defined as

$$E(\Omega):=\{v\in W^{1,p}(\Omega): v=0 \text{ on } \Gamma_1 \text{ in the sense of traces}\}. \tag{2.2}$$

Using the regularity of Ω, one can show that $(\int_\Omega|\nabla u|^p)^{1/p}$ is a norm on $E(\Omega)$ which is equivalent to the $W^{1,p}(\Omega)$ norm (cf. [9–11]).

Solutions of (1.1) are always understood in the weak sense, that is, $u\in E(\Omega)$ with the property

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v=\lambda\int_\Omega\left[m(u^+)^{p-1}-n(u^-)^{p-1}\right]v,\quad \forall\, v\in E(\Omega). \tag{2.3}$$

We will denote by

$$\lambda_1(m):=\inf\left\{\int_\Omega|\nabla u|^p: u\in E(\Omega)\,,\ \int_\Omega m|u|^p=1\right\} \tag{2.4}$$

the positive principal eigenvalue of the p-Laplacian with weight m under the above mixed boundary conditions. Arguing as in the Dirichlet and Neumann cases (see, e.g., [12–14] for bounded weights and [15–17] for unbounded weights), one can show that $\lambda_1(m)$ is

nonnegative, simple, and admits an eigenfunction $\varphi_{\mathrm{m}} \in E(\Omega) \cap C^{\alpha}_{\mathrm{loc}}(\Omega)$ with $\varphi_m(x) > 0$ in Ω and $\int_\Omega m\varphi_m^p = 1$. In the case $m^- \not\equiv 0$, the negative principal eigenvalue is obtained by reversing the sign of the weight: $\lambda_{-1}(m) = -\lambda_1(-m)$.

3. Construction of a nontrivial eigenvalue

In this section, we look for eigenvalues λ of (1.1) with $\lambda > 0$. Clearly, (1.1) with $\lambda > 0$ has a nontrivial solution u which does not change sign if and only if $\lambda = \lambda_1(m)$ or $\lambda = \lambda_1(n)$. Moreover, choosing u^+ and u^- as test functions in (2.3), one easily sees that if (1.1) with $\lambda > 0$ has a solution which changes sign, then $\lambda > \max\{\lambda_1(m), \lambda_1(n)\}$. Looking for such a solution which changes sign is our purpose in this section. Note that condition (2.1) is necessary for (1.1) with $\lambda > 0$ to have a solution which changes sign.

We will use a variational approach and consider the functionals

$$A(u) = \int_\Omega |\nabla u|^p, \qquad B_{m,n}(u) = \int_\Omega \left[m(u^+)^p + n(u^-)^p\right], \tag{3.1}$$

which are C^1 functionals on $E(\Omega)$. We are interested in the critical points of the restriction $\widetilde{A}$ of A to the manifold:

$$M_{m,n} = \{u \in E(\Omega) : B_{m,n}(u) = 1\}. \tag{3.2}$$

Note that 1 is a regular value of $B_{m,n}$ and $\varphi_m, -\varphi_n \in M_{m,n}$. Using (2.1), one can construct $u \in C_c^\infty(\Omega)$ which changes sign such that $\int_\Omega m(u^+)^p > 0$ and $\int_\Omega n(u^-)^p > 0$. Consequently, $u/B_{m,n}(u)^{1/p}$ belongs to $M_{m,n}$.

By Lagrange's multiplier rule, $u \in M_{m,n}$ is a critical point of $\widetilde{A}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $A'(u) = \lambda B'_{m,n}(u)$, that is,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \lambda \int_\Omega \left[m(u^+)^{p-1} - n(u^-)^{p-1}\right]v \tag{3.3}$$

for all $v \in E(\Omega)$. This means that u is a solution of (1.1) in the sense of (2.3). Moreover, taking $v = u$ in (3.3), one sees that the Lagrange multiplier λ is equal to the critical value $\widetilde{A}(u)$. So, problem (1.1) is transformed into the problem of finding critical points and critical values of $\widetilde{A}$.

The following proposition can be proved by a simple adaptation of the arguments in [1].

Proposition 3.1. (i) *φ_m or $-\varphi_n$ is a global minimum of $\widetilde{A}$.*

(ii) *φ_m and $-\varphi_n$ are strict local minima of $\widetilde{A}$, with $\lambda_1(m)$ or $\lambda_1(n)$ as corresponding critical values.*

(iii) *$\widetilde{A}$ satisfies the Palais-Smale condition on $M_{m,n}$.*

(iv) *The set $\Gamma := \{\gamma \in C([-1,1], M_{m,n}) : \gamma(-1) = \varphi_m$ and $\gamma(1) = -\varphi_n\}$ is nonempty.*

(v) *The minimum value*

$$c(m,n) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([-1,1])} \widetilde{A}(u) \tag{3.4}$$

is a critical value of $\widetilde{A}$, with $c(m,n) > \max\{\lambda_1(m), \lambda_1(n)\}$.

(vi) *$c(m,n)$ is the first nonprincipal positive eigenvalue of (1.1); that is, there is no eigenvalue of (1.1) between* $\max\{\lambda_1(m),\lambda_1(n)\}$ *and* $c(m,n)$.

Remark 3.2. The construction of the eigenvalue $c(m,n)$ in (3.4) is made by applying a version of the mountain pass theorem on the C^1 manifold $M_{m,n}$ as given in [1, Proposition 4].

4. Some properties of $c(m,n)$

In this section, we study the dependence of $c(m,n)$ with respect to the weights m and n; namely, continuity, monotonicity, and homogeneity properties will be discussed.

We start by modifying a little bit the variational characterization (3.4) of $c(m,n)$ in order to allow a larger family of paths, which in addition depends a little less on the weights. The proof of the following proposition is similar to that of [1, Proposition 21].

Proposition 4.1. *One has*

$$c(m,n) = \inf_{\gamma\in\Gamma_1} \max_{u\in\gamma([-1,1])} \widetilde{A}(u), \tag{4.1}$$

where $\Gamma_1 := \{\gamma \in C([-1,1],M_{m,n}) : \gamma(-1) \geq 0$ *and* $\gamma(1) \leq 0\}$.

The following proposition gives some properties of the eigenvalue $c(m,n)$ and can be proved by simple adaptations of arguments in [1].

Proposition 4.2. (i) *If* $(m_k,n_k)\to(m_0,n_0)$ *in* $L^r(\Omega)\times L^r(\Omega)$, *then* $c(m_k,n_k)\to c(m_0,n_0)$.

(ii) *If* $m \leq \widehat{m}$ *and* $n \leq \widehat{n}$, *then* $c(m,n) \geq c(\widehat{m},\widehat{n})$.

(iii) *If* $m \leq \widehat{m}$ *and* $n \leq \widehat{n}$ *with, in addition,*

$$\int_\Omega (\widehat{m}-m)(u^+)^p + \int_\Omega (\widehat{n}-n)(u^-)^p > 0, \tag{4.2}$$

for at least one eigenfunction u associated to $c(m,n)$, then $c(m,n) > c(\widehat{m},\widehat{n})$.

(iv) *If* $0 < s < \widehat{s}$, *then* $c(sm,n) > c(\widehat{s}m,n)$ *and* $c(m,sn) > c(m,\widehat{s}n)$.

To conclude this section, let us observe that definition (3.4) clearly implies that $c(m,n)$ is homogeneous of degree -1:

$$c(sm,sn) = \frac{1}{s}c(m,n) \quad \text{for } s > 0. \tag{4.3}$$

5. Fučik spectrum with weights

Let $m,n \in L^r(\Omega)$ with $r > N/p$ if $N \geq p$ and $r = 1$ if $p > N$. Unless otherwise stated, we also assume (2.1). The Fučik spectrum $\Sigma = \Sigma(m,n)$ clearly contains the lines $\lambda_1(m)\times\mathbb{R}$ and $\mathbb{R}\times\lambda_1(n)$, and also, if $m^- \not\equiv 0$ (resp., $n^- \not\equiv 0$), $\lambda_{-1}(m)\times\mathbb{R}$ (resp., $\mathbb{R}\times\lambda_{-1}(n)$). These lines are in fact exactly made of those $(\alpha,\beta) \in \Sigma$ for which (1.2) admits a solution which does not change sign. We denote below by $\Sigma^* = \Sigma^*(m,n)$ the set Σ minus those trivial lines.

Let us start by looking at the part of Σ^* which lies in $\mathbb{R}^+\times\mathbb{R}^+$. From the properties of the first eigenvalue recalled in Section 2, it follows that if $(\alpha,\beta) \in \Sigma^* \cap (\mathbb{R}^+\times\mathbb{R}^+)$, then $\alpha > \lambda_1(m)$ and $\beta > \lambda_1(n)$.

Theorem 5.1. *For any $s > 0$, the line $\beta = s\alpha$ in the (α,β) plane intersects $\Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. Moreover, the first point in this intersection is given by $\alpha(s) = c(m,sn)$, $\beta(s) = s\alpha(s) = c(m/s,n)$, where $c(\cdot,\cdot)$ is defined in (3.4).*

Proof. It is an easy consequence of Proposition 3.1 (cf. (v) and (vi)). □

Letting $s > 0$ vary, we get a first curve $\mathscr{C} := \{(\alpha(s),\beta(s)) : s > 0\}$ in $\Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. Here are some properties of this curve.

Proposition 5.2. *The functions $\alpha(s)$ and $\beta(s)$ in Theorem 5.1 are continuous. Moreover, $\alpha(s)$ is strictly decreasing and $\beta(s)$ is strictly increasing. One also has $\alpha(s) \to +\infty$ if $s \to 0$ and $\beta(s) \to +\infty$ if $s \to +\infty$.*

Proof. The first two statements are direct consequences of Proposition 4.2. To show that $\alpha(s) \to +\infty$ as $s \to 0$, let us assume by contradiction that $\alpha(s)$ remains bounded as $s \to 0$. Then, $\beta(s) = s\alpha(s) \to 0$ as $s \to 0$, which is impossible since $\beta(s) > \lambda_1(n)$ for all $s > 0$. Similar argument holds for the behavior of $\beta(s)$ as $s \to +\infty$. □

We now investigate the asymptotic values $\alpha_\infty := \lim_{s\to+\infty}\alpha(s)$ and $\beta_\infty := \lim_{s\to 0}\beta(s)$ of the first curve $\mathscr{C}$. We will limit ourselves below to the study of α_∞. One has a similar result for the asymptotic value β_∞ by interchanging the roles of m and n.

The following proposition is the main result of this section.

Proposition 5.3. (i) *$\alpha_\infty = \lambda_1(m)$ if $p \le N$ or $p > N$, with* supp $n^+ \cap \Gamma_1 \not\equiv \varnothing$.

(ii) *$\alpha_\infty > \lambda_1(m)$ if $p > N$, with* supp n^+ *compact in Ω.*

Proof. The proof borrows ideas from [1], but new difficulties arise here in connection with the boundary conditions.

We start by introducing

$$\overline{\alpha} := \inf\left\{\int_\Omega |\nabla u^+|^p : u \in E(\Omega), \int_\Omega m(u^+)^p = 1, \int_\Omega n(u^-)^p > 0\right\} \tag{5.1}$$

and show that $\alpha_\infty = \overline{\alpha}$. The proof of this equality is a direct adaptation of [1].

Clearly, $\overline{\alpha} \ge \lambda_1(m)$. One first considers the case $N \ge p$. Adapting the arguments of [1], one easily obtains $\overline{\alpha} = \lambda_1(m)$.

We now consider the case where $p > N$ and the fact that the support of n^+ intersects Γ_1.

For $\epsilon > 0$ sufficiently small, let us take a nonempty neighborhood of Γ_1, denoted by the set

$$\widetilde{\Omega}_\epsilon = \{x \in \Omega; \operatorname{dist}(x,\Gamma_1) < \epsilon\} \tag{5.2}$$

such that

$$\widehat{\Omega}_\epsilon = \operatorname{int}(\Omega/\widetilde{\Omega}_\epsilon) \tag{5.3}$$

is a smooth bounded domain. On $\widehat{\Omega}_\varepsilon$, we consider the boundary conditions of Neumann on Γ_2 and those of Dirichlet on $\partial\widehat{\Omega}_\epsilon \setminus \Gamma_2$. We denote by $\lambda_1(m,\widehat{\Omega}_\varepsilon)$ the corresponding

principal eigenvalue of $-\Delta_p$ with the weight m, and by $\varphi_m(\widehat{\Omega}_\varepsilon)$ the associated normalized positive eigenfunction. Note that these are well defined for $\varepsilon > 0$ sufficiently small since $m^+ \not\equiv 0$ in Ω (cf. Section 2). By the regularity of $\widehat{\Omega}_\varepsilon$, extending $\widehat{\varphi}_m(\widehat{\Omega}_\varepsilon)$ by zero on $\Omega/\widehat{\Omega}_\varepsilon$ yields a function in $E(\Omega)$, which we still denote by $\widehat{\varphi}_m(\widehat{\Omega}_\varepsilon)$. Moreover, the argument of [18, Lemma 3.1] immediately extends to the present situation and shows that as $\varepsilon \to 0$, $\lambda_1(m,\widehat{\Omega}_\varepsilon) \to \lambda_1(m)$ and $\varphi_m(\widehat{\Omega}_\varepsilon) \to \varphi_m$ in $E(\Omega)$. Then, the argument [1, page 601] can be adapted to obtain $\overline{\alpha} = \lambda_1(m)$

Let us finally consider the case where $p > N$ and the fact that the support of n^+ is compact in Ω. Assume by contradiction that $\overline{\alpha} = \lambda_1(m)$ and let u_k be a minimizing sequence in definition (5.1) of $\overline{\alpha}$. It follows by standard arguments that for a subsequence, u_k^+ converges to φ_m weakly in $E(\Omega)$ and uniformly on supp n^+. Since there exists $\varepsilon > 0$ such that $\varphi_m(x) \geq \varepsilon$ on the compact set supp n^+, we deduce that $u_k^+ \geq \varepsilon/2$ on supp n^+ for k sufficiently large. Consequently, for those k, $u_k^- = 0$ on supp n^+, which implies that

$$\int_\Omega n(u_k^-)^p = \int_\Omega n^+(u_k^-)^p - \int_\Omega n^-(u_k^-)^p = -\int_\Omega n^-(u_k^-)^p \leq 0, \tag{5.4}$$

which is a contradiction since u_k is admissible in definition (5.1) of $\overline{\alpha}$. □

Remark 5.4. If $\varphi_m \in C^1(\overline{\Omega})$, the assumptions of Proposition 5.3(ii) can be weakened and this proposition can be stated as follows.

Proposition 5.5. (i) $\alpha_\infty = \lambda_1(m)$ *if* $p \leq N$ *or* $p > N$, *with* supp $n^+ \cap \Gamma_1 \neq \varnothing$.

(ii) $\alpha_\infty > \lambda_1(m)$ *if* $p > N$, *with* supp $n^+ \cap \Gamma_1 = \varnothing$.

Proof. The proof of (i) is similar to the corresponding part in Proposition 5.3. The proof of (ii) can be simplified since $\varphi_m \in C^1(\overline{\Omega})$. In fact, from the maximum principle of Vázquez (cf. [19]), one deduces that $\varphi_m(x) > 0$ for all $x \in \Omega \cup \Gamma_2$. If supp $n^+ \cap \Gamma_1 = \varnothing$, then there exists $\varepsilon > 0$ such that $\varphi_m(x) \geq \varepsilon$ on supp $n^+ \subset \Omega \cup \Gamma_2$. Hence, one can use the same arguments as in the proof of Proposition 5.3. □

We finally observe that the distribution of Σ^* in the other quadrants of $\mathbb{R} \times \mathbb{R}$ can be studied in a way similar to that of [1].

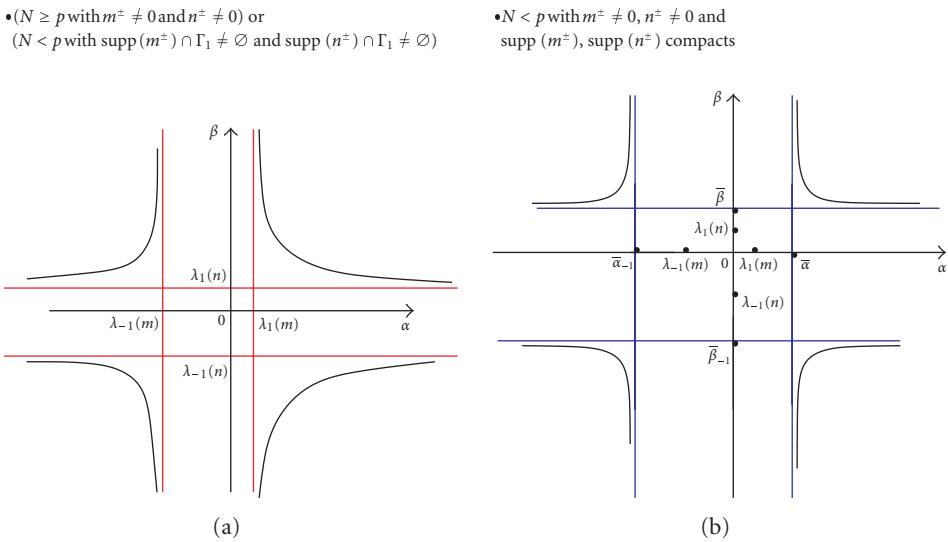
The figures illustrate the result of Proposition 5.3. In Figure 5.1(b), $\overline{\alpha}_{-1}$ is defined by

$$\overline{\alpha}_{-1} = \inf\left\{\int_\Omega |\nabla u^+|^p,\ u \in E(\Omega),\ \int_\Omega m(u^+)^p = -1,\ \int_\Omega n(u^-)^p < 0\right\}, \tag{5.5}$$

and $\overline{\beta}$ and $\overline{\beta}_{-1}$ are deduced from (5.1) and (5.5) by interchanging the roles of the weights m and n.

6. The spectra in dimension 1

In this section, we give a full description of the classical spectrum and the Fučik spectrum $\Sigma = \Sigma(1,1)$ in case of dimension 1. In particular, the first curve in $\Sigma(1,1)$ has been clearly specified.



(a) (b)

Figure 5.1. Mixed Fučik spectrum with weights.

Let $\Omega =]0,\pi[$ and $m \equiv n \equiv 1$ in Ω. Then, (1.2) holds in this case as

$$\begin{gathered} -(\Phi_p(u'))' = \alpha\Phi_p(u^+) - \beta\Phi_p(u^-) \quad \text{in }]0,\pi[, \\ u(0) = u'(\pi) = 0, \end{gathered} \tag{6.1}$$

where $\Phi_p(x) = |x|^{p-2}x$ and $1 < p < +\infty$.

It is well known (cf. [20]) that any weak solution of (6.1) belongs to $C^1([0,\pi])$. Thus, by a solution of (6.1), we mean a function $u \in C^1([0,\pi])$ such that $\Phi_p(u')$ is continuously differentiable in $[0,\pi]$ and satisfies (6.1).

When $\alpha = \beta = \lambda$ in (6.1), one recovers the classical eigenvalue problem

$$\begin{gathered} -(\Phi_p(u'))' = \lambda\Phi_p(u) \quad \text{in }]0,\pi[, \\ u(0) = u'(\pi) = 0. \end{gathered} \tag{6.2}$$

As in Section 2, we look for eigenvalues and nontrivial solutions of (6.1) and (6.2).

The following result gives the description of the spectrum of (6.2).

Theorem 6.1. (i) *The eigenvalues of (6.2) are of the form*

$$\lambda_{k,p} := \left(\frac{2k+1}{2}\frac{\pi_p}{\pi}\right)^p, \quad k \in \mathbb{N} \cup \{0\}, \tag{6.3}$$

where $\pi_p := (2(p-1)^{1/p}/p)(\pi/\sin(\pi/p))$ *and* $\mathbb{N} = \{1,2,3,\dots\}$.

(ii) *The eigenfunctions associated to* $\lambda_{k,p}$, $k \geq 0$, *are the constant multiples of*

$$u(t) = \sin_p(\lambda_{k,p}^{1/p} t), \tag{6.4}$$

where $y(s) = \sin_p(s)$ *is the function implicitly defined by the equation*

$$s = \int_0^y \frac{dt}{(1 - t^p/(p-1))^{1/p}} \tag{6.5}$$

for $s \in [0, \pi_p/2]$, *extended by symmetry on* $[\pi_p/2, \pi_p]$ *and by* π_p*-periodicity on* $\mathbb{R}$ *(cf. [21, 22]).*

Proof. Let us consider the function v defined by $v(t) = \lambda^{-1/p} \sin_p(\lambda^{1/p} t)$, with $\lambda \in \mathbb{R}^*$ and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

It is well known (cf. [22]) that the function v is π_p-periodic and is the unique solution of the equation in (6.2), satisfying the initial value conditions $v(0) = 0$ and $v'(0) = 1$. Moreover, any nontrivial solution u of (6.2) is such that $u'(0) \neq 0$ (cf. [20]) and accordingly u should be of the form

$$u(t) = u'(0)[\lambda^{-1/p} \sin_p(\lambda^{1/p} t)], \tag{6.6}$$

with $\lambda \in \mathbb{R}^*$ to be specified with respect to the second boundary condition $u'(\pi) = 0$.

To go further, let us consider the well known beta function defined by

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt. \tag{6.7}$$

Since it verifies $\beta(z, 1-z) = \pi/\sin(\pi z)$, one deduces from the definition of π_p that

$$\begin{aligned} \frac{\pi_p}{2} &= \frac{(p-1)^{1/p}}{p} \beta\left(\frac{1}{p}, 1 - \frac{1}{p}\right) \\ &= \frac{(p-1)^{1/p}}{p} \int_0^1 z^{1/p-1}(1-z)^{-1/p} dz \\ &= \int_0^{(p-1)^{1/p}} \left(1 - \frac{t^p}{p-1}\right)^{-1/p} dt. \end{aligned} \tag{6.8}$$

Moreover, $(p-1)^{1/p} \in]0, \pi_p/2[$ (using the definition of π_p). Thus, one deduces from (6.5) that

$$\sin_p\left(\frac{\pi_p}{2}\right) = (p-1)^{1/p}. \tag{6.9}$$

On the other hand, one has

$$y'(s) = \frac{dy}{ds}(s) = \left[1 - \frac{sin_p^p(s)}{p-1}\right]^{1/p}, \quad \forall\, s \in \left[0, \frac{\pi_p}{2}\right] \tag{6.10}$$

and the derivative function y' is also π_p-periodic. By solving $y'(s) = 0$, one has

$$\sin_p(s) = (p-1)^{1/p} = \sin_p\left(\frac{\pi_p}{2}\right). \tag{6.11}$$

Hence,

$$y'(s) = 0 \quad \text{iff } s = \frac{\pi_p}{2} + k\pi_p = \frac{(2k+1)\pi_p}{2},\ k \in \mathbb{N} \cup \{0\}. \tag{6.12}$$

Using (6.12) and the fact that $u(t) = u'(0)\lambda^{-1/p} y(\lambda^{1/p} t)$, one deduces that

$$u'(\pi) = 0 \quad \text{iff } \lambda^{1/p}\pi = \left(\frac{2k+1}{2}\right)\pi_p,\ k \in \mathbb{N} \cup \{0\}, \tag{6.13}$$

that is, $\lambda = [((2k+1)/2)(\pi_p/\pi)]^p$, and then the conclusions of Theorem 6.1 follow. □

Remark 6.2. Let

$$\lambda_k := \lambda_{k-1,p}, \quad k \in \mathbb{N}. \tag{6.14}$$

One derives from Theorem 6.1 that the first eigenvalue is defined by

$$\lambda_1 = \lambda_{0,p} = \left(\frac{\pi_p}{2\pi}\right)^p, \tag{6.15}$$

and the spectrum of the p-Laplacian (with mixed boundary conditions) is defined by the sequence

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots. \tag{6.16}$$

The following theorem gives the full description of the Fučik $\Sigma = \Sigma(1,1)$.

Theorem 6.3. *The Fučik spectrum $\Sigma = \Sigma(1,1)$ of (6.1) is composed of two trivial lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$, and the sequence of curves*

$$\begin{aligned}
\mathscr{C}^1_{2k} &= \left\{(\alpha,\beta) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ : \frac{k}{\alpha^{1/p}} + \frac{2k-1}{2\beta^{1/p}} = \frac{\pi}{\pi_p}\right\},\\
\mathscr{C}^2_{2k} &= \left\{(\alpha,\beta) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ : \frac{2k-1}{2\alpha^{1/p}} + \frac{k}{\beta^{1/p}} = \frac{\pi}{\pi_p}\right\},\\
\mathscr{C}^1_{2k+1} &= \left\{(\alpha,\beta) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ : \frac{2k+1}{2\alpha^{1/p}} + \frac{k}{\beta^{1/p}} = \frac{\pi}{\pi_p}\right\},\\
\mathscr{C}^2_{2k+1} &= \left\{(\alpha,\beta) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ : \frac{k}{\alpha^{1/p}} + \frac{2k+1}{2\beta^{1/p}} = \frac{\pi}{\pi_p}\right\}, \quad k \ge 1.
\end{aligned} \tag{6.17}$$

Remark 6.4. The curves $\mathscr{C}^1_{2k}$ and $\mathscr{C}^2_{2k}$ pass through the points $\lambda_{2k}, \lambda_{2k}$; the curves $\mathscr{C}^1_{2k+1}$ and $\mathscr{C}^2_{2k+1}$ pass through the points $\lambda_{2k+1}, \lambda_{2k+1}$.

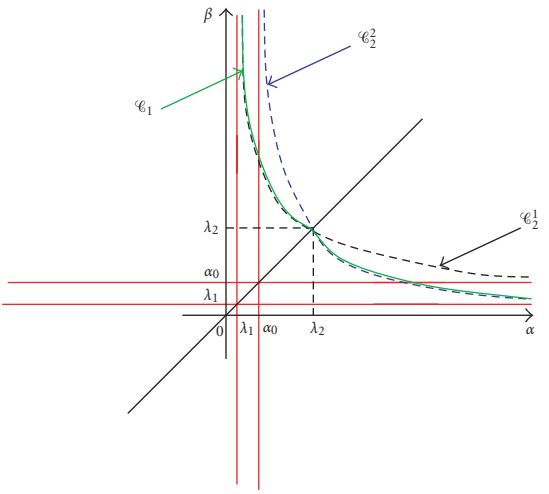


FIGURE 6.1. The first curve in the Fučik spectrum.

Proof of Theorem 6.3. Let us define the functions $u_i, i = 1,2$, by

$$u_1(t) = \begin{cases} \alpha^{-1/p}\sin_p(\alpha^{1/p}t) & \text{if } t \in]0,\alpha^{-1/p}\pi_p], \\ -\beta^{-1/p}\sin_p[\beta^{1/p}(t-\alpha^{-1/p}\pi_p)] & \text{if } t \in [\alpha^{-1/p}\pi_p,(\alpha^{-1/p}+\beta^{-1/p})\pi_p]; \end{cases}$$
$$u_2(t) = \begin{cases} \beta^{-1/p}\sin_p(\beta^{1/p}t) & \text{if } t \in]0,\beta^{-1/p}\pi_p], \\ -\alpha^{-1/p}\sin_p[\alpha^{1/p}(t-\beta^{-1/p}\pi_p)] & \text{if } t \in [\beta^{-1/p}\pi_p,(\alpha^{-1/p}+\beta^{-1/p})\pi_p]. \end{cases} \tag{6.18}$$

Extending these functions by symmetry and by $(\alpha^{-1/p}+\beta^{-1/p})\pi_p$-periodicity, one gets functions $\widetilde{u}_i$ defined on $]0,\pi[$ which verify (6.1). Hence, the conclusion follows by using the fact that $\widetilde{u}_i(0) = \widetilde{u}_i'(\pi) = 0$. □

COROLLARY 6.5. *A first curve $\mathscr{C}_1$ in the Fučik spectrum $\Sigma = \Sigma(1,1)$ of (6.1) is defined by (see Figure 6.1)*

$$\mathscr{C}_1 = \min(\mathscr{C}_2^1, \mathscr{C}_2^2). \tag{6.19}$$

The asymptotic values here are

$$\alpha_\infty = \beta_\infty = \left(\frac{\pi_p}{2\pi}\right)^p, \qquad \alpha_0 = \left(\frac{\pi_p}{\pi}\right)^p. \tag{6.20}$$

Acknowledgment

This work was partially carried out during the stay of L. Leadi at Université Libre de Bruxelles (ULB). He would like to express his gratitude to Professor J. P. Gossez for his constructive suggestions and to Mathematics Department of ULB for its support.

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