Research Article

Sufficient Conditions for Janowski Starlikeness

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Let $A, B, D, E \in [-1, 1]$ and let $p(z)$ be an analytic function defined on the open unit disk, $p(0) = 1$. Conditions on A, B, D , and E are determined so that $1 + \beta zp'(z)$ being subordinated to $(1 + Dz)/(1 + Ez)$ implies that $p(z)$ is subordinated to $(1 + Az)/(1 + Bz)$. Similar results are obtained by considering the expressions $1 + \beta(zp'(z)/p(z))$ and $1 + \beta(zp'(z)/p^2(z))$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

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1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $S^*[A, B]$ denote the class of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1). \quad (1.1)$$

Functions in $S^*[A, B]$ are called the *Janowski starlike functions* ([1, 2]). Certain well-known subclasses of starlike functions are special cases of the class $S^*[A, B]$ for suitable choices of the parameters A and B . For example, when $0 \leq \alpha < 1$, $S^*[1 - 2\alpha, -1] =: S_\alpha^*$ is the familiar class of starlike functions of order α and $S^*[1 - \alpha, 0] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \alpha \quad (z \in U)\} =: S^*(\alpha)$. For $0 < \alpha \leq 1$, let $S^*[\alpha, -\alpha] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \alpha |zf'(z)/f(z) + 1| \quad (z \in U)\} =: S^*[\alpha]$.

Silverman [3], Obradowic and Tuneski [4], and many others (see [5–9]) have studied properties of functions defined in terms of the quotient $(1 + zf''(z)/f'(z))/(zf'(z)/f(z))$. In fact, Silverman [3] has obtained the order of starlikeness for the functions in the class

G_b defined by

$$G_b := \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, 0 < b \leq 1, z \in U \right\} \quad (1.2)$$

and Obradowic and Tuneski [4] improved the result of Silverman [3] by showing $G_b \subset S^*[0, -b] \subset S^*(2/(1 + \sqrt{1 + 8b}))$. Later, Tuneski [10] obtained conditions for the inclusion $G_b \subset S^*[A, B]$ to hold. If we let $zf'(z)/f(z) =: p(z)$, then $G_b \subset S^*[A, B]$ becomes

$$1 + \frac{zp'(z)}{p(z)^2} < 1 + bz \implies p(z) < \frac{1 + Az}{1 + Bz}. \quad (1.3)$$

Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. Frasin and Darus [11] have shown that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} < \frac{(1 - \alpha)z}{2 - \alpha} \implies \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha. \quad (1.4)$$

Again by writing $z^2 f'(z)/(f(z))^2$ as $p(z)$, we see that the above implication is special case of

$$1 + \beta \frac{zp'(z)}{p(z)} < \frac{1 + Dz}{1 + Ez} \implies p(z) < \frac{1 + Az}{1 + Bz}. \quad (1.5)$$

Another special case of the above implications was considered by Ponnusamy and Rajasekaran [12].

Nunokawa et al. [13] have shown that if $p(z)$ is analytic in U , $p(0) = 1$ and $1 + zp'(z) < 1 + z$, then $p(z) < 1 + z$. Using this, they have obtained a criterion for a normalized analytic function to be univalent. In this paper, we extend the result by replacing the subordinate function $1 + z$ by a function of the form $(1 + Dz)/(1 + Ez)$. In fact, we determine conditions on $A, B, D, E \in [-1, 1]$ so that

$$1 + \beta zp'(z) < \frac{1 + Dz}{1 + Ez} \implies p(z) < \frac{1 + Az}{1 + Bz}. \quad (1.6)$$

Similar results are obtained by considering the expressions $1 + \beta(zp'(z)/p^2(z))$, $1 + \beta(zp'(z)/p(z))$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

2. Differential subordination

LEMMA 2.1. Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$, and $\beta \neq 0$. Assume that

$$(A - B)|\beta| \geq (D - E)(1 + B^2) + |2B(D - E) - E\beta(A - B)|. \quad (2.1)$$

If $p(z)$ is analytic in U with $p(0) = 1$ and

$$1 + \beta z p'(z) \prec \frac{1 + Dz}{1 + Ez}, \quad (2.2)$$

then

$$p(z) \prec \frac{1 + Az}{1 + Bz}. \quad (2.3)$$

Proof. Define the function $P(z)$ by

$$P(z) := 1 + \beta z p'(z), \quad (2.4)$$

and the function $w(z)$ by

$$w(z) := \frac{p(z) - 1}{A - Bp(z)}, \quad (2.5)$$

or equivalently by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.6)$$

Then $w(z)$ is meromorphic in U and $w(0) = 0$. We need to show that $|w(z)| < 1$ in U . By a computation, we get

$$P(z) = \frac{(1 + Bw(z))^2 + (A - B)\beta zw'(z)}{(1 + Bw(z))^2}. \quad (2.7)$$

Therefore

$$\frac{P(z) - 1}{D - EP(z)} = \frac{(A - B)\beta zw'(z)}{(D - E)(1 + Bw(z))^2 - E(A - B)\beta zw'(z)}. \quad (2.8)$$

Assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \quad (2.9)$$

Then by [14, Lemma 1.3, page 28], there exists $k \geq 1$ such that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{i\theta}$. For this z_0 , we have

$$\begin{aligned} \left| \frac{P(z_0) - 1}{D - EP(z_0)} \right| &= \frac{(A - B)k|\beta|}{[I^2 + (H - J)^2 + 4HJt^2 + 4I(H + J)t]^{1/2}} \\ &\geq \frac{(A - B)k|\beta|}{\max_{-1 \leq t \leq 1} \{[I^2 + (H - J)^2 + 4HJt^2 + 4I(H + J)t]^{1/2}\}}, \end{aligned} \quad (2.10)$$

where $I := 2B(D - E) - k\beta E(A - B)$, $J := (D - E)B^2$, $H := (D - E)$, and $t := \cos \theta$. A computation shows that

$$\left| \frac{P(z_0) - 1}{D - EP(z_0)} \right| \geq \frac{(A - B)|\beta|k}{H + |I| + J}. \quad (2.11)$$

Yet another calculation shows that the function $\psi(k) := (A - B)|\beta|k/(H + |I| + J)$ is an increasing function of k . Since $k \geq 1$, we have $\psi(k) \geq \psi(1)$ and therefore

$$\left| \frac{P(z_0) - 1}{D - EP(z_0)} \right| \geq \frac{(A - B)|\beta|}{(D - E)(1 + B^2) + |2B(D - E) - E\beta(A - B)|}, \quad (2.12)$$

which by (2.1) is greater than or equal to 1. This contradicts $P(z) \prec (1 + Dz)/(1 + Ez)$ and completes the proof. $\square$

Remark 2.2. When $\beta = 1$, $E = 0 = B$, and $D = 1 = A$, Lemma 2.1 reduces to [13, Lemma 1, page 1035]. Further if $p(z) = z^2 f'(z)/f(z)^2$, Lemma 2.1 reduces to [13, Theorem 1, page 1036].

By taking $p(z) = z f'(z)/f(z)$ in Lemma 2.1, we have the following result.

THEOREM 2.3. *Let the conditions of Lemma 2.1 hold. If $f \in \mathcal{A}$ satisfies*

$$1 + \beta \frac{z f'(z)}{f(z)} \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) \prec \frac{1 + Dz}{1 + Ez}, \quad (2.13)$$

then $f \in \mathcal{S}^[A, B]$.*

By taking $\beta = 1$, $A = \alpha = -B$, and $D = -E = \delta$ ($0 < \alpha, \delta \leq 1$) in Theorem 2.3, we have the following result.

COROLLARY 2.4. *Let $0 < \alpha \leq 1$ and $\delta = \alpha/(1 + \alpha)^2$. If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{z f'(z)}{f(z)} \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) \right| < \delta \left| 2 + \frac{z f'(z)}{f(z)} \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) \right|, \quad (2.14)$$

then $f(z) \in \mathcal{S}^[\alpha]$.*

By taking $\beta = 1$, $A = 1 - 2\alpha$, $B = -1$, $D = (1 - \alpha)/2$, and $E = 0$ ($0 \leq \alpha < 1$) in Theorem 2.3, we have the following result.

COROLLARY 2.5. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{z f'(z)}{f(z)} \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) \right| < \frac{1 - \alpha}{3} \quad (0 \leq \alpha < 1), \quad (2.15)$$

then $f(z) \in \mathcal{S}_\alpha^$.*

By replacing $p(z)$ by $1/p(z)$, $\beta = -1$, A by $-B$, and B by $-A$ in Lemma 2.1, we have the following result.

LEMMA 2.6. Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$. Assume that

$$(A - B) \geq (D - E)(1 + A^2) + |E(A - B) - 2A(D - E)|. \quad (2.16)$$

If $p(z)$ is analytic in U with $p(0) = 1$ and

$$1 + \frac{zp'(z)}{p^2(z)} < \frac{1 + Dz}{1 + Ez}, \quad (2.17)$$

then

$$p(z) < \frac{1 + Az}{1 + Bz}. \quad (2.18)$$

When $p(z) = zf'(z)/f(z)$, in Lemma 2.6, we have the following theorem.

THEOREM 2.7. Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$. Assume that (2.16) holds. If $f \in \mathcal{A}$ satisfies

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < \frac{1 + Dz}{1 + Ez}, \quad (2.19)$$

then $f \in S^*[A, B]$.

Example 2.8. If $f \in G_{1-\alpha/(2-\alpha)^2}$ ($0 \leq \alpha < 1$), then $f \in S^*(\alpha)$. If $f \in \mathcal{A}$ satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \beta \left| 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \right| \quad \left(\beta = \frac{\alpha}{1 + 3\alpha + \alpha^2}, 0 < \alpha \leq 1 \right), \quad (2.20)$$

then $f \in S^*[\alpha]$. Similarly if (2.20) holds with $\beta = (1 - \alpha)/[1 + (1 - 2\alpha)^2 + |5\alpha - 3|]$ ($0 \leq \alpha < 1$), then $f \in S_\alpha^*$.

Remark 2.9. When $E = 0$ and $D = b$ ($0 < b \leq 1$), Corollary 2.5 reduces to [10, Corollary 2.6, page 203]. When $A = 0 = E$ and $D = -B = b$ ($0 < b \leq 1$), Corollary 2.5 reduces to [4, Theorem 1, page 61]. When $A = 0 = E$ and $D = -B = 1$, Corollary 2.5 reduces to [3, Corollary 1, page 76].

LEMMA 2.10. Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$, $AB \geq 0$, and $\beta \neq 0$. Assume that

$$|\beta|(A - B) \geq (D - E)(1 + AB) + |(D - E)(A + B) - E\beta(A - B)|. \quad (2.21)$$

Let $p(z)$ be analytic in U with $p(0) = 1$ and

$$1 + \beta \frac{zp'(z)}{p(z)} < \frac{1 + Dz}{1 + Ez}, \quad (2.22)$$

then

$$p(z) < \frac{1 + Az}{1 + Bz}. \quad (2.23)$$

Proof. The proof is similar to the proof of Lemma 2.1. □

Remark 2.11. When $E\beta \leq 0$, $AB \leq 0$, Lemma 2.10 is valid provided the following conditions hold:

$$(1 - A\beta)^2 \{2E\beta(A + B)(D - E) - (A - B)[(D - E)^2 + (E\beta)^2]\} \geq 4\beta^2(A - B)AB \quad (2.24)$$

instead of (2.21).

Remark 2.12. When $\beta = -1$, $A = \lambda = E$, and $D = B = 0$ ($|\lambda| \leq 1$), Lemma 2.10 reduces to [12, Theorem 1(iii), page 195].

Example 2.13. By taking $\beta = 1$, $B = 0$, $D = A/(1 + A)$, and $E = 0$ in Lemma 2.10, we have the following result. Let $0 < A \leq 1$. Let $p(z)$ be analytic in U with $p(0) = 1$. If $|zp'(z)/p(z)| < A/(1 + A)$, then $p(z) \prec 1 + Az$. When $p(z) = zf'(z)/f(z)$, $A = 1 - \alpha$, we have the following result.

If $f(z) \in \mathcal{A}$ satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \frac{1 - \alpha}{2 - \alpha} \quad (0 \leq \alpha < 1), \quad (2.25)$$

then $f(z) \in \mathcal{G}^*(\alpha)$.

By taking $p(z) = z^2 f'(z)/f^2(z)$ in Lemma 2.10, we have the following result.

THEOREM 2.14. *Let the conditions of Lemma 2.10 hold. If $f \in \mathcal{A}$ satisfies*

$$1 + \beta \left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) \prec \frac{1 + Dz}{1 + Ez}, \quad (2.26)$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec \frac{1 + Az}{1 + Bz}. \quad (2.27)$$

Remark 2.15. When $\beta = 1$, $A = \alpha$, $B = 0$, $E = 0$, and $D = (1 - \alpha)/(2 - \alpha)$ ($0 \leq \alpha < 1$), Theorem 2.14 reduces to [11, Theorem 2.4, page 307].

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References

- [1] W. Janowski, "Some extremal problems for certain families of analytic functions. I," *Annales Polonici Mathematici*, vol. 28, pp. 297–326, 1973.
- [2] Y. Polatoğlu and M. Bolcal, "The radius of convexity for the class of Janowski convex functions of complex order," *Matematichki Vesnik*, vol. 54, no. 1-2, pp. 9–12, 2002.
- [3] H. Silverman, "Convex and starlike criteria," *International Journal of Mathematics and Mathematical Sciences*, vol. 22, no. 1, pp. 75–79, 1999.

- [4] M. Obradowi and N. Tuneski, “On the starlike criteria defined by Silverman,” *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka*, no. 24, pp. 59–64, 2000.
- [5] M. Nunokawa, S. Owa, H. Saitoh, A. Ikeda, and N. Koike, “Some results for strongly starlike functions,” *Journal of Mathematical Analysis and Applications*, vol. 212, no. 1, pp. 98–106, 1997.
- [6] M. Nunokawa, S. Owa, H. Saitoh, and N. Takahashi, “On a strongly starlikeness criteria,” *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 31, no. 3, pp. 195–199, 2003.
- [7] M. Obradowi and S. Owa, “On some criterions for starlikeness of order α ,” *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, vol. 8, no. 2, pp. 283–289, 1988.
- [8] V. Ravichandran and M. Darus, “On a criteria for starlikeness,” *International Mathematical Journal*, vol. 4, no. 2, pp. 119–125, 2003.
- [9] V. Ravichandran, M. Darus, and N. Seenivasagan, “On a criteria for strong starlikeness,” *The Australian Journal of Mathematical Analysis and Applications*, vol. 2, no. 1, pp. 1–12, 2005, article 6.
- [10] N. Tuneski, “On the quotient of the representations of convexity and starlikeness,” *Mathematische Nachrichten*, vol. 248–249, no. 1, pp. 200–203, 2003.
- [11] B. A. Frasin and M. Darus, “On certain analytic univalent functions,” *International Journal of Mathematics and Mathematical Sciences*, vol. 25, no. 5, pp. 305–310, 2001.
- [12] S. Ponnusamy and S. Rajasekaran, “New sufficient conditions for starlike and univalent functions,” *Soochow Journal of Mathematics*, vol. 21, no. 2, pp. 193–201, 1995.
- [13] M. Nunokawa, M. Obradowi, and S. Owa, “One criterion for univalence,” *Proceedings of the American Mathematical Society*, vol. 106, no. 4, pp. 1035–1037, 1989.
- [14] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, vol. 83 of *Seminar on Higher Mathematics*, Presses de l’Universit de Montral, Montreal, Quebec, Canada, 1982.

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