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Research Article

On Semiabelian π-Regular Rings

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A ring R is defined to be *semiabelian* if every idempotent of R is either right semicentral or left semicentral. It is proved that the set $N(R)$ of nilpotent elements in a π-regular ring R is an ideal of R if and only if $R/J(R)$ is abelian, where $J(R)$ is the Jacobson radical of R. It follows that a semiabelian ring R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular, which extends the fundamental result of Badawi (1997). Moreover, several related results and examples are given.

1. Introduction

Throughout this paper, rings are associative with unity and modules are unitary. Given a ring R, we use the symbol $\mathrm{Id}(R)$ to denote the set of idempotents in R, $U(R)$ its unit group. The Jacobson radical, the prime radical, and the set of nilpotent elements of a ring R are denoted by $J(R)$, $P(R)$, and $N(R)$, respectively. The symbol $\mathrm{Max}(R)$ (resp., $\mathrm{Max}_r(R)$) stands for the set of maximal (resp., maximal right) ideals of a ring R. As usual, the symbol $M_n(R)$ denotes the ring of $n\times n$ matrices over a ring R, $\mathrm{UTM}_n(R)$ denotes the ring of $n\times n$ upper triangular matrices over R, and E_{ij} $(1\le i,\ j\le n)$ denotes the $n\times n$ matrix units over R. Let M be an R-R bimodule and $A=(a_{ij})_{n\times n}\in M_n(R)$, we write $MA=\{(ma_{ij})_{n\times n}\mid m\in M\}$, and write $V=\sum_{i=1}^{n-1}E_{i,i+1}$ for $n\ge 2$. And we use the symbol $T_n(R,M)$ to denote the ring of $n\times n$ upper triangular matrices whose principal diagonal elements are identical and belong to R and the other elements belong to M, and write $V_n(R,M)=RI_n+MV+\cdots+MV^{n-1}$ for $n\ge 2$ where I_n is the $n\times n$ identity matrix over R. Moreover, we use the symbol $\mathbb{Z}_p$ to denote the ring of integers modulo a prime p.

Following [1], an idempotent e in a ring R is called right (resp., left) semicentral if for every $x\in R$, $ex=exe$ (resp., $xe=exe$). And the set of right (resp., left) semicentral

idempotents of R is denoted by $S_r(R)$ (resp., $S_l(R)$). We define a ring R to be *semiabelian* if $\mathrm{Id}(R) = S_r(R) \cup S_l(R)$, this notion is a proper generalization of that of an abelian ring.

Recall that a ring R is called π-regular if for every $x \in R$, there exist an element $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. In the case of $n = 1$ for all $x \in R$, then R is regular. An element a in a ring R is strongly π-regular if there exist $b \in R$ and a positive integer n such that $a^n = a^{n+1}b$ with $ab = ba$. And a ring R is strongly π-regular if every element of R is strongly π-regular. Clearly, a strongly π-regular ring is a π-regular ring. A ring is called right (resp., left) quasiduo if every maximal right (resp., left) ideal is an ideal. And a ring is quasiduo if it is right and left quasiduo. A ring R is called an exchange ring if for every $a \in R$, there exists $e \in \mathrm{Id}(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. It is known that a π-regular ring is an exchange ring (see [2, Example 2.3]). A ring is reduced if it has no nonzero nilpotent elements. And a ring is abelian if every idempotent is central. It is well known that a reduced ring is an abelian ring. For the above notions we refer the reader to [3, 4].

In [5], Badawi studied abelian π-regular rings and obtained some interesting results. The fundamental result is that an abelian ring R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. In this paper, we study semiabelian π-regular rings, extending some of the main results of [5]. It is proved that for every such ring R, $N(R)$ is an ideal of R if and only if $R/J(R)$ is abelian. It follows that if R is a semiabelian ring, then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. Moreover, several related results and examples are given.

2. Extensions of semiabelian rings

We start this section with the following definition.

Definition 2.1. A ring R is called *semiabelian* if $\mathrm{Id}(R) = S_r(R) \cup S_l(R)$.

Clearly, an abelian ring is semiabelian. But the converse is not true in general as the following example shows.

Example 2.2. Let R be any ring for which $\mathrm{Id}(R) = \{0, 1\}$ (e.g., a local ring). Then $\mathrm{UTM}_2(R)$ is a semiabelian ring which is not abelian.

Proof. Clearly, $\mathrm{UTM}_2(R)$ is not abelian. And it is quite easy to check that

$$\mathrm{Id}(\mathrm{UTM}_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \mid a, b \in R \right\}, \tag{2.1}$$

and that $\left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$ is a left semicentral idempotent and $\left(\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right)$ is a right semicentral idempotent for any $a, b \in R$. Hence, $\mathrm{UTM}_2(R)$ is semiabelian.

One may expect that the conclusion of Example 2.2 is true for $n \geq 3$, but this is not the case. In fact, for any ring R, idempotent $E_{11} + E_{33}$ is neither right nor left semicentral in $\mathrm{UTM}_3(R)$. This implies that for any $n \geq 3$, $\mathrm{UTM}_n(R)$ is not semiabelian. Also the direct sum of two nonabelian semiabelian rings is not semiabelian. Now let R_1 and R_2 be semiabelian rings which are not abelian. Take $e_1 \in R_1$ to be a right semicentral idempotent which is not central and $e_2 \in R_2$ to be a left semicentral idempotent which is not central, then the idempotent (e_1, e_2) is neither right nor left semicentral in $R_1 \bigoplus R_2$.

In view of this situation, it is necessary for us to study how to obtain more examples of nonabelian semiabelian rings from a given nonabelian semiabelian ring. Clearly, the direct sum of an abelian ring and a nonabelian semiabelian ring is a semiabelian ring which is not abelian. Next we consider several extensions of semiabelian rings. □

Theorem 2.3. *A ring R is semiabelian if and only if the ring $R[[x]]$ of formal power series over R is semiabelian.*

Proof. Assume that R is semiabelian and $f(x) \in \mathrm{Id}(R[[x]])$. Then $f(x) = e + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ with $e \in \mathrm{Id}(R)$. Now we prove that if $e \in S_r(R)$, then $f(x) \in S_r(R[[x]])$. Since $f(x)^2 = f(x)$, we have $ea_1 + a_1e = a_1$ by comparing the coefficients of x in the equation $f(x)^2 = f(x)$. Multiplying two sides of the equation $ea_1 + a_1e = a_1$ by e, we have $ea_1e = 0$, which gives $ea_1e = ea_1 = 0$ and so $a_1 = a_1e$. Assume that $ea_i = 0$ and $a_i = a_ie$ hold for all $1 \le i \le n-1$. We claim that $ea_n = 0$ and $a_ne = a_n$. In fact, comparing the coefficients of x^n in the equation $f(x)^2 = f(x)$, we have $ea_n + a_1a_{n-1} + \cdots + a_ia_{n-i} + \cdots + a_{n-1}a_1 + a_ne = a_n$. Since $a_i = a_ie$ for $1 \le i \le n-1$ in the above expression, we have $a_ia_{n-i} = a_iea_{n-i}e = 0$. It follows that $ea_n + a_ne = a_n$. Multiplying both sides of this equation by e, then $ea_ne = ea_n = 0$, which gives $a_n = a_ne$. By induction, we have $f(x) = f(x)e$ and $ef(x) = e$. Now for any $g(x) \in R[[x]]$, then $eg(x)e = eg(x)$ since $e \in S_r(R)$, and hence $f(x)g(x)f(x) = f(x)eg(x)f(x) = f(x)eg(x)ef(x) = f(x)eg(x)e = f(x)eg(x) = f(x)g(x)$. Similarly, if $e \in S_l(R)$, then $f(x) \in S_l(R[[x]])$ holds. Hence $R[[x]]$ is semiabelian. The only if part of the proof is trivial since the subring of a semiabelian ring is semiabelian. And the proof is complete. □

Corollary 2.4. *A ring R is semiabelian if and only if the ring $R[x]$ of polynomials over R is semiabelian.*

It is known by [6, Propositions 2.4 and 2.5] that if $f(x) = e + \sum_{i=1}^{\infty} a_ix^i \in S_l(R[[x]])$, then $e \in S_l(R)$, $ef(x) = f(x)$, and $f(x)e = e$. This is true, in particular, for a polynomial $f(x) = e + \sum_{i=1}^{n} a_ix^i \in S_l(R[[x]])$. Similarly, if $f(x) = e + \sum_{i=1}^{\infty} a_ix^i \in S_r(R[[x]])$, then $e \in S_r(R)$, $f(x)e = f(x)$, and $ef(x) = e$. And this is true especially when $f(x) \in S_r(R[x])$.

From the proof of Theorem 2.3 and the above argument, we obtain a characterization of left (resp., right) semicentral idempotents in $R[[x]]$ and $R[x]$.

Proposition 2.5. *Let $f(x)$ be in $R[[x]]$ (resp., $R[x]$) with the constant term e. Then one has the following conclusions:*

(1) *$f(x) \in S_l(R[[x]])$ (resp., $S_l(R[x])$) if and only if $e \in S_l(R)$, $ef(x) = f(x)$ and $f(x)e = e$;*
(2) *$f(x) \in S_r(R[[x]])$ (resp., $S_r(R[x])$) if and only if $e \in S_r(R)$, $f(x)e = f(x)$ and $ef(x) = e$.*

Similar to the proof of Theorem 2.3, it is easy to prove the next theorem.

Theorem 2.6. *A ring R is semiabelian if and only if the group ring RC_∞ is semiabelian, where C_∞ is the infinite cyclic group.*

Theorem 2.7. *A ring R is semiabelian if and only if $T_n(R,M)$ is semiabelian, where M is an R-R bimodule.*

Proof. Assume that R is a semiabelian ring and $E_n \in \mathrm{Id}(T_n(R,M))$. Then

$$E_n = \begin{pmatrix} e & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & e & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & e & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & e \end{pmatrix}, \tag{2.2}$$

where $e \in \mathrm{Id}(R)$ and $a_{ij} \in M$. We claim that if $e \in S_r(R)$, then $E_n \in S_r(T_n(R,M))$. First we prove that $E_n = E_n e$ is true by induction on n. This is trivial in the case of $n = 1$. Assume that $E_{n-1} = E_{n-1}e$ holds for any $n \geq 2$. In the case of n, then $E_n = \left(\begin{smallmatrix} E_{n-1} & \alpha \\ 0 & e \end{smallmatrix}\right)$ where $\alpha = (a_{1n}, a_{2n}, \ldots, a_{n-1,n})^T$. Since E_n is an idempotent, we have $E_{n-1}\alpha + \alpha e = \alpha$, which gives $E_{n-1}\alpha e = 0$. On the other hand, since $E_{n-1}\alpha e = E_{n-1}e\alpha e = E_{n-1}e\alpha = E_{n-1}\alpha$, then $E_{n-1}\alpha = 0$. Therefore $\alpha = \alpha e$; this implies $E_n = E_n e$. Using this fact, we prove that $E_n B_n E_n = E_n B_n$ is true for any $B_n \in T_n(R,M)$. This is trivial in the case of $n = 1$. Assume that $E_{n-1}B_{n-1}E_{n-1} = E_{n-1}B_{n-1}$ holds for any $n \geq 2$ and $B_{n-1} \in T_{n-1}(R,M)$. In the case of n, we write $B_n = \left(\begin{smallmatrix} B_{n-1} & \beta \\ 0 & b_{nn} \end{smallmatrix}\right)$ where $B_{n-1} \in T_{n-1}(R,M)$ and $b_{nn} \in R$. Hence we have the following equations:

$$\begin{gathered} E_n B_n = \begin{pmatrix} E_{n-1}B_{n-1} & E_{n-1}\beta + \alpha b_{nn} \\ 0 & e b_{nn} \end{pmatrix}, \\ E_n B_n E_n = \begin{pmatrix} E_{n-1}B_{n-1}E_{n-1} & E_{n-1}B_{n-1}\alpha + E_{n-1}\beta e + \alpha b_{nn} e \\ 0 & e b_{nn} e \end{pmatrix}. \end{gathered} \tag{2.3}$$

By the assumption, $E_{n-1}B_{n-1}E_{n-1} = E_{n-1}B_{n-1}$ and $eb_{nn}e = eb_{nn}$ hold. Also, $E_n = E_n e$ implies $\alpha = \alpha e$. It follows that $\alpha b_{nn} e = \alpha e b_{nn} e = \alpha e b_{nn} = \alpha b_{nn}$ and $E_{n-1}\beta e = E_{n-1}e\beta e = E_{n-1}e\beta = E_{n-1}\beta$. Moreover, we have $E_{n-1}B_{n-1}\alpha = E_{n-1}B_{n-1}E_{n-1}\alpha = 0$ since $E_{n-1}\alpha = 0$. Hence, $E_n B_n E_n = E_n B_n$ and so $E_n \in S_r(T_n(R,M))$. Similarly, it can be proved that if $e \in S_l(R)$, then $E_n \in S_l(T_n(R,M))$. Therefore $T_n(R,M)$ is semiabelian. The only if part of the proof is trivial. □

Corollary 2.8. *A ring R is semiabelian if and only if the trivial extension $T_2(R,M)$ is semiabelian, where M is an R-R bimodule.*

Corollary 2.9. *A ring R is semiabelian if and only if $R[x]/(x^n)$ is semiabelian, where (x^n) is an ideal generated by x^n in $R[x]$.*

Proof. It is trivial in the case of $n = 1$. If $n \geq 2$, then there exists a ring isomorphism θ: $V_n(R,R) = RI_n + RV + \cdots + RV^{n-1} \to R[x]/(x^n)$ defined by $\theta(r_0 I_n + r_1 V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} + (x^n)$. By Theorem 2.7, $V_n(R,R)$ is semiabelian, so $R[x]/(x^n)$ is semiabelian. □

3. Semiabelian π-regular rings

For convenience of the reader, we list some known facts which are necessary for the study of π-regularity of rings.

Lemma 3.1 (see [7, Lemma 2.1]). *For an idempotent e in a ring R, the following conditions are equivalent:*

(1) $e \in S_r(R)$;
(2) $1 - e \in S_l(R)$;
(3) $(1 - e)Re = 0$.

Lemma 3.2 (see [4, Theorem 23.2]). *The following conditions are equivalent for a ring R:*

(1) *R is strongly π-regular;*
(2) *every prime factor ring of R is strongly π-regular;*
(3) *$R/P(R)$ is strongly π-regular;*

The next theorem extends [8, Theorem 1].

Theorem 3.3. *Let R be a semiabelian exchange ring. Then R/P is a local ring for every prime ideal of R.*

Proof. According to [9, Theorem 1], an exchange ring with only two idempotents is a local ring. And by [10, Lemma 4.2], a prime ring is semicentral reduced, that is, it has only 0 and 1 for its semicentral idempotents. Because R/P is a prime semiabelian exchange ring, R/P is a local ring. □

Corollary 3.4 (see [8, Theorem 1]). *Let R be an abelian exchange ring. Then R/P is a local ring for every prime ideal of R.*

Corollary 3.5. *Let R be a semiabelian exchange ring. Then R/P is a division ring for every right (resp., left) primitive ideal of R.*

Proof. Since R/P is a local ring and $J(R/P) = 0$, R/P is a division ring. □

Stock [2, Lemma 4.10] proved that if R is an exchange ring with $J(R) = 0$, then R is abelian if and only if it is reduced. Since a semiprime semiabelian ring is abelian (see [1, page 569]), we get the following lemma immediately.

Lemma 3.6. *Let R be an exchange ring with $J(R) = 0$. Then the following conditions are equivalent:*

(1) *R is reduced;*
(2) *R is abelian;*
(3) *R is semiabelian.*

Lemma 3.7. *Let R be a semiabelian exchange ring. Then so is every homomorphic image of R, and $R/J(R)$ is an abelian exchange ring.*

Proof. The first assertion is easy to prove since any homomorphic image of an exchange ring is also an exchange ring and idempotents can be lifted modulo every right ideal of R (cf. [11]). Now since $R/J(R)$ is a semiabelian exchange ring with the Jacobson radical zero, it is an abelian exchange ring by Lemma 3.6. □

Theorem 3.8. *Let R be an exchange ring with $J(R)$ nil. Then $N(R)$ is an ideal of R if and only if $R/J(R)$ is an abelian ring.*

Proof. ($\Rightarrow$) Since $N(R)$ is a nil ideal of R, $N(R) \subseteq J(R)$ holds. On the other hand, we have $J(R) \subseteq N(R)$ by the assumption. It follows that $J(R) = N(R)$ and so $R/J(R)$ is reduced and hence it is abelian.

($\Leftarrow$) Because R is an exchange ring and $R/J(R)$ is abelian, $R/J(R)$ is an abelian exchange ring and so it is reduced by Lemma 3.6. Hence $N(R) \subseteq J(R)$. On the other hand, $J(R) \subseteq N(R)$ by the assumption. So $N(R) = J(R)$ is an ideal of R.

It is known and easy to prove that the Jacobson radical of a π-regular ring is nil. Hence Theorem 3.8 implies that for a π-regular ring R, $N(R)$ is an ideal of R if and only if $R/J(R)$ is abelian. And [2, Example 4.16] shows that the class of exchange rings with $J(R)$ nil properly contains the class of π-regular rings.

Badawi [5, Theorem 2] proved that if R is an abelian π-regular ring, then $N(R)$ is an ideal of R. In fact, the similar result is true for a right (resp., left) quasiduo π-regular ring. □

Corollary 3.9. *If R is a right (resp., left) quasiduo π-regular ring, then $N(R)$ is an ideal of R.*

Proof. Since R is a right (resp., left) quasiduo ring, $R/J(R)$ is reduced by [12, Corollary 2] and hence it is abelian. And since R is π-regular, it is an exchange ring with $J(R)$ nil. Hence, $N(R)$ is an ideal of R by Theorem 3.8. □

Corollary 3.10. *If R is a semiabelian π-regular ring, then $N(R)$ is an ideal of R.*

Proof. Clearly, R is a semiabelian exchange ring with $J(R)$ nil, and $R/J(R)$ is abelian by Lemma 3.7. By Theorem 3.8, $N(R)$ is an ideal of R. □

Theorem 3.11. *Let R be a semiabelian ring. Then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular.*

Proof. ($\Rightarrow$) Suppose that R is π-regular. By Corollary 3.10, $N(R)$ is an ideal of R and so $R/N(R)$ is reduced and π-regular. Let $\bar{x} \in R/N(R)$. Then there exist $\bar{y} \in R/N(R)$ and a positive integer n such that $\bar{x}^n = \bar{x}^n \bar{y} \bar{x}^n$. Write $\bar{e} = \bar{x}^n \bar{y}$. Then $\bar{e} \in \mathrm{Id}(R/N(R))$ and $[(\bar{1} - \bar{e})\bar{x}]^n = \bar{0} = (\bar{1} - \bar{e})\bar{x}$ since $R/N(R)$ is reduced. Hence $R/N(R)$ is regular.

($\Leftarrow$) Assume that $N(R)$ is an ideal of R and $\widetilde{R} = R/N(R)$ is regular. Then $\widetilde{R}$ is abelian regular (and hence unit regular) since it is reduced. To prove R is π-regular, it is sufficient to prove that R/P is strongly π-regular for every prime ideal P of R by Lemma 3.2. For any $x \in R$, then $\widetilde{x} = x + N(R) \in \widetilde{R}$ is unit regular. So we have $\widetilde{x} = \widetilde{e}\widetilde{u} = \widetilde{u}\widetilde{e}$ with $e \in \mathrm{Id}(R)$ and $u \in U(R)$ since idempotents and units of $\widetilde{R}$ can be lifted modulo $N(R)$. Hence, $x = eu + w_1 = ue + w_2$ where $w_1, w_2 \in N(R)$, which implies $ex = e(u + w_1)$ and $xe = (u + w_2)e$, and $(1 - e)x = x - ex = (1 - e)w_1 \in N(R)$, $x(1 - e) = x - xe = w_2(1 - e) \in N(R)$. So there exists a positive integer n such that $[(1 - e)x]^n = [x(1 - e)]^n = 0$. Now if $e \in S_l(R)$, then $1 - e \in S_r(R)$ by Lemma 3.1. Equation $[(1 - e)x]^n = 0$ implies $(1 - e)x^n = 0$, and hence $x^n = ex^n$. If $e \in P$, then $x^n \in P$ and $\bar{x} = x + P \in N(R/P)$, so $\bar{x}$ is strongly π-regular in R/P. If $e \notin P$, then $0 = eR(1 - e) \subseteq P$ by Lemma 3.1, which gives $1 - e \in P$ and so $\bar{e} = \bar{1}$ in R/P. This implies $\bar{x} = \bar{e}\bar{x} = \overline{e(u + w_1)} = \overline{u + w_1}$ in R/P. Hence, $\bar{x}$ is a unit and so it is a strongly π-regular element in R/P. If $e \in Sr(R)$, then $1 - e \in S_l(R)$. Equation $[x(1 - e)]^n = 0$ implies $x^n(1 - e) = 0$, and hence $x^n = x^n e$. Note that $xe = (u + w_2)e$. Similar to the above

proof, it can be shown that $\bar{x}$ is a nilpotent element or a unit in R/P. And the proof is completed. □

It is known by Example 2.2 that $\mathrm{UTM}_2(R)$ is a nonabelian semiabelian π-regular ring for any π-regular local ring (e.g., an artinian local ring by Lemma 3.2) R. From this we can construct more nonabelian semiabelian π-regular rings by using Theorems 2.7 and 3.11.

Corollary 3.12 (see [5, Theorem 3]). *Let R be an abelian ring. Then R is π-regular if and only if $N(R)$ is an ideal of R and $R/(R)$ is regular.*

The following corollary is an immediate result of Theorem 3.11.

Corollary 3.13. *Let R be a semiabelian π-regular ring. Then for any prime ideal P of R, every element in R/P is either a nilpotent element or a unit, and hence R is strongly π-regular with $J(R) = N(R)$.*

In light of Theorem 3.11, we naturally ask the following question.

Question 3.14. Let R be any ring. If $N(R)$ is an ideal of R and $R/N(R)$ is regular, then is R π-regular?

There are many partially positive solutions to this question (see [13–15] for the details). For a ring R with bounded index (i.e., there exists a positive integer n such that $a^n = 0$ for all $a \in N(R)$), the answer is also positive.

Proposition 3.15. *Let R be a ring with bounded index. If $N(R)$ is an ideal of R and $R/N(R)$ is regular, then R is strongly π-regular.*

Proof. It is proved in [16, Lemma 11] that if I is a right ideal of a ring R and n is a positive integer such that $a^n = 0$ for all $a \in I$, then $a^{n-1}Ra^{n-1} = 0$. Now since $R/N(R)$ is reduced and regular, it is strongly π-regular. By Lemma 3.2, it is sufficient to prove that $N(R) = P(R)$. Let m be the bounded index (the least positive integer m such that $a^m = 0$ for all $a \in N(R)$) of R. If $m = 1$, then $P(R) = N(R) = 0$. If $m \geq 2$, then $N(R) \neq 0$. We claim that $P(R) = N(R)$ is also true. If not, then $N(R)/P(R)$ is a nonzero nil ideal of $\overline{R} = R/P(R)$ with the bounded index $n \geq 2$. Thus there exists a nonzero element $\bar{a} \in N(R)/P(R)$ such that $\bar{a}^n = \bar{0}$ and $\bar{a}^{n-1} \neq 0$, so $\overline{R}\bar{a}^{n-1}\overline{R} \neq \bar{0}$. By [16, Lemma 11], $\bar{a}^{n-1}\overline{R}\bar{a}^{n-1} = \bar{0}$ and so $(\overline{R}\bar{a}^{n-1}\overline{R})^2 = \bar{0}$, which is impossible since $R/P(R)$ is a semiprime ring. So $P(R) = N(R)$, and the proof is completed. □

Theorem 3.16. *Let R be a semiabelian ring. Then R is π-regular if and only if there exists a nil ideal I of R such that R/I is π-regular.*

Proof. ($\Rightarrow$) If R is π-regular, then $I = N(R)$ is an ideal of R and R/I is regular by Theorem 3.11, and so we are done.

($\Leftarrow$) If R/I is π-regular for some nil ideal I of R, then R/I is semiabelian π-regular by Lemma 3.7. According to Theorem 3.11, $N(R/I) = N(R)/I$ is an ideal of R/I. So $N(R)$ is a nil ideal of R. Since R/I is π-regular, $R/N(R)$ is π-regular. And since $R/N(R)$ is reduced and π-regular, $R/N(R)$ is regular by [4, Proposition 23.5]. Therefore R is π-regular by Theorem 3.11. □

A consequence of the above theorem is the following corollary.

Corollary 3.17. *Let R be a semiabelian ring. Then R is π-regular if and only if $R/P(R)$ is π-regular.*

Recall [17] that a ring R is said to have stable range one if whenever $aR+bR=R$ for $a,b\in R$, there exists $y\in R$ such that $a+by\in U(R)$. In [18], a ring R is said to satisfy the unit 1-stable condition if for any $a,b,c\in R$ with $ab+c=1$, there exists $u\in U(R)$ such that $au+c\in U(R)$. Combining [18, Corollary 4.2] with Corollary 3.5, we have the following proposition which extends [5, Theorem 6].

Proposition 3.18. *For a semiabelian exchange ring (in particular, a semiabelian π-regular ring) R, the following statements are equivalent:*

(1) *every element of R is a sum of two units;*
(2) *R satisfies the unit 1-stable range condition;*
(3) *for any factor ring R_1 of R, every element of R_1 is a sum of two units;*
(4) *$\mathbb{Z}_2$ is not a homomorphic image of R.*

4. Some remarks

In the final section, we give some remarks upon the previous results.

Remark 4.1. Every semiabelian exchange ring (in particular, a semiabelian π-regular ring) R is a quasiduo ring.

Proof. According to Theorem 3.3, a semiabelian exchange ring R is a right pm-ring in the sense that every prime ideal of R is contained in a unique maximal right ideal, equivalently, every prime ideal is contained in a unique maximal ideal. By [19], if R is a right pm-ring, then $\mathrm{Max}(R)=\mathrm{Max}_r(R)$ and hence R is a right quasiduo ring. And by [3, Theorem 4.6], an exchange ring R is right quasiduo if and only if it is left quasiduo. So every semiabelian exchange ring R is a quasiduo ring. But the converse is not true in general. □

Remark 4.2. There exists a quasiduo π-regular ring R which is not a semiabelian π-regular ring.

Proof. Let $R_1=R_2=\mathrm{UTM}_2(\mathbb{Z}_2)$ and $R=R_1\bigoplus R_2$. Then R is clearly π-regular. Since $R/J(R)\cong R_1/J(R_1)\bigoplus R_2/J(R_2)=\mathbb{Z}_2\bigoplus\mathbb{Z}_2\bigoplus\mathbb{Z}_2\bigoplus\mathbb{Z}_2$ is a commutative ring, it is a quasiduo ring and so is R. But R is not semiabelian. In fact, (E_{11},E_{22}) is neither right nor left semicentral idempotent in R where E_{11} and E_{22} are the 2×2 matrix units over $\mathrm{UTM}_2(\mathbb{Z}_2)$.

In [20], a ring R is called unit π-regular if for every $a\in R$, there exist $u\in U(R)$ and a positive integer n such that $a^n=a^nua^n$. By [21, page 3584], a strongly π-regular ring is unit π-regular, but the converse is not true in general. □

Remark 4.3. There exists a unit regular ring R which is not a strongly π-regular ring.

Proof. Let F be a field and $R=\prod_{n=1}^{\infty}M_n(F)$. Then R is unit regular since every $M_n(F)$ is unit regular. We prove that R is not strongly π-regular. Assume to the contrary, then $a=(a_1,a_2,\dots,a_n,\dots)$ is strongly π-regular, where for any positive integer n, $a_n=(a_{ij})_{n\times n}\in M_n(F)$ with $a_{ij}=0$ when $i\geq j$, and $a_{ij}=1$ when $i<j$. Hence there exist $b\in R$ and a

positive integer m such that $a^m = a^{2m}b$. It follows that $a^m_{m+1} \neq 0$ and $a^{2m}_{m+1} = 0$, which is impossible.

Ara proved in [17] that a strongly π-regular ring has stable range one. In light of Remark 4.3, we naturally ask the following question with which we conclude this paper. □

Question 4.4. Does a unit π-regular ring R have stable range one?

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