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Research Article

δ-Small Submodules and δ-Supplemented Modules

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Let R be a ring and M a right R-module. It is shown that (1) $\delta(M)$ is Noetherian if and only if M satisfies ACC on δ-small submodules; (2) $\delta(M)$ is Artinian if and only if M satisfies DCC on δ-small submodules; (3) M is Artinian if and only if M is an amply δ-supplemented module and satisfies DCC on δ-supplement submodules and on δ-small submodules.

1. Introduction and preliminaries

In this note, all rings are associative with identity and all modules are unital right modules unless otherwise specified.

Let R be a ring and M a module. The concept of δ-small submodules was introduced by Zhou in [1]. Motivated by [2–4], we study modules with ACC (resp., DCC) on δ-small submodules and prove that $\delta(M)$ is Noetherian (resp., Artinian) if and only if M satisfies ACC (resp., DCC) on δ-small submodules in Section 2. In Section 3, we give the concepts of (amply) δ-supplemented modules via δ-small submodules. It is shown that M is Artinian if and only if M is an amply δ-supplemented module and satisfies DCC on δ-supplement submodules and on δ-small submodules. In Section 4, we introduce the concept of δ-semiperfect modules and investigate the connections between δ-supplemented modules and δ-semiperfect modules.

Let M be a module and $N \leq M$. N is said to be δ-small in M (see [5]) if, whenever $N+X=M$ with M/X singular, we have $X=M$. $\delta(M)=\mathrm{Rej}_M(\wp)=\cap\{N \leq M \mid M/N \in \wp\}$, where $\wp$ be the class of all singular simple modules. M is called an *amply supplemented* module if for any two submodules A and B of M with $A+B=M$, B contains a supplement of A. M is called a *supplemented module* if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll B$. The notions which are not explained here will be found in [6].

Lemma 1.1 (see [7, Proposition 5.20]). *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \leq_e M_1 \oplus M_2$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.*

2. Modules with chain conditions on δ-small submodules

In this section, we study modules with chain conditions on δ-small submodules and prove that $\delta(M)$ is Noetherian (resp., Artinian) if and only if M satisfies ACC (resp., DCC) on δ-small submodules. Let us start with the following.

Lemma 2.1 (see [1, Lemma 1.3]). *Let M be a module.*

(i) *For submodules N, K, L of M with $K \leq N$,*
  (1) *$N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$;*
  (2) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*

(ii) *If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \leq N$, then $K \ll_\delta N$.*

(iii) *Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

Lemma 2.2 (see [1, Lemma 1.5]). *Let M and N be modules.*

(1) *$\delta(M) = \Sigma\{L \leq M \mid L$ is a δ-small submodule of $M\}$.*
(2) *If $f : M \to N$ is a homomorphism, then $f(\delta(M)) \leq \delta(N)$.*
(3) *If $M = \bigoplus_{i\in I} M_i$, then $\delta(M) = \bigoplus_{i\in I} \delta(M_i)$.*
(4) *If every proper submodule of M is contained in a maximal submodule of M, then $\delta(M)$ is the unique largest δ-small submodule of M.*

Theorem 2.3. *Let M be a module. Then $\delta(M)$ is Noetherian if and only if M satisfies ACC on δ-small submodules.*

Proof. "⇒" It is clear by Lemma 2.2.

"⇐" Suppose that $\delta(M)$ is not Noetherian. Let $A_1 \leq A_2 \leq \cdots$ be an infinite ascending chain of submodules of $\delta(M)$. Let $a_1 \in A_1$ and $a_j \in A_j - A_{j-1}$ for each $j > 1$. For any $k \geq 1$, let $N_k = \Sigma_{j=1}^{k} a_j R$. Then N_k is finitely generated and $N_k \leq \delta(M)$. Hence $N_k \ll_\delta M$. It is clear that $N_1 \leq N_2 \leq \cdots$ and so M fails to satisfy ACC on δ-small submodules. This completes the proof. □

Recall that a module M has finite uniform dimension k, for some nonnegative k, if M does not contain any infinite direct sum of nonzero submodules and k is the maximal number of summands in a direct sum of nonzero submodules of M. In this case, we call k the uniform dimension of M, and write $\operatorname{udim} M = k$.

Proposition 2.4. *Let M be a module. Then the following statements are equivalent.*

(1) *$\delta(M)$ has finite uniform dimension.*
(2) *Every δ-small submodule of M has finite uniform dimension and there exists a positive integer k such that $\operatorname{udim} N \leq k$ for any $N \ll_\delta M$.*
(3) *M does not contain an infinite direct sum of nonzero δ-small submodules.*

Proof. "(1)⇒(2)" It is obvious because $\operatorname{udim} N \leq \operatorname{udim} \delta(M)$ for any $N \ll_\delta M$.

"(2)⇒(3)" Let $N_1 \oplus N_2 \oplus \cdots$ be an infinite direct sum of nonzero δ-small submodules of M. Then $N_1 \oplus \cdots \oplus N_{k+1}$ is a δ-small submodule of M and $\operatorname{udim}(N_1 \oplus \cdots \oplus N_{k+1}) \geq k+1$. This is a contradiction.

"(3)⇒(1)" Let $N_1 \oplus N_2 \oplus \cdots$ be an infinite direct sum of nonzero submodules of $\delta(M)$. For every $i \geq 1$, let n_i be a nonzero element of N_i. Then $n_iR \ll_\delta M$. Thus $n_1R + n_2R + \cdots$ is an infinite direct sum of nonzero δ-small submodules of M. This is a contradiction and so $\delta(M)$ has finite uniform dimension. □

Theorem 2.5. *Let M be a module. Then the following statements are equivalent.*

(1) *$\delta(M)$ is Artinian.*

(2) *Every δ-small submodule of M is Artinian.*

(3) *M satisfies DCC on δ-small submodules.*

Proof. "(1)⇒(2)⇒(3)" They are clear.

"(3)⇒(1)" It suffices to prove that any factor module of $\delta(M)$ is finitely cogenerated. If there exists a factor module of $\delta(M)$ that is not finitely cogenerated, then the set Ω of submodules of $\delta(M)$, such that $\delta(M)/L$ is not finitely cogenerated, is nonempty. Let $\{L_\lambda : \lambda \in \Lambda\}$ be any chain of submodules in Ω. Let $L = \bigcap_{\lambda\in\Lambda} L_\lambda$. If $L \overline{\in} \Omega$, then $\delta(M)/L$ is finitely cogenerated and hence $L = L_\lambda$ for some $\lambda \in \Lambda$. Thus $L \in \Omega$. By Zorn's lemma, Ω has a minimal member A. □

Let N be a finitely generated submodule of $\delta(M)$. Then N is a δ-small submodule of M and hence Artinian by hypothesis. Thus $\delta(M)$ is locally Artinian. Now let $x \in \delta(M)$, $x \overline{\in} A$. Then xR is Artinian and $(xR+A)/A \simeq xR/(xR\cap A)$. So $(xR+A)/A$ is a nonzero Artinian module and hence $\delta(M)/A$ has essential socle. Let S denote the submodule of $\delta(M)$, containing A, such that S/A is the socle of $\delta(M)/A$. Thus S/A is not finitely generated by [7, Proposition 10.7].

Next we show that $A \ll_\delta M$. If $M = A + B$ for some $B \leq M$ and M/B is singular, then $S = A + (S\cap B)$. Suppose that $A\cap B \neq A$. Then $\delta(M)/(A\cap B)$ is finitely cogenerated by the choice of A. But $S/A = (A + (S\cap B))/A \simeq (S\cap B)/(A\cap B) \leq \operatorname{Soc}(\delta(M)/(A\cap B))$ and hence S/A is finitely generated. This is a contradiction. Thus $A = A\cap B \leq B$ and we have $M = A + B = B$. So $A \ll_\delta M$.

Now suppose that $M = S + V$ of some submodule V of M and M/V is singular. Then $M/(A+V) = (S+V)/(A+V) \simeq S/(A+(S\cap V))$. Thus $M/(A+V)$ is semisimple. If $M \neq A+V$, then there exists a maximal submodule W of M such that $A+V \leq W$. But $S \leq \delta(M) \leq W$ since M/W is a singular simple module and this gives the contradiction $M = W$. Thus $M = A+V$, hence $M = V$ since $A \ll_\delta M$. Thus $S \ll_\delta M$ and hence S is Artinian by hypothesis. It follows that S/A is Artinian, and, in particular, S/A is finitely generated. This is a contradiction. Thus $\delta(M)$ is Artinian.

Example 2.6. Let $R = \mathbb{Z}$, p is a prime and $M = \mathbb{Z}_{(p^\infty)}$, the Prüfer p-group, then every proper submodule of M is Noetherian, but M is not Neotherian. Indeed, every proper submodule of M is δ-small. Moreover, $M = \delta(M)$. Thus every δ-small submodule of M is Noetherian, but $\delta(M)$ is not Noetherian.

Corollary 2.7. *Let R be a ring which satisfies DCC on δ-small right ideals. Then R satisfies ACC on δ-small right ideals.*

Let $N \leq M$. N is called a δ-semimaximal submodule of M if $N = \bigcap_{i=1}^{n} L_i$ with M/L_i singular simple for any $i = 1,\dots,n$.

Proposition 2.8. *Let M be a module. Then the following statements are equivalent.*

(1) *M is Artinian.*

(2) *M satisfies DCC on δ-small submodules and on δ-semimaximal submodules.*

(3) *M satisfies DCC on δ-small submodules and $\delta(M)$ is a δ-semimaximal submodule.*

Proof. "(1)⇒(2)" It is clear.

"(2)⇒(3)" Suppose that M satisfies DCC on δ-semimaximal submodules. Let N be a minimal δ-semimaximal submodule of M. Clearly $\delta(M) \leq N$. If $M = \delta(M)$, then $\delta(M) = N$. Suppose that $M \neq \delta(M)$. If P is a maximal submodule of M with M/P singular, then $N \cap P$ is a δ-semimaximal submodule of M and hence $N = N \cap P$, so that $N \leq P$. It follows that $N \leq \delta(M)$. Hence $N = \delta(M)$. Thus $\delta(M)$ is a δ-semimaximal submodule of M.

"(3)⇒(1)" It is clear $\delta(M)$ is Artinian. If $M = \delta(M)$, then M is Artinian. Suppose that $M \neq \delta(M)$. Then $\delta(M) = P_1 \cap P_2 \cap \cdots \cap P_n$, where M/P_i is singular simple for any $i = 1,\dots,n$. It follows that $M/\delta(M)$ embeds in the finitely generated semisimple module $M/P_1 \oplus \cdots \oplus M/P_n$. Hence $M/\delta(M)$ is Artinian and so M is Artinian. □

3. δ-supplemented modules

Let M be a module. Let N and L be submodules of M. N is called a δ-supplement of L if $M = N + L$ and $N \cap L \ll_\delta N$. N is called a δ-supplement submodule if N is a δ-supplement of some submodule of M. M is called a δ-supplemented module if every submodule of M has a δ-supplement. On the other hand, M is called an amply δ-supplemented module if for any submodules A, B of M with $M = A + B$ there exists a δ-supplement P of A such that $P \leq B$. Clearly, supplemented modules are δ-supplemented modules and every amply δ-supplemented module is δ-supplemented. But the converses are not true.

Lemma 3.1. *Let M be a δ-supplemented module. Then*

(1) *$M/\delta(M)$ is semisimple;*

(2) *L a submodule of M with $L \cap \delta(M) = 0$, then L is semisimple.*

Proof. (1) Let N be any submodule of M containing $\delta(M)$. Then there exists a δ-supplement K of N in M, that is, $M = N + K$ and $N \cap K \ll_\delta K$. Thus $M/\delta(M) = N/\delta(M) \oplus (K + \delta(M))/\delta(M)$, and so every submodule of $M/\delta(M)$ is a direct summand. Therefore $M/\delta(M)$ is semisimple.

(2) It is clear by (1), since $L \cong L \oplus \delta(M)/\delta(M) \leq M/\delta(M)$. □

Proposition 3.2. *Let M be an amply δ-supplemented module. Then homomorphic images are amply δ-supplemented modules.*

Proof. Assume M is amply δ-supplemented and $f : M \to N$ is any epimorphism. We want to show that N is amply δ-supplemented. Let $N = A + B$. Then $M = f^{-1}(A) + f^{-1}(B)$.

Since M is amply δ-supplemented, there exists a submodule X of M such that $M = f^{-1}(A)+X$, $f^{-1}(A)\cap X \ll X \leq f^{-1}(B)$. Now, $N = A + f(X)$ and $A \cap f(X) = f(f^{-1}(A)\cap X) \ll_\delta f(X)$. Clearly $f(X) \leq B$. □

Proposition 3.3. *Let M be a δ-supplemented module. Then $M = N \oplus L$ for some semisimple module N and some module L with $\delta(L) \leq_e L$.*

Proof. For $\delta(M)$, there exists $N \leq M$ such that $N \cap \delta(M) = 0$ and $N \oplus \delta(M) \leq_e M$. Since M is a δ-supplemented module, there exists $L \leq M$ such that $N + L = M$ and $N \cap L \ll_\delta L$. Since $N \cap L = N \cap (N \cap L) \leq N \cap \delta(L) \leq N \cap \delta(M) = 0$, $M = N \oplus L$. By Lemma 3.1, N is semisimple. Thus $\delta(M) = \delta(N) \oplus \delta(L)$. Since $N \oplus \delta(L) \leq_e M = N \oplus L$, $\delta(L) \leq_e L$ by Lemma 1.1. This completes the proof. □

Lemma 3.4. *Let $M_1, U \leq M$ and let M_1 be a δ-supplemented module. If $M_1 + U$ has a δ-supplement in M, then so does U.*

Proof. Since $M_1 + U$ has a δ-supplement in M, there exists $X \leq M$ such that $X + (M_1 + U) = M$ and $X \cap (M_1 + U) \ll_\delta X$. For $(X + U) \cap M_1$, since M_1 is a δ-supplemented module, there exists $Y \leq M_1$ such that $(X + U) \cap M_1 + Y = M_1$ and $(X + U) \cap Y \ll_\delta Y$. Thus we have $X + U + Y = M$ and $(X + U) \cap Y \ll_\delta Y$, that is, Y is a δ-supplement of $X + U$ in M. Next, we will show that $X + Y$ is a δ-supplement of U in M. It is clear that $(X + Y) + U = M$, so it suffices to show that $(X + Y) \cap U \ll_\delta X + Y$. Since $Y + U \leq M_1 + U$, $X \cap (Y + U) \leq X \cap (M_1 + U) \ll_\delta X$. Thus $(X + Y) \cap U \leq X \cap (Y + U) + Y \cap (X + U) \ll_\delta X + Y$ by Lemma 2.1, as required. □

Proposition 3.5. *Let M_1 and M_2 be δ-supplemented modules. If $M = M_1 + M_2$, then M is a δ-supplemented module.*

Proof. Let U be a submodule of M. Since $M_1 + M_2 + U = M$ trivially has a δ-supplement in M, $M_2 + U$ has a δ-supplement in M by Lemma 3.4. Thus U has a δ-supplement in M by Lemma 3.4 again. So M is a δ-supplemented module. □

Proposition 3.6. *If M is a δ-supplemented module, then every finitely M-generated module is a δ-supplemented module.*

Proof. From Proposition 3.5, we know that every finite sum of δ-supplemented modules is a δ-supplemented module. Next we will show that every factor module of a δ-supplemented module is again a δ-supplemented module.

Let M be a δ-supplemented module and M/N any factor module of M. For any submodule L of M containing N, since M is a δ-supplemented module, there exists $K \leq M$ such that $L + K = M$ and $L \cap K \ll_\delta K$. Thus $M/N = L/N + (N + K)/N$ and $(L/N) \cap ((N + K)/N) = (N + (L \cap K))/N \ll_\delta (N + K)/N$, that is, $(N + K)/N$ is a δ-supplement of L/N in M/N, as required. □

Proposition 3.7. *Let M be a module. If every submodule of M is a δ-supplemented module, then M is an amply δ-supplemented module.*

Proof. Let $L, N \leq M$ and $M = N + L$. By assumption, there is $H \leq L$ such that $(L \cap N) + H = L$ and $(L \cap N) \cap H = N \cap H \ll_\delta H$. Thus $H + N \geq H + (L \cap N) = L$ and hence $H + N \geq (N + L) = M$. Therefore, $M = H + N$ as desired. □

Corollary 3.8. *Let R be any ring. Then the following statements are equivalent.*

(1) *Every module is an amply δ-supplemented module.*

(2) *Every module is a δ-supplemented module.*

A module M is said to be *π-projective* if for every two submodules U, V of M with $U+V=M$ there exists $f \in \mathrm{End}(M)$ with $\mathrm{Im}\, f \leq U$ and $\mathrm{Im}(1-f) \leq V$.

Theorem 3.9. *Let M be a module. If M is a π-projective δ-supplemented module, then M is an amply δ-supplemented module.*

Proof. Let A, B be submodules of M such that $M=A+B$. Since M is π-projective, there exists an endomorphism e of M such that $e(M) \leq A$ and $(1-e)(M) \leq B$. Note that $(1-e)(A) \leq A$. Let C be a δ-supplement of A in M. Then $M = e(M)+(1-e)(M) = e(M)+(1-e)(A+C) \leq A+(1-e)(C) \leq M$, so that $M = A+(1-e)(C)$. Note that $(1-e)(C)$ is a submodule of B. Let $y \in A \cap (1-e)(C)$. Then $y \in A$ and $y=(1-e)(x) = x-e(x)$ for some $x \in C$. Next $x = y+e(x) \in A$, so that $y \in (1-e)(A\cap C)$. But $A \cap C \ll_\delta C$ gives that $A \cap (1-e)(C) = (1-e)(A \cap C) \ll_\delta (1-e)(C)$. Thus $(1-e)(C)$ is a δ-supplement of A in M. It follows that M is an amply δ-supplemented module. □

Theorem 3.10. *Let M be a module. Then M is Artinian if and only if M is an amply δ-supplemented module and satisfies DCC on δ-supplement submodules and on δ-small submodules.*

Proof. The necessity is clear. Conversely, suppose that M is an amply δ-supplemented module which satisfies DCC on δ-supplement submodules and on δ-small submodules. Then $\delta(M)$ is Artinian by Theorem 2.5. Next, it suffices to show that $M/\delta(M)$ is Artinian. It is clear that $M/\delta(M)$ is semisimple by Lemma 3.1.

Now suppose that $\delta(M) \leq N_1 \leq N_2 \leq N_3 \leq \cdots$ is an ascending chain of submodules of M. Because M is an amply δ-supplemented module, there exists a descending chain of submodules $K_1 \geq K_2 \geq \cdots$ such that K_i is a δ-supplement of N_i in M for each $i \geq 1$. By hypothesis, there exists a positive integer t such that $K_t = K_{t+1} = K_{t+2} = \cdots$. Because $M/\delta(M) = N_i/\delta(M) \oplus (K_i+\delta(M))/\delta(M)$ for all $i \geq t$, it follows that $N_t = N_{t+1} = \cdots$. Thus $M/\delta(M)$ is Noetherian, and hence finitely generated. So $M/\delta(M)$ is Artinian, as desired. □

Example 3.11. For $\mathbb{Z}_\mathbb{Z}$, the only δ-supplement submodules are 0 and $\mathbb{Z}$ and the only δ-small submodule is 0, but $\mathbb{Z}_\mathbb{Z}$ is not Artinian.

Corollary 3.12. *Let M be a finitely generated δ-supplemented module. Then M is Artinian if and only if M satisfies DCC on δ-small submodules.*

Proof. "$\Leftarrow$" Since $M/\delta(M)$ is semisimple and M is finitely generated, $M/\delta(M)$ is Artinian. Now that M satisfies DCC on δ-small submodules, $\delta(M)$ is Artinian by Theorem 2.5. Thus M is Artinian.

"$\Rightarrow$" It is clear. □

Remark 3.13. Let R be a ring. If R_R is an amply δ-supplemented module, then R is a right Artinian ring if and only if R satisfies DCC on δ-small right ideals. Thus a right perfect ring which satisfies DCC on δ-small right ideals is a right Artinian ring.

Let us end this section with the following.

Proposition 3.14. *If M is a δ-supplemented module and satisfies DCC on δ-small submodules, then so does M/A for any submodule A of M.*

Proof. Let A be any submodule of M and $B_1/A \leq B_2/A \leq \cdots$ where each $B_i/A \ll_\delta M/A$. Let C be a δ-supplement of A in M. Then $M/A = (A+C)/A \simeq C/A \cap C$. Since B_i/A is δ-small in M/A, $B_i/A \simeq D_i/A \cap C \ll C/A \cap C$ for some D_i. Next we prove that $D_i \ll_\delta M$. Let $D_i + E = M$ with M/E singular. Then $(D_i + (E + A \cap C))/A \cap C = M/A \cap C$. Hence $E + A \cap C = M$ and $E = M$. Thus we have $D_1 \leq D_2 \leq \cdots$. Since M satisfies ACC on δ-small submodules, there exists n such that $D_k = D_{k+1}$ for all $k \geq n$. Thus $B_k/A = B_{k+1}/A$ for all $k \geq n$. Therefore M/A satisfies ACC on δ-small submodules, as required. □

4. δ-semiperfect modules

In this section, we introduce the concept of δ-semiperfect modules and investigate the interconnections between δ-supplemented modules and δ-semiperfect modules. Let P and M be modules, we call an epimorphism $f : P \to M$ a δ-cover in case $\operatorname{Ker} f \ll_\delta P$. A δ-cover $f : P \to M$ is called a projective δ-cover in case P is a projective module.

Definition 4.1. A module M is called a δ-semiperfect module if any homomorphic image of M has a projective δ-cover.

Proposition 4.2. *If $f : M \to N$ is an epimorphism with $\operatorname{Ker} f \leq \delta(M)$, then $\delta(N) = f(\delta(M))$.*

Proof. It follows from [7, Corollary 8.17]. □

Lemma 4.3. *If both $f : P \to M$ and $g : M \to N$ are δ-covers, then $gf : P \to N$ is a δ-cover.*

Proof. If both $f : P \to M$ and $g : M \to N$ are δ-covers, then $\operatorname{Ker} f \ll_\delta P$ and $\operatorname{Ker} g \ll_\delta M$. We want to show that $\operatorname{Ker} gf \ll_\delta P$. Let $P = \operatorname{Ker} gf + L$ with P/L singular. Then $M = \operatorname{Ker} g + f(L)$. Since $M/f(L)$ is singular, $M = f(L)$. This implies that $P = L$ since P/L is singular and $\operatorname{Ker} f \ll_\delta P$, as desired. □

Lemma 4.4. *If each $f_i : P_i \to M_i$ $(i = 1, 2, \ldots, n)$ is a δ-cover, then $\bigoplus_{i=1}^n f_i : \bigoplus_{i=1}^n P_i \to \bigoplus_{i=1}^n M_i$ is a δ-cover.*

Proof. It is straightforward. □

Theorem 4.5. *Let M be a module and $U \leq M$. Then the following statements are equivalent.*

(1) *M/U has a projective δ-cover.*

(2) *If $V \leq M$ and $M = U + V$, then U has a δ-supplement $U' \leq V$ such that U' has a projective δ-cover.*

(3) *U has a δ-supplement U' which has a projective δ-cover.*

Proof. "(1)⇒(2)" Let $f : P \to M/U$ be a projective δ-cover. Since $M = U + V$, $g : V \to M/U$ via $v \mapsto v + U$ is an epimorphism. Since P is projective, there is a homomorphism $h : P \to V$ such that $f = gh$. It is easy to see that $M = U + h(P)$, where $h(P) \leq V$. Now $\operatorname{Ker} f \ll_\delta P$, so we have $U \cap h(P) = h(\operatorname{Ker} f) \ll_\delta h(P)$ and $h(P)$ is a δ-supplement of U in M. Since $\operatorname{Ker} h \subseteq \operatorname{Ker} f \ll_\delta P$, $h : P \to h(P)$ is a projective δ-cover.

"(2)⇒(3)" It is obvious.

"(3)⇒(1)" Let $f : P \to U'$ be a projective δ-cover. Since U' is a δ-supplement of U, the natural epimorphism $g : U' \to U'/U \cap U' \simeq U + U'/U = M/U$ is a δ-cover. Hence $hgf : P \to M/U$ is a projective δ-cover by Lemma 4.3, where $h : U'/U \cap U' \simeq U + U'/U$ is an isomorphism □

Theorem 4.6. *Let M be a module. Then the following statements are equivalent.*

(1) *M is δ-semiperfect.*

(2) *M is amply δ-supplemented by δ-supplements which have projective δ-covers.*

(3) *M is δ-supplemented by δ-supplements which have projective δ-covers.*

Proof. It is clear from Theorem 4.5. □

Example 4.7. A δ-semiperfect module is not necessarily semiperfect. Let $Q = \Pi_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R_R is δ-semiperfect but not semiperfect. It is also seen that R_R is a δ-supplemented module but not a supplemented module (see [1, Example 4.1]).

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