Research Article

On the Existence of Positive Solutions of a Nonlinear Differential Equation

Sonia Ben Othman, Habib Mâagli, and Noureddine Zeddini

Received 7 October 2006; Revised 20 December 2006; Accepted 5 January 2007

Recommended by Thomas P. Witelski

We study some existence results for the nonlinear equation $(1/A)(Au')' = u\psi(x, u)$ for $x \in (0, \omega)$ with different boundary conditions, where $\omega \in (0, \infty]$, A is a continuous function on $[0, \omega)$, positive and differentiable on $(0, \omega)$, and ψ is a nonnegative function on $(0, \omega) \times [0, \infty)$ such that $t \mapsto t\psi(x, t)$ is continuous on $[0, \infty)$ for each $x \in (0, \omega)$. We give asymptotic behavior for positive solutions using a potential theory approach.

Copyright © 2007 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

In this paper, we study the following nonlinear equation:

$$\frac{1}{A}(Au')' = f(x, u), \quad \text{in } (0, \omega), \quad (1.1)$$

where $\omega \in (0, \infty]$ and A is a continuous function on $[0, \omega)$, which is positive and differentiable on $(0, \omega)$.

Several results have been obtained for (1.1) with different boundary conditions (see [1–7] and references therein).

In [5], Mâagli and Zeddini generalize the result of Taliaferro [7] who took $A(t) = 1$. Indeed, they studied (1.1) with the following boundary conditions $u(0) = u(1) = 0$ and a nonlinear term $f(x, u) = -\varphi(x, u)$, where φ is a nonnegative continuous function on $(0, 1) \times (0, \infty)$, nonincreasing with respect to the second variable and the function A satisfies $\int_0^1 (1/A(t))dt < \infty$.

Usually $A(t) = t^{n-1}$, $n \geq 2$, so the integral $\int_0^1 (1/A(t))dt$ diverges. The condition $\int_0^1 (1/A(t))dt < \infty$ seems to be too restrictive from an application view point.

Our aim in this paper is to study (1.1) with a nonlinear term $f(x, u) = u\psi(x, u)$ and two boundary conditions. More precisely, we assume that $x \mapsto 1/A(x)$ is integrable in the

neighborhood of ω and the integral $\int_0^\omega (1/A(t))dt$ may diverges and we search a positive continuous solution u of (1.1).

Our paper is organized as follows. In Section 2, we give some properties of the Green's function $G(x, y)$ of the operator $u \mapsto -(1/A)(Au')'$ with $Au'(0) = 0$ and $u(\omega) = 0$, which will be used later. We recall (see [4]) that for x, y in $[0, \omega)$, we have

$$G(x, y) = A(y) \int_{x \vee y}^\omega \frac{1}{A(t)} dt. \quad (1.2)$$

We refer in this paper to Vf , the potential of a measurable nonnegative function f defined on $(0, \omega)$ by

$$Vf(x) = \int_0^\omega G(x, y) f(y) dy. \quad (1.3)$$

Note that Vf is a lower semicontinuous function on $(0, \omega)$. Moreover, for two nonnegative measurable functions f and g with $f \leq g$ and Vg is continuous, we have Vf is continuous.

In Section 3, we are interested to the following problem:

$$\begin{aligned} \frac{1}{A}(Au')' &= u\psi(x, u), \quad \text{a.e in } (0, \omega), \\ u &> 0, \\ \lim_{x \rightarrow 0} \frac{u(x)}{\rho(x)} &= c > 0, \\ u(\omega) &:= \lim_{x \rightarrow \omega} u(x) = 0, \end{aligned} \quad (P_1)$$

where $\rho(x) = \int_x^\omega (1/A(t))dt$.

We assume that ρ and ψ satisfy the following conditions.

(H₀) The function $t \mapsto t\psi(x, t)$ is continuous on $[0, \infty)$ for each $x \in (0, \omega)$.

(H₁) The integral $\int_0^\omega (1/A(t))dt$ diverges.

(H₂) For each $a > 0$, there exists $q_a = q \in K$ such that for $0 \leq s \leq t \leq a$ and $x \in (0, \omega)$, we have

$$t\psi(x, t\rho(x)) - s\psi(x, s\rho(x)) \leq q(x)(t - s), \quad (1.4)$$

where K is the set of nonnegative Borel measurable functions q on $(0, \omega)$ satisfying $\int_0^\omega G(0, y)q(y)dy < \infty$.

Under these hypotheses, we prove the following result.

THEOREM 1.1. *Assume (H₀)–(H₂), then the problem (P₁) has a positive solution $u \in C^1((0, \omega))$ satisfying*

$$c_1\rho(x) \leq u(x) \leq c\rho(x), \quad (1.5)$$

where c_1 is a positive constant.

If we replace hypothesis (H_2) by the following condition:

(H_3) for each $a > 0$, there exists $q_a = q \in K$ such that for $0 \leq s \leq t \leq a$ and $x \in (0, \omega)$, we have

$$t\psi(x, t) - s\psi(x, s) \leq q(x)(t - s), \quad (1.6)$$

we obtain the following result.

THEOREM 1.2. *Under hypotheses (H_0) and (H_3) , the problem*

$$\begin{aligned} \frac{1}{A}(Au')' &= u\psi(x, u), \quad a.e \text{ in } (0, \omega), \\ u &> 0, \\ Au'(0) &:= \lim_{x \rightarrow 0} Au'(x) = 0, \\ u(\omega) &:= \lim_{x \rightarrow \omega} u(x) = c > 0 \end{aligned} \quad (P_2)$$

has a positive bounded solution $u \in C([0, \omega]) \cap C^1((0, \omega))$ satisfying

$$c_1 \leq u(x) \leq c, \quad \forall x \in (0, \omega), \quad (1.7)$$

where c_1 is a positive constant.

In order to simplify our statements, we adopt the following notation.

Notation.

- (i) $\mathcal{B}((0, \omega))$ denotes the set of Borel measurable functions on $(0, \omega)$.
- (ii) $\mathcal{B}^+((0, \omega))$ is the set of nonnegative Borel measurable functions on $(0, \omega)$.
- (iii) We denote by $C([0, \omega]) := \{u \in C((0, \omega)), \lim_{x \rightarrow 0} u(x), \text{ and } \lim_{x \rightarrow \omega} u(x) \text{ exist}\}$, and by $C^+([0, \omega])$ the set of nonnegative ones.
- (iv) Let f and g be two positive functions defined on a set S .
 - (a) We call $f \preceq g$, if there exists a constant $c > 0$, such that

$$f(x) \leq cg(x), \quad \forall x \in S. \quad (1.8)$$

- (b) We call $f \sim g$, if there exists a constant $c > 0$ such that

$$\frac{1}{c}g(x) \leq f(x) \leq cg(x), \quad \forall x \in S. \quad (1.9)$$

- (v) For $s, t \in [0, \omega)$, we denote $s \vee t = \max(s, t)$.

- (vi) We denote

$$K = \left\{ q \in \mathcal{B}^+((0, \omega)), \int_0^\omega G(0, y)q(y)dy < \infty \right\}. \quad (1.10)$$

2. Properties of Green's function

In the sequel, we denote

$$\begin{aligned}\Gamma(x, y) &= \int_{x \vee y}^{\omega} \frac{1}{A(t)} dt, \quad \text{for } x, y \in [0, \omega), \\ \delta(x) &= \min \left(1, \int_x^{\omega} \frac{1}{A(t)} dt \right), \quad \text{for } x \in (0, \omega).\end{aligned}\tag{2.1}$$

Let $a \in (0, \omega)$, then for each $x \in (0, \omega)$, we have

$$\Gamma(x, a) \sim \delta(x).\tag{2.2}$$

Indeed, the result follows from the following inequalities:

$$\min(\alpha, 1) \min(1, \beta) \leq \min(\alpha, \beta) \leq \max(\alpha, 1) \min(1, \beta), \quad \text{for } \alpha, \beta > 0.\tag{2.3}$$

First, we give the following version of a comparison principle.

PROPOSITION 2.1. *The following properties hold.*

(1) Let $f \in \mathcal{B}^+((0, \omega))$, then for a fixed $a \in (0, \omega)$,

$$Vf(x) \geq Vf(a) \frac{\Gamma(x, a)}{\Gamma(a, a)}, \quad \forall x \in [0, \omega).\tag{2.4}$$

(2) The function $x \mapsto \Gamma(x, 0)/\delta(x)$, is nonincreasing on $(0, \omega)$.

(3) For each $x, y \in (0, \omega)$,

$$\frac{\delta(y)}{\delta(x)} G(x, y) \leq G(0, y).\tag{2.5}$$

Proof. (1) Let $x, y \in [0, \omega)$ and $a \in (0, \omega)$, then we have

$$\Gamma(x, y) \Gamma(a, a) \geq \Gamma(x, a) \Gamma(a, y),\tag{2.6}$$

which implies the result.

(2) It follows from the fact that $x \mapsto \Gamma(x, 0)$ is nonincreasing and

$$\frac{\Gamma(x, 0)}{\delta(x)} = \max(1, \Gamma(x, 0)).\tag{2.7}$$

(3) For $x, y \in (0, \omega)$, we distinguish the following cases:

(i) if $y \leq x$, then

$$\begin{aligned}\frac{\delta(y)}{\delta(x)} G(x, y) &= \delta(y) A(y) \frac{\Gamma(x, y)}{\delta(x)} \\ &\leq \delta(y) A(y) \frac{\Gamma(x, 0)}{\delta(x)} \leq A(y) \Gamma(y, 0) = G(0, y);\end{aligned}\tag{2.8}$$

(ii) if $y \geq x$, then $\delta(y)/\delta(x) \leq 1$, which implies that

$$\frac{\delta(y)}{\delta(x)} G(x, y) \leq G(x, y) \leq G(0, y), \quad (2.9)$$

and this completes the proof. $\square$

Next, we will give some inequalities satisfied by Green's function.

THEOREM 2.2 (3G-theorem). *For each $x, y, z \in [0, \omega)$,*

$$\frac{G(x, z)G(z, y)}{G(x, y)} \leq \frac{\delta(z)}{\delta(x)} G(x, z) + \frac{\delta(z)}{\delta(y)} G(y, z). \quad (2.10)$$

Proof. We remark that assertion (2.10) is equivalent to

$$\frac{\Gamma(x, z)\Gamma(z, y)}{\Gamma(x, y)} \leq \frac{\delta(z)}{\delta(x)} \Gamma(x, z) + \frac{\delta(z)}{\delta(y)} \Gamma(z, y). \quad (2.11)$$

Since $\Gamma(x, y)$ is symmetric in x, y , we will discuss three cases.

- (i) If $z \leq x \leq y$, then $\Gamma(x, z) = \int_x^\omega (1/A(t))dt$, $\Gamma(z, y) = \int_y^\omega (1/A(t))dt$, and $\Gamma(x, y) = \int_y^\omega (1/A(t))dt$. Since $\delta(z)/\delta(x) \geq 1$, then we have the result.
- (ii) If $x \leq y \leq z$, then we obtain $\Gamma(x, z) = \Gamma(z, y) = \Gamma(z, 0)$ and $\Gamma(x, y) = \Gamma(y, 0)$. Hence, we have

$$(2.11) \iff \frac{\Gamma(z, 0)}{\delta(z)} \leq \frac{\Gamma(y, 0)}{\delta(x)} + \frac{\Gamma(y, 0)}{\delta(y)}. \quad (2.12)$$

Now, using the second assertion of Proposition 2.1, we obtain the result.

- (iii) If $x \leq z \leq y$, then we obtain $\Gamma(x, z) = \int_z^\omega (1/A(t))dt$, $\Gamma(z, y) = \int_y^\omega (1/A(t))dt$, and $\Gamma(x, y) = \int_y^\omega (1/A(t))dt$. So if $\delta(z) = 1$, then $\delta(x) = 1$, and if $\delta(z) = \int_z^\omega (1/A(t))dt$, then $\delta(y) = \int_y^\omega (1/A(t))dt$.

This proves (2.11). $\square$

In the sequel, for a fixed $q \in \mathcal{B}^+((0, \omega))$, we put

$$\begin{aligned} \|q\| &= \sup_{x \in (0, \omega)} \int_0^\omega \frac{\delta(y)}{\delta(x)} G(x, y) q(y) dy, \\ \alpha_q &= \sup_{x, y \in (0, \omega)} \int_0^\omega \frac{G(x, z)G(z, y)}{G(x, y)} q(z) dz. \end{aligned} \quad (2.13)$$

Then, we have the following result.

PROPOSITION 2.3. *Let $q \in K$, then*

$$\|q\| \leq Vq(0) \leq \alpha_q \leq 2\|q\|. \quad (2.14)$$

Proof. By Proposition 2.1, we have $(\delta(y)/\delta(x))G(x, y) \leq G(0, y)$, which implies that $\|q\| \leq Vq(0)$.

On the other hand, using Lebesgue's theorem and Fatou's lemma, we obtain that

$$\begin{aligned}
 Vq(0) &= \int_0^\omega G(0,z)q(z)dz = \sup_{x \in (0,\omega)} \int_0^\omega G(x,z)q(z)dz \\
 &= \sup_{x \in (0,\omega)} \int_0^\omega \lim_{y \rightarrow \omega} \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz \leq \sup_{x \in (0,\omega)} \liminf_{y \rightarrow \omega} \int_0^\omega \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz \\
 &\leq \sup_{x,y \in (0,\omega)} \int_0^\omega \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz = \alpha_q.
 \end{aligned} \tag{2.15}$$

Now, by (2.10) we deduce that

$$\alpha_q \leq 2\|q\|. \tag{2.16}$$

This completes the proof. $\square$

Remark 2.4. It is clear that if $q \in K$, then the function

$$x \mapsto Vq(x) = \int_x^\omega \frac{1}{A(t)} \left(\int_0^t A(s)q(s)ds \right) dt \tag{2.17}$$

is continuous on $[0, \omega)$

In the next two propositions, we will give some estimates on the potential Vq , for a convenient function q .

PROPOSITION 2.5. *Let $\lambda \geq 0$, $\alpha < \min(\lambda + 1, 2)$, and $\beta < 2$. Put $A(x) = x^\lambda$ and $q(x) = 1/x^\alpha(1-x)^\beta$, for $x \in (0, 1)$. Then*

$$Vq(x) \sim \begin{cases} (1-x)^{2-\beta} & \text{if } 1 < \beta < 2, \\ (1-x) \log\left(\frac{2}{1-x}\right) & \text{if } \beta = 1, \\ (1-x) & \text{if } \beta < 1. \end{cases} \tag{2.18}$$

Proof. Since the function $x \mapsto Vq(x)$ is continuous and positive on $[0, 1/2]$, then we deduce that $Vq(x) \sim 1$, for $x \in [0, 1/2]$.

Now, assume that $x \in [1/2, 1)$. Using the fact that for $t \in [x, 1)$, we have $1/2 \leq t \leq 1$, then we obtain

$$Vq(x) \sim \int_x^1 \left(\int_0^t \frac{s^{\lambda-\alpha}}{(1-s)^\beta} ds \right) dt. \tag{2.19}$$

Since $\alpha < \min(\lambda + 1, 2)$ and $\beta < 2$, then for each $t \in [x, 1)$, we have

$$\begin{aligned}
 \int_0^t \frac{s^{\lambda-\alpha}}{(1-s)^\beta} ds &\sim \left(\int_0^{1/2} s^{\lambda-\alpha} ds + \int_{1/2}^t (1-s)^{-\beta} ds \right) \\
 &\sim \left(1 + \int_{1/2}^t (1-s)^{-\beta} ds \right).
 \end{aligned} \tag{2.20}$$

- (i) If $\beta < 1$, then since $\int_{1/2}^t (1-s)^{-\beta} ds = (1/(1-\beta))((1/2)^{1-\beta} - (1-t)^{1-\beta})$, we deduce that $1 + \int_{1/2}^t (1-s)^{-\beta} ds \sim 1$. So $Vq(x) \sim 1 - x$.
- (ii) If $\beta > 1$, then since $\int_{1/2}^t (1-s)^{-\beta} ds \sim (1-t)^{1-\beta} - 2^{\beta-1}$, we deduce that $1 + \int_{1/2}^t (1-s)^{-\beta} ds \sim (1-t)^{1-\beta}$. So $Vq(x) \sim \int_x^1 (1-t)^{1-\beta} dt \sim (1-x)^{2-\beta}$.
- (iii) If $\beta = 1$, then since $\int_{1/2}^t (1-s)^{-1} ds = \log(1/2(1-t))$, we deduce that $\int_0^t (s^{\lambda-\alpha}/(1-s)^\beta) ds \sim \log(e/2(1-t))$.

Now using the fact that for $\mu \in \mathbb{R}$ and $\sigma > 1$,

$$\int_x^{+\infty} \frac{(\log(t))^\mu}{t^\sigma} dt \sim \frac{(\log(x))^\mu}{(\sigma-1)x^{\sigma-1}}, \quad \text{as } x \rightarrow \infty, \quad (2.21)$$

we deduce that $\int_x^1 \log(e/2(1-t)) dt \sim (1-x) \log(e/2(1-x))$, as $x \rightarrow 1$.

So $Vq(x) \sim (1-x) \log(e/2(1-x))$.

Thus, by combination of the two cases we obtain the result. $\square$

The following results will be used later.

Let $q \in K$ and $\varphi \in C^+([0, \omega]) \cap C^1((0, \omega))$ be the solution of the problem

$$\begin{aligned} \frac{1}{A}(Au')' - qu &= 0 \quad \text{in } (0, \omega), \\ Au'(0) &= 0, \quad u(0) = 1. \end{aligned} \quad (Q)$$

Then we have the following.

PROPOSITION 2.6 (see [4]). (i) φ is nondecreasing on $[0, \omega)$.

(ii) For each $x \in [0, \omega)$,

$$1 \leq \varphi(x) \leq e^{(Vq(0) - Vq(x))}. \quad (2.22)$$

In the sequel, we denote by

$$G_q(x, y) = A(y)\varphi(y)\varphi(x) \int_{x \vee y}^\omega \frac{1}{A(t)\varphi^2(t)} dt \quad (2.23)$$

the Green's function of the operator $u \mapsto -(1/A)(Au')' + qu$, with $Au'(0) = 0$ and $u(\omega) = 0$. Let $V_q f(x) = \int_0^\omega G_q(x, y)f(y)dy$, for $f \in \mathcal{B}^+((0, \omega))$. Then we have the following.

PROPOSITION 2.7. Let $q \in K$, then the following resolvent equation holds:

$$V = V_q + V_q(qV) = V_q + V(qV_q). \quad (2.24)$$

Moreover, for each $u \in \mathcal{B}^+((0, \omega))$ such that $V(qu) < \infty$,

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \quad (2.25)$$

For each $x, y \in [0, \omega)$,

$$e^{-Vq(0)}G(x, y) \leq G_q(x, y) \leq G(x, y), \quad \forall x, y \in (0, \omega), \quad (2.26)$$

$$1 - V_q(q)(x) \geq e^{-Vq(0)}, \quad \forall x \in [0, \omega), \quad (2.27)$$

$$(\rho - V_q(q\rho))(x) \geq \rho(x)e^{-Vq(0)}, \quad (2.28)$$

where $\rho(x) = \Gamma(x, 0) = \int_x^\omega (1/A(t))dt$.

Proof. The proofs of (2.25) and (2.26) can be found in [4, Theorem 4].

For each $x \in (0, \omega)$, we obtain by Fubini-Tonelli's theorem that

$$\begin{aligned} V_q(q)(x) &= \int_0^\omega G_q(x, y)q(y)dy \\ &= \int_0^\omega A(y)\varphi(y)\varphi(x) \left(\int_{x \vee y}^\omega \frac{1}{A(t)\varphi^2(t)} dt \right) q(y)dy \\ &= \varphi(x) \int_x^\omega \frac{1}{A(t)\varphi^2(t)} \left(\int_0^t A(y)\varphi(y)q(y)dy \right) dt. \end{aligned} \quad (2.29)$$

Now using that φ is the solution of the problem (Q), and by integrating by parts, we have

$$V_q(q)(x) = 1 - \frac{\varphi(x)}{\varphi(\omega)}. \quad (2.30)$$

On the other hand, we deduce from (2.22) that

$$0 < \varphi(\omega) \leq e^{Vq(0)}, \quad (2.31)$$

which proves (2.27).

For each $x \in (0, \omega)$, we obtain by Fubini-Tonelli's theorem that

$$\begin{aligned} V_q(q\rho)(x) &= \int_0^\omega G_q(x, y)q(y)\rho(y)dy \\ &= \int_0^\omega A(y)\varphi(y)\varphi(x) \left(\int_{x \vee y}^\omega \frac{1}{A(t)\varphi^2(t)} dt \right) q(y)\rho(y)dy \\ &= \varphi(x) \int_x^\omega \frac{1}{A(t)\varphi^2(t)} \left(\int_0^t A(y)\varphi(y)q(y)\rho(y)dy \right) dt. \end{aligned} \quad (2.32)$$

Now using that φ is the solution of the problem (Q) and ρ is differentiable on $(0, \omega)$, we obtain by integrating by parts that

$$\rho(x) - V_q(q\rho)(x) = \varphi(x) \int_x^\omega \frac{1}{A(t)\varphi^2(t)} dt = \frac{G_q(0, x)}{A(x)}. \quad (2.33)$$

Hence, from the lower inequality of (2.26), we deduce that

$$\rho(x) - V_q(q\rho)(x) \geq e^{-Vq(0)} \frac{G(0, x)}{A(x)} = e^{-Vq(0)} \Gamma(x, 0) = e^{-Vq(0)} \rho(x). \quad (2.34)$$

This completes the proof of (2.28). $\square$

3. Proofs of the main results

In this section, we aim at proving Theorems 1.1 and 1.2.

We recall that $\rho(x) = \int_x^\omega (1/A(t))dt$.

Proof of Theorem 1.1. Let $c > 0$ and $q \in K$ satisfying (H_2) . We denote by

$$\Lambda := \{u \in \mathcal{B}^+((0, \omega)); c\rho e^{-Vq(0)} \leq u \leq c\rho\} \quad (3.1)$$

the nonempty convex set of $\mathcal{B}^+((0, \omega))$, and we define the operator T on Λ by

$$Tu(x) := c(\rho(x) - V_q(q\rho)(x)) + V_q(u(q - \psi(\cdot, u)))(x), \quad \forall x \in (0, \omega). \quad (3.2)$$

We claim that $T\Lambda \subset \Lambda$. Indeed, for $u \in \Lambda$ we have by (H_2) ,

$$Tu \leq c\rho - cV_q(q\rho) + cV_q(\rho(q - \psi(\cdot, c\rho))) \leq c\rho - cV_q(\psi(\cdot, c\rho)) \leq c\rho. \quad (3.3)$$

On the other hand, by using (2.28), we obtain that

$$Tu \geq c(\rho - V_q(q\rho)) \geq c\rho e^{-Vq(0)}. \quad (3.4)$$

Hence $T\Lambda \subset \Lambda$. Next, we prove that the operator T is nondecreasing on Λ . Let $u_1, u_2 \in \Lambda$ such that $u_1 \leq u_2$, then from (H_2) , we have for each $x \in (0, \omega)$,

$$Tu_2(x) - Tu_1(x) = V_q[q(u_2 - u_1) + u_1\psi(\cdot, u_1) - u_2\psi(\cdot, u_2)](x) \geq 0. \quad (3.5)$$

Now, we consider the sequence $(u_j)_j$ defined by $u_0(x) = c(\rho(x) - V_q(q\rho)(x))$ and $u_{j+1}(x) = Tu_j(x)$, for $j \in \mathbb{N}$ and $x \in (0, \omega)$. Then since Λ is invariant under T , we have obviously $u_1 = Tu_0 \geq u_0$ and so from the monotonicity of T , we deduce that

$$u_0 \leq u_1 \leq \dots \leq u_j \leq c\rho. \quad (3.6)$$

Hence, the sequence $(u_j)_j$ converges on $(0, \omega)$ to a function $u \in \Lambda$. Now, using (H_0) , (H_2) , and the dominated convergence theorem, we deduce that $(Tu_j)_j$ converges to Tu on $(0, \omega)$. Consequently, we have

$$u(x) = c(\rho(x) - V_q(c\rho)(x)) + V_q(u(q - \psi(\cdot, u)))(x), \quad (3.7)$$

or equivalently

$$u(x) - V_q(qu)(x) = c\rho(x) - V_q(c\rho + u\psi(\cdot, u))(x). \quad (3.8)$$

Applying the operator $(I + V(q, \cdot))$ on both sides of the above equality and using (2.25), we deduce that u satisfies

$$u = c\rho - V(u\psi(\cdot, u)). \quad (3.9)$$

Finally, we need to verify that u is a positive continuous solution for the problem (P_1) . Indeed, by (H_2) we have $u\psi(\cdot, u) \leq c\rho$, then using the fact that $q \in K$ and ρ is bounded

on each interval $[x_0, \omega)$ with $x_0 > 0$, we deduce the continuity of $V(cq\rho)$, which implies the continuity of u on $(0, \omega)$. Now since $q \in K$ and for each $x, y \in (0, \omega)$, we have

$$A(y) \frac{\rho(y)\rho(x \vee y)}{\rho(x)} \leq A(y)\rho(y)q(y) = G(0, y)q(y), \quad (3.10)$$

then we obtain by (H_1) and the dominated convergence theorem that

$$\lim_{x \rightarrow 0} \frac{V(q\rho)(x)}{\rho(x)} = 0, \quad (3.11)$$

which implies that $\lim_{x \rightarrow 0} (u(x)/\rho(x)) = c$. This completes the proof. $\square$

Example 3.1. Let $\gamma > 1$ and let p be a nonnegative Borel measurable function on $(0, 1)$ such that $\int_0^1 \gamma p(y)(\log(1/y))^\gamma dy < \infty$. Then the problem

$$\begin{aligned} \frac{1}{x}(xu')' - p(x)u^\gamma &= 0, \quad \text{in } (0, 1), \\ u &> 0, \\ \lim_{x \rightarrow 0} \frac{u(x)}{\log(1/x)} &= c > 0, \quad u(1) = 0, \end{aligned} \quad (3.12)$$

has a positive solution $u \in C^2((0, 1))$ satisfying

$$u(x) \sim \log\left(\frac{1}{x}\right). \quad (3.13)$$

Example 3.2. Let $\gamma > 1, \lambda > 1$, and put $A(x) = x^\lambda$. Let p be a nonnegative Borel measurable function on $(0, \infty)$ such that $\int_0^\infty (p(y)/y^{(\lambda-1)(\gamma-1)-1}) dy < \infty$. Then the following problem:

$$\begin{aligned} \frac{1}{x^\lambda}(x^\lambda u')' - p(x)u^\gamma &= 0, \quad \text{in } (0, \infty), \\ u &> 0, \\ \lim_{x \rightarrow 0} \frac{u(x)}{x^{1-\lambda}} &= c > 0, \quad \lim_{x \rightarrow \infty} u(x) = 0, \end{aligned} \quad (3.14)$$

has a positive solution $u \in C((0, \infty))$ satisfying

$$u(x) \sim x^{1-\lambda}. \quad (3.15)$$

In the next, we will give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $c > 0$, then by hypothesis (H_3) , there exists $q \in K$ such that the function $t \mapsto t(\psi(x, t) - q(x))$ is nonincreasing on $[0, c]$. We consider the nonempty closed convex set Λ given by

$$\Lambda = \{u \in C([0, \omega]); ce^{-Vq(0)} \leq u(x) \leq c\}, \quad (3.16)$$

and we define the operator T on Λ by

$$Tu := c(1 - V_q(q)) + V_q((q - \psi(\cdot, u))u), \quad (3.17)$$

and $Tu(\omega) := \lim_{x \rightarrow \omega} Tu(x) = c$.

Now, by similar arguments as in the proof of Theorem 1.1, we obtain that $T\Lambda \subset \Lambda$ and T is an increasing operator on Λ . Let $(u_n)_n$ be the sequence of functions defined by

$$\begin{aligned} u_0 &= c(1 - V_q(q)), \\ u_{n+1} &= Tu_n, \quad \text{for } n \in \mathbb{N}. \end{aligned} \quad (3.18)$$

Then the sequence $(u_n)_n$ converges to a function $u = \sup_n u_n \in \Lambda$, satisfying

$$u = c - V(u\psi(\cdot, u)). \quad (3.19)$$

Since we have $\psi(\cdot, u) \leq q$ and $Vq \in C^+([0, \omega))$, then $V(qu) \in C^+([0, \omega))$ and consequently $V(u\psi(\cdot, u)) \in C^+([0, \omega))$. Hence, u is a positive continuous solution of the problem (P_2) . $\square$

Example 3.3. Let $\gamma, \lambda \geq 0$, $\alpha < \min(\lambda + 1, 2)$, and $\beta < 2$. Put $A(x) = x^\lambda$, for $x \in (0, 1)$. Then the problem

$$\begin{aligned} \frac{1}{A}(Au')' - \frac{u^{\gamma+1}(x)}{x^\alpha(1-x)^\beta} &= 0, \quad \text{in } (0, 1), \\ Au'(0) &= 0, \quad u(1) = c > 0 \end{aligned} \quad (3.20)$$

has a positive solution $u \in C([0, 1]) \cap C^1((0, 1))$ satisfying for each x in $(0, 1)$

$$0 \leq c - u(x) \leq \begin{cases} (1-x)^{2-\beta} & \text{if } 1 < \beta < 2, \\ (1-x) \log\left(\frac{2}{1-x}\right) & \text{if } \beta = 1, \\ (1-x) & \text{if } \beta < 1. \end{cases} \quad (3.21)$$

References

- [1] R. P. Agarwal and D. O'Regan, "Twin solutions to singular Dirichlet problems," *Journal of Mathematical Analysis and Applications*, vol. 240, no. 2, pp. 433–445, 1999.
- [2] R. P. Agarwal and D. O'Regan, "Existence theory for single and multiple solutions to singular positone boundary value problems," *Journal of Differential Equations*, vol. 175, no. 2, pp. 393–414, 2001.
- [3] R. Kannan and D. O'Regan, "A note on singular boundary value problems with solutions in weighted spaces," *Nonlinear Analysis*, vol. 37, no. 6, pp. 791–796, 1999.
- [4] H. Mâagli and S. Masmoudi, "Sur les solutions d'un opérateur différentiel singulier semi-linéaire," *Potential Analysis*, vol. 10, no. 3, pp. 289–304, 1999.
- [5] H. Mâagli and N. Zeddini, "Positive solutions for a singular nonlinear Dirichlet problem," *Nonlinear Studies*, vol. 10, no. 4, pp. 295–306, 2003.

- [6] S. Masmoudi and N. Yazidi, "On the existence of positive solutions of a singular nonlinear differential equation," *Journal of Mathematical Analysis and Applications*, vol. 268, no. 1, pp. 53–66, 2002.
- [7] S. D. Taliaferro, "A nonlinear singular boundary value problem," *Nonlinear Analysis*, vol. 3, no. 6, pp. 897–904, 1979.

Sonia Ben Othman: Département de Mathématiques, Faculté des Sciences de Tunis,
Campus Universitaire, Tunis 2092, Tunisia
Email address: sonia.benothman@fsb.rnu.tn

Habib Mâagli: Département de Mathématiques, Faculté des Sciences de Tunis,
Campus Universitaire, Tunis 2092, Tunisia
Email address: habib.maagli@fst.rnu.tn

Noureddine Zeddini: Département de Mathématiques, Faculté des Sciences de Tunis,
Campus Universitaire, Tunis 2092, Tunisia
Email address: noureddine.zeddini@ipein.rnu.tn