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Research Article

Approximation of a Common Random Fixed Point for a Finite Family of Random Operators

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We construct implicit random iteration process with errors for a common random fixed point of a finite family of asymptotically quasi-nonexpansive random operators in uniformly convex Banach spaces. The results presented in this paper extend and improve the corresponding results of Beg and Abbas in 2006 and many others.

1. Introduction

Random approximations and random fixed point theorems are stochastic generalizations of classical approximations and random fixed point theorems. The study of random fixed points forms a central topic in this area. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Špacček [1] and Hanš [2, 3]. Subsequently, Bharucha-Reid [4] had given sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for a random operator. Random fixed point theorems for multivalued random contraction mappings on separable complete metric spaces were first proved by Itoh [5]. Now, this theory has become a full-fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear system (cf. Itoh [5]). In an attempt to construct iterations for finding fixed points of random operators defined on linear spaces, random Ishikawa iteration was introduced in [6]. This iteration and some other random iterations based on the same idea have been applied for finding solutions of random operator equations and fixed points of random operators (see [6]).

Recently, Beg [7], Beg and Shahzad [8], Choudhury [9], Duan and Li [10], Li and Duan [11], Yuan et al. [12], and many others have studied fixed point of random operators. In 2005, Beg and Abbas [13] studied different random iterative algorithms for

weakly contractive and asymptotically nonexpansive random operators on arbitrary Banach spaces. They also established the convergence of an implicit random iterative process to a common random fixed point for a finite family of asymptotically quasi-nonexpansive operators.

In 2005, Fukhar-Ud-Din and Khan [14] proved weak and strong convergence of an implicit iterative process with errors, in the sense of Xu [15], for a finite family of asymptotically quasi-nonexpansive mappings on a closed convex unbounded set in a real uniformly convex Banach space.

It is our purpose in this paper to construct an implicit random iteration process with errors which converges strongly to a common random fixed point of a finite family of asymptotically quasi-nonexpansive random operators on an unbounded set in uniformly convex Banach spaces. Our results extend and improve the corresponding ones announced by Beg and Abbas [13], and many others.

2. Preliminaries

Let (Ω,Σ) be a measurable space with Σ being a sigma-algebra of subsets of Ω and let C be a nonempty subset of a Banach space X. A mapping $\xi:\Omega\to X$ is *measurable* if $\xi^{-1}(U)\in\Sigma$ for each open subset U of X. The mapping $T:\Omega\times C\to C$ is a *random map* if and only if for each fixed $x\in C$, the mapping $T(\cdot,x):\Omega\to C$ is measurable, and it is *continuous* if for each $\omega\in\Omega$, the mapping $T(\omega,\cdot):C\to X$ is continuous. A measurable mapping $\xi:\Omega\to X$ is a *random fixed point* of the random map $T:\Omega\times C\to X$ if and only if $T(\omega,\xi(\omega))=\xi(\omega)$ for each $\omega\in\Omega$. We denote by $\mathrm{RF}(T)$ the set of all random fixed points of a random map T and $T^n(\omega,x)$ the nth iteration $T(\omega,T(\omega,T(,\dots,T(\omega,x))))$ of T. The letter I denotes the random mapping $I:\Omega\times C\to C$ defined by $I(\omega,x)=x$ and $T^0=I$.

Definition 2.1. Let C be a nonempty subset of a separable Banach space X and let $T:\Omega\times C\to C$ be a random map. The map T is said to be

(a) a *nonexpansive random operator* if for arbitrary $x,y\in C$, one has

$$\|T(\omega,x)-T(\omega,y)\|\le\|x-y\| \tag{2.1}$$

for each $\omega\in\Omega$;

(b) an *asymptotically nonexpansive random operator* if there exists a sequence of measurable mappings $r_n:\Omega\to[0,\infty)$ with for each $\omega\in\Omega$, $\lim_{n\to\infty}r_n(\omega)=0$, such that for arbitrary $x,y\in C$, one has

$$\|T^n(\omega,x)-T^n(\omega,y)\|\le(1+r_n(\omega))\|x-y\|\quad\text{for each }\omega\in\Omega; \tag{2.2}$$

(c) an *asymptotically quasi-nonexpansive random operator* if there exists a sequence of measurable mappings $r_n:\Omega\to[0,\infty)$ with for each $\omega\in\Omega$, $\lim_{n\to\infty}r_n(\omega)=0$, such that

$$\|T^n(\omega,\eta(\omega))-\xi(\omega)\|\le(1+r_n(\omega))\|\eta(\omega)-\xi(\omega)\|\quad\text{for each }\omega\in\Omega, \tag{2.3}$$

where $\xi:\Omega\to C$ is a random fixed point of T and $\eta:\Omega\to C$ is any measurable map;

(d) a *completely continuous random operator* if the sequence $\{x_n\}$ in C converges weakly to x_0 implies that $\{T(\omega,x_n)\}$ converges strongly to $T(\omega,x_0)$ for each $\omega \in \Omega$;

(e) a *uniformly L-Lipschitzian random operator* if for arbitrary $x, y \in C$, one has

$$\|T^n(\omega,x) - T^n(\omega,y)\| \leq L\|x-y\|, \quad n = 1,2,\ldots, \tag{2.4}$$

where L is a positive constant;

(f) a *semicompact random operator* if for any sequence of measurable mappings $\{\xi_n\}$ from Ω to C, with $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega,\xi_n(\omega))\| = 0$, for every $\omega \in \Omega$, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ which converges pointwise to ξ, where $\xi : \Omega \to C$ is a measurable mapping.

Definition 2.2. A family $\{T_i : i \in I\}$ of N-mappings on C with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy *condition* (B) on C if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\max_{1\leqslant l\leqslant N} \{\|x - T_l x\|\} \geqslant f(d(x,F)) \quad \text{for all } x \in C. \tag{2.5}$$

Definition 2.3. Let $\{T_1, T_2, T_3, \ldots, T_N\}$ be a family of N-random operators from $\Omega \times C$ to C, where C is a nonempty closed convex subset of a separable Banach space X satisfying $C + C \subset C$ for each $\omega \in \Omega$ and let $\{f_n\}$ be a sequence of measurable mappings from Ω to C. Let $\xi_0 : \Omega \to C$ be a measurable mapping. Following Sun [16], define the random iteration process with errors $\{\xi_n\}$, in the sense of Liu [17], as follows:

$$\begin{aligned}
\xi_1(\omega) &= \alpha_1\xi_0(\omega) + (1-\alpha_1)T_1(\omega,\xi_1(\omega)) + f_1(\omega),\\
\xi_2(\omega) &= \alpha_2\xi_1(\omega) + (1-\alpha_2)T_2(\omega,\xi_2(\omega)) + f_2(\omega),\\
&\vdots\\
\xi_N(\omega) &= \alpha_N\xi_{N-1}(\omega) + (1-\alpha_N)T_N(\omega,\xi_N(\omega)) + f_N(\omega),\\
\xi_{N+1}(\omega) &= \alpha_{N+1}\xi_N(\omega) + (1-\alpha_{N+1})T_1^2(\omega,\xi_{N+1}(\omega)) + f_{N+1}(\omega),\\
&\vdots\\
\xi_{2N}(\omega) &= \alpha_{2N}\xi_{2N-1}(\omega) + (1-\alpha_{2N})T_N^2(\omega,\xi_{2N}(\omega)) + f_{2N}(\omega),\\
\xi_{2N+1}(\omega) &= \alpha_{2N+1}\xi_{2N}(\omega) + (1-\alpha_{2N+1})T_1^3(\omega,\xi_{2N+1}(\omega)) + f_{2N+1}(\omega),\\
&\vdots
\end{aligned} \tag{2.6}$$

where $\{\alpha_n\}$ is an appropriate real sequence in $[0,1]$. In the compact form, we have

$$\xi_n(\omega) = \alpha_n\xi_{n-1}(\omega) + (1-\alpha_n)T_{i(n)}^{k(n)}(\omega,\xi_n(\omega)) + f_n(\omega), \tag{2.7}$$

where $n = (k-1)N + i$, $k = k(n)$, $i = i(n)$, and each $\{f_n(\omega)\}$ is summable sequence in C, that is, $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$.

Remark 2.4. Let $\{T_1, T_2, T_3, \ldots, T_N\}$ be a family of N asymptotically quasi-nonexpansive continuous random operators with sequences of measurable mappings $\{r_n^i(\omega)\}$ for $i = 1, 2, \ldots, N$. If $n = (k-1)N + i$, $i \in \{1, 2, \ldots, N\} = J$, then there exists a measurable mapping $r_k(\omega) = \max\{r_k^1(\omega), r_k^2(\omega), \ldots, r_k^N(\omega)\}$ with for all $\omega \in \Omega$, $\lim_{k\to\infty} r_k(\omega) = 0$, such that

$$\left\| T_{i(n)}^{k(n)}(\omega, \eta(\omega)) - \xi(\omega) \right\| \le (1 + r_k(\omega)) \left\| \eta(\omega) - \xi(\omega) \right\| \quad \text{for each } \omega \in \Omega, \tag{2.8}$$

where $\xi : \Omega \to C$ is a random fixed point of T and $\eta : \Omega \to C$ is any measurable map.

In the sequel, we will need the following lemmas.

Lemma 2.5 (see [18]). *Let the nonnegative number sequences $\{a_n\}, \{b_n\}$, and $\{d_n\}$ satisfy the following:*

$$a_{n+1} \le (1 + b_n) a_n + d_n, \quad \forall n = 1, 2, \ldots, \quad \sum_{n=1}^{\infty} b_n < \infty, \quad \sum_{n=1}^{\infty} d_n < \infty. \tag{2.9}$$

Then

(1) $\lim_{n\to\infty} a_n$ *exists*

(2) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (see [19]). *Let X be a uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in X, $\alpha, \beta \in (0,1)$, $a \ge 0$, and let $\{\alpha_n\}$ be a real sequence number satisfying the following:*

(i) $0 < \alpha \le \alpha_n \le \beta < 1$, *for all $n \ge n_0$, and for some $n_0 \in \mathbb{N}$;*

(ii) $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$;

(iii) $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = a$.

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. An implicit random iterative process

In this section, we present an implicit random iterative process with errors for a finite family of asymptotically quasi-nonexpansive mappings. We also establish the necessary and sufficient condition for the convergence of this process to the common random fixed point of the finite family mentioned before. Our results can also be seen as an extension to the results of Beg and Abbas [13] and Chang et al. [20].

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex separable Banach space X. Let $\{T_i : i \in J\}$ be N asymptotically quasi-nonexpansive random operators from $\Omega \times C$ to C with sequence of measurable mappings $r_n(\omega) : \Omega \to [0, +\infty)$ satisfying $\sum_{n=1}^{\infty} r_n(\omega) < \infty$ for each $\omega \in \Omega$ and for all $i \in J$ and $F = \bigcap_{i=1}^{N} RF(T_i) \ne \emptyset$. Let ξ_0 be a measurable mapping from Ω to C, then the sequence of random implicit iteration with errors (2.7) converges to a common random fixed point of random operators $\{T_i : i \in J\}$ in C if and only if $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$, where $\{\alpha_n\}$ is a sequence of real numbers in $(s, 1-s)$ for some $s \in (0,1)$ and $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$.*

Proof. The sufficient condition is obvious. Conversely, for any measurable mapping $\xi \in F$, we have

$$\|\xi_n(\omega) - \xi(\omega)\| = \|\alpha_n \xi_{n-1}(\omega) + (1-\alpha_n) T_n^k(\omega, \xi(\omega)) + f_n(\omega) - \xi(\omega)\|, \tag{3.1}$$

where $n = (k-1)N + i$, $k = k(n)$, and $T_n = T_{i(\bmod N)} = T_i$. This implies that

$$\begin{aligned}\|\xi_n(\omega) - \xi(\omega)\| &\le \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n)\|T_i^k(\omega, \xi_n(\omega)) - \xi(\omega)\| + \|f_n(\omega)\| \\ &\le \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n)(1 + r_k(\omega))\|\xi_n(\omega) - \xi(\omega)\| + \|f_n(\omega)\| \\ &\le \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n + r_k(\omega))\|\xi_n(\omega) - \xi(\omega)\| + \|f_n(\omega)\|.\end{aligned} \tag{3.2}$$

Thus, we have

$$\|\xi_n(\omega) - \xi(\omega)\| \le \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{r_k(\omega)}{\alpha_n}\|\xi_n(\omega) - \xi(\omega)\| + \frac{\|f_n(\omega)\|}{\alpha_n}. \tag{3.3}$$

Since $0 < s < \alpha_n < 1 - s < 1$, it follows from (3.3) that

$$\|\xi_n(\omega) - \xi(\omega)\| \le \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{r_k(\omega)}{s}\|\xi_n(\omega) - \xi(\omega)\| + \frac{\|f_n(\omega)\|}{s}. \tag{3.4}$$

Since $\sum_{k=1}^{\infty} r_k(\omega) < \infty$ for each $\omega \in \Omega$, there exists a positive integer n_0 such that $s - r_n(\omega) > 0$ and $r_n(\omega) < s/2$ for each $\omega \in \Omega$ and for all $n = (k-1)N + i \ge n_0$.

Thus, we have

$$\|\xi_n(\omega) - \xi(\omega)\| \le \left(1 + \frac{r_k(\omega)}{s - r_k(\omega)}\right)\|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{\|f_n(\omega)\|}{s - r_k(\omega)}. \tag{3.5}$$

It follows from (3.5) that, for each $n = (k-1)N + i \ge n_0$, we have

$$\|\xi_n(\omega) - \xi(\omega)\| \le \left(1 + \frac{2r_n(\omega)}{s}\right)\|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{2\|f_n(\omega)\|}{s} \quad \text{for each } \xi \in F. \tag{3.6}$$

Setting $b_n(\omega) = 2r_k(\omega)/s$, where $n = (k-1)N + i$, $i \in J$ and $k \ge 1$, then we obtain

$$d(\xi_n(\omega), F) \le (1 + v_k(\omega)) d(\xi_{n-1}(\omega), F) + \frac{2\|f_n(\omega)\|}{s} \quad \text{for each } \omega \in \Omega \text{ and for all } n \ge n_0. \tag{3.7}$$

Taking $a_{n+1}(\omega) = d(\xi_n(\omega), F)$, $d_n(\omega) = 2\|f_n(\omega)\|/s$ in Lemma 2.5 and using conditions $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$ and $\sum_{n=1}^{\infty} r_n(\omega) < \infty$, it is easy to see that $\sum_{n=1}^{\infty} \|b_n(\omega)\| < \infty$, $\sum_{n=1}^{\infty} \|c_n(\omega)\| < \infty$. It follows from Lemma 2.5 that $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$ for each $\omega \in \Omega$. Let $b_n(\omega) = v_{i_k}(\omega)$, and by (3.6) since $b_n(\omega) = 2r_k(\omega)/s$, where $n = (k-1)N + i$, $i \in J$. Notice that when $x > 0$, $1 + x \le \exp(x)$, and

$$\|\xi_n(\omega) - \xi(\omega)\| \le (1 + b_n(\omega))\|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{2\|f_n(\omega)\|}{s}, \quad \forall \xi \in F. \tag{3.8}$$

Then

$$
\begin{aligned}
&\|\xi_{n+m}(\omega)-\xi(\omega)\| \\
&\quad\le (1+b_{n+m}(\omega))\|\xi_{n+m-1}(\omega)-\xi(\omega)\|+\frac{2\|f_{n+m}(\omega)\|}{s} \\
&\quad\le \exp(b_{n+m}(\omega))\|\xi_{n+m-1}(\omega)-\xi(\omega)\|+\frac{2\|f_{n+m}(\omega)\|}{s} \\
&\quad\le \exp(b_{n+m}(\omega))(1+b_{n+m-1}(\omega))\left(\|\xi_{n+m-2}(\omega)-\xi(\omega)\|+\frac{2\|f_{n+m-1}(\omega)\|}{s}\right) \\
&\qquad+\frac{2\|f_{n+m}(\omega)\|}{s} \\
&\qquad\qquad\vdots \\
&\quad\le \exp\left(\sum_{k=n}^{n+m} b_k(\omega)\right)\|\xi_n(\omega)-\xi(\omega)\|+\frac{2}{s}\exp\left(\sum_{k=n}^{n+m-1} b_k\|f_k(\omega)\|\right)+\frac{2\|f_{n+m}(\omega)\|}{s} \\
&\quad\le \exp\left(\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{i_k}(\omega)\right)\|\xi_n(\omega)-\xi(\omega)\|+\frac{2}{s}\exp\left(\sum_{i=1}^{N-1}\sum_{k=1}^{\infty} v_{i_k}\|f_n(\omega)\|\right)+\frac{2\|f_{n+m}(\omega)\|}{s}
\end{aligned}
\tag{3.9}
$$

for each $\omega\in\Omega$, and for all natural numbers m,n. Put $M=\exp(\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{i_k}(\omega))+1<\infty$. Since $\sum_{n=1}^{\infty}\|f_n(\omega)\|<\infty$, it follows that

$$
\|\xi_{n+m}(\omega)-\xi(\omega)\|\le M\|\xi_n(\omega)-\xi(\omega)\|,\quad \forall\,\omega\in\Omega. \tag{3.10}
$$

Let $\varepsilon>0$. Since $\lim_{n\to\infty} d(\xi_n(\omega),F)=0$ for each $\omega\in\Omega$, there exists a natural number n_1 such that $d(\xi_n,F)<\varepsilon/2M$ for each $\omega\in\Omega$ and for all $n\ge n_1$. In particular, there exists a point $\xi(\omega)\in F$ such that $\|\xi_{n_1}(\omega)-\xi(\omega)\|\le\varepsilon/2M$. Now, for each $n\ge n_1$ and for all $m\ge 1$, we have

$$
\begin{aligned}
\|\xi_{n+m}(\omega)-\xi_n(\omega)\| &\le \|\xi_{n+m}(\omega)-\xi(\omega)\|+\|\xi_n(\omega)-\xi(\omega)\| \\
&\le M\|\xi_n(\omega)-\xi(\omega)\|+\|\xi_n(\omega)-\xi(\omega)\| \\
&\le \varepsilon.
\end{aligned}
\tag{3.11}
$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega\in\Omega$. Therefore, $\xi_n(\omega)\to p(\omega)$ for each $\omega\in\Omega$, where $p:\Omega\to F$, being the limit of the measurable mappings, is also measurable. Now, $\lim_{n\to\infty} d(\xi_n(\omega),F)=0$, for each $\omega\in\Omega$, and the set F is closed; we have $p\in F$, that is, p is a common random fixed point of $\{T_i : i\in J\}$. □

Lemma 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex separable Banach space X. Let $\{T_i : i\in J\}$ be N uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operators from $\Omega\times C\to C$, with sequence of measurable mappings*

$r_n(\omega) : \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} r_n(\omega) < \infty$ for each $\omega \in \Omega$ and for each $i \in J$ and $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let ξ_0 be a measurable mapping from Ω to C and let the implicit random iterative process with errors be as in (2.7). If $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$, where $\{\alpha_n\}$ is a sequence of real numbers in an open interval $(s, s-1)$ for some $s \in (0,1)$ and $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$, then

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.12}$$

for each $\omega \in \Omega$.

Proof. It follows from (3.6), and Lemma 2.6, that $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ exists for any $\xi \in F$. Since $\{\xi_n(\omega) - \xi(\omega)\}$ is a convergent sequence, without loss of generality, we can assume that

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = d_\omega, \tag{3.13}$$

where $d_\omega \geq 0$. Observe that

$$\begin{aligned} \|\xi_n(\omega) - \xi(\omega)\| = \|&\alpha_n(\xi_{n-1}(\omega) - \xi(\omega) + f_n(\omega)) \\ &+ (1-\alpha_n)(T_i^k(\omega, \xi_{n-1}(\omega)) - \xi(\omega) + f_n(\omega))\|. \end{aligned} \tag{3.14}$$

From $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$ and (3.13), it follows that

$$\begin{aligned} &\limsup_{n\to\infty} \|\xi_{n-1}(\omega) - \xi(\omega) + f_n(\omega)\| \\ &\quad \leq \limsup_{n\to\infty} \|\xi_{n-1}(\omega) - \xi(\omega)\| + \limsup_{n\to\infty} \|f_n(\omega)\| \leq d_\omega \end{aligned} \tag{3.15}$$

and hence

$$\begin{aligned} &\limsup_{n\to\infty} \|T_n^k(\omega, \xi_n(\omega)) - \xi(\omega) + f_n(\omega)\| \\ &\quad \leq \limsup_{n\to\infty} r_n(\omega)\|\xi_n(\omega) - \xi(\omega)\| + \limsup_{n\to\infty} \|f_n(\omega)\| \leq d_\omega, \end{aligned} \tag{3.16}$$

where $n = (k-1)N + i$.

Therefore from (3.13)–(3.16) and Lemma 2.6, we have that

$$\lim_{n\to\infty} \|T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)\| = 0, \quad \forall \omega \in \Omega. \tag{3.17}$$

Moreover, since

$$\begin{aligned} \|\xi_n(\omega) - \xi_{n-1}(\omega)\| &= \|(1-\alpha_n) T_n^k(\omega, \xi_n(\omega)) - (1-\alpha_n)\xi_{n-1}(\omega) + f_n(\omega)\| \\ &\leq (1-\alpha_n)\|T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)\| + \|f_n(\omega)\|, \end{aligned} \tag{3.18}$$

hence by (3.17),

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi_{n-1}(\omega)\| = 0 \tag{3.19}$$

for each $\omega \in \Omega$ and $\|\xi_n(\omega) - \xi_{n+l}(\omega)\| \to 0$, for each $\omega \in \Omega$, and for all $l < 2N$. Now, for $n > N$, we have

$$\begin{aligned}
&\|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| \\
&\quad \le \|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\| + \|T_n^k(\omega, \xi_n(\omega)) - T_n(\omega, \xi_n(\omega))\| \\
&\quad \le \|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\| + L\|T_n^{k-1}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| \\
&\quad \le \|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\| + L\{\|T_n^{k-1}(\omega, \xi_n(\omega)) - T_{n-N}^{k-1}(\omega, \xi_{n-N}(\omega))\|\} \\
&\qquad + L\{\|T_{n-N}^{k-1}(\omega, \xi_{n-N}(\omega)) - \xi_{(n-N)-1}(\omega)\| + \|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\|\}.
\end{aligned} \tag{3.20}$$

Since for each $n > N$, $n \equiv (n - N) \bmod N$. Thus $T_n = T_{n-N}$, therefore

$$\begin{aligned}
&\|\xi_{n-1}(\omega)) - T_n(\omega, \xi_n(\omega))\| \\
&\quad \le \|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\| + L^2\|\xi_n(\omega) - \xi_{n-N}(\omega)\| \\
&\qquad + L\|T_{n-N}^{k-1}(\omega, \xi_{n-N}(\omega)) - \xi_{(n-N)-1}(\omega)\| + L\|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\|.
\end{aligned} \tag{3.21}$$

This implies that

$$\lim_{n\to\infty} \|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.22}$$

for each $\omega \in \Omega$. Now

$$\|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| \le \|\xi_{n-1}(\omega) - \xi_n(\omega)\| + \|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\|. \tag{3.23}$$

Hence

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.24}$$

for each $\omega \in \Omega$. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a uniformly convex separable Banach space X. Let $\{T_i : i \in J\}$ be N uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operators from $\Omega \times C \to C$, with sequence of measurable mappings $r_n(\omega) : \Omega \to [0, +\infty)$ satisfying $\sum_{n=1}^{\infty} r_n(\omega) < \infty$ for each $\omega \in \Omega$ and for each $i \in J$. Suppose that $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$, and there is one member T in the family $\{T_i : i \in J\}$ which is a semicompact random operator. Let ξ_0 be a measurable mapping from Ω to C. Then, the implicit random iterative process with errors (2.7) converges to a common random fixed point of random operators $\{T_i,\ i \in J\}$, where $\{\alpha_n\}$ is a sequence of real numbers in $(s,\ s-1)$ for some $s \in (0,1)$ and $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$.*

Proof. For any given $\xi(\omega) \in F$, we note that

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = d_\omega, \tag{3.25}$$

where $d_\omega \geq 0$. By Lemma 3.2, we have

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.26}$$

for each $\omega \in \Omega$. Consequently, for any $j = 1, 2, \ldots, N$,

$$\begin{aligned}
\|\xi_n(\omega) - T_{n+j}(\omega, \xi_n(\omega))\| &\leq \|\xi_n(\omega) - \xi_{n+j}(\omega)\| + \|\xi_{n+j}(\omega) - T_{n+j}(\omega, \xi_{n+j}(\omega))\| \\
&\quad + \|T_{n+j}(\omega, \xi_{n+j}(\omega)) - T_{n+j}(\omega, \xi_n(\omega))\| \\
&\leq (1+L)\|\xi_n(\omega) - \xi_{n+j}(\omega)\| + \|\xi_{n+j}(\omega) - T_{n+j}(\omega, \xi_{n+j}(\omega))\| \\
&\longrightarrow 0,
\end{aligned} \tag{3.27}$$

as $n \to \infty$ for each $\omega \in \Omega$ and $j \in J$.

Consequently, $\|\xi_n(\omega) - T_j(\omega, \xi_n(\omega))\| \to 0$ as $n \to \infty$ for each $\omega \in \Omega$ and $j \in J$. Assume that T_1 is a semicompact random operator. Therefore, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi_0 : \Omega \to C$ such that ξ_{n_k} converges pointwise to ξ_0. Now

$$\lim_{n\to\infty} \|\xi_{n_k}(\omega) - T_j(\omega, \xi_{n_k}(\omega))\| = \|\xi_0(\omega) - T_j(\omega, \xi_0(\omega))\| = 0 \tag{3.28}$$

for each $\omega \in \Omega$, and $j \in J$. It implies that $\xi_0 \in F$, and so $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$. Hence, by Theorem 3.1, we obtain that $\{\xi_n\}$ converges to a point in F. □

Corollary 3.4 (cf. [13, Theorem 4.2]). *Let C be a nonempty closed bounded and convex subset of a uniformly convex separable Banach space X. Let $\{T_i : i \in J\}$ be N uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operator from $\Omega \times C \to C$, with sequence of measurable mappings $r_{i_n}(\omega) : \Omega \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n}(\omega) < \infty$, for each $\omega \in \Omega$ and for each $i \in J$. Let $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$, and there is one member T in the family $\{T_i : i \in J\}$ which is a semicompact random operator. Let ξ_0 be a measurable mapping from Ω to C. Then the sequence $\{\xi_n\}$ defined by*

$$\xi_n(\omega) = \alpha_n \xi_{n-1}(\omega) + (1 - \alpha_n) T_i^k(\omega, \xi_n(\omega)), \tag{3.29}$$

where $n = (k-1)N + i$, $i \in \{1, 2, \ldots, N\} = J$, converges to a common random fixed point of random operators $\{T_i,\ i \in J\}$, where $\{\alpha_n\}$ is a sequence of real numbers in an open interval $(s,\ s-1)$ for some $s \in (0, 1)$.

Proof. Taking $f_n(\omega) = 0$, for all $n \geq 1$, for each $\omega \in \Omega$ in Theorem 3.3, the conclusion of the corollary is immediate. □

THEOREM 3.5. *Let C be a nonempty closed and convex subset of a uniformly convex separable Banach space X. Let $\{T_i : i \in J\}$ be N uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operator from $\Omega \times C \to C$, with sequence of measurable mappings $r_{i_n}(\omega) : \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n}(\omega) < \infty$, for each $\omega \in \Omega$ and for each $i \in J$. Suppose that $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$, and the family $\{T_i : i \in J\}$ satisfies the condition (B). Let ξ_0 be a measurable mapping from Ω to C. Then the implicit random iterative process with errors (2.7) converges to a common random fixed point of random operators $\{T_i,\ i \in J\}$, where $\{\alpha_n\}$ is a sequence of real numbers in $(s,\ s-1)$ for some $s \in (0,1)$ and $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$.*

Proof. Let $\xi(\omega) \in F$. Then it follows from (3.25) and the condition (B) that

$$d(\xi_n(\omega),F) \leq \max_{1\leq l\leq N} \{||\xi_n(\omega) - T_l(\omega,\xi_n(\omega))||\} \tag{3.30}$$

for each $n \geq 1$, and $\omega \in \Omega$. Since $\|\xi_n(\omega) - T_j(\omega,\xi_n(\omega))\| \to 0$ as $n \to \infty$, for each $\omega \in \Omega$ and $j \in J$, we have

$$\lim_{n\to\infty} f(d(\xi_n(\omega),F)) = 0. \tag{3.31}$$

Since f is nondecreasing on $[0,\infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0,\infty)$, it follows that

$$\lim_{n\to\infty} d(\xi_n(\omega),F) = 0. \tag{3.32}$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(\xi_n(\omega),F) = 0$, for each $\omega \in \Omega$, there exists a natural number n_1 such for $n \geq n_1$, $d(\xi_n,F) \leq \varepsilon/2M$, for each $\omega \in \Omega$. In particular, there exists a point $\xi(\omega) \in F$ such that $\|\xi_{n_1}(\omega) - \xi(\omega)\| \leq \varepsilon/2M$. Now for $n \geq n_1$ and for all $m \geq 1$, consider

$$\begin{aligned} ||\xi_{n+m}(\omega) - \xi_n(\omega)|| &\leq ||\xi_{n+m}(\omega) - \xi(\omega)|| + ||\xi_n(\omega) - \xi(\omega)|| \\ &\leq M||\xi_n(\omega) - \xi(\omega)|| + ||\xi_n(\omega) - \xi(\omega)||. \end{aligned} \tag{3.33}$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega \in \Omega$. By the completeness of the space X, there exists a measurable mapping $p : \Omega \to C$ such that

$$\lim_{n\to\infty} \xi_n(\omega) = p(\omega), \quad \forall \omega \in \Omega. \tag{3.34}$$

Next, we prove that $p(\omega) \in F$. To this end, we let $\varepsilon > 0$ be given, there exists $n_1 \in \mathbb{N}$ such that $\|\xi_n(\omega) - p(\omega)\| < \varepsilon/4$ for all $n \geq n_1$. Since $\lim_{n\to\infty} d(\xi_n(\omega),F) = 0$, there is $n_2 \geq n_1$

such that $d(\xi_{n_2}(\omega), F) < \varepsilon/4$. This implies that there exists $q(\omega) \in F$ such that $\|\xi_{n_2}(\omega) - q(\omega)\| < \varepsilon/4$. Then for each $i = 1, 2, 3, \ldots, N$ and $n \geq n_2$, we have

$$\begin{aligned} \|T_l(\omega, p(\omega)) - p(\omega)\| &\leq \|T_l(\omega, q(\omega)) - p(\omega)\| + \|q(\omega) - p(\omega)\| \\ &\leq 2\|q(\omega) - p(\omega)\| \\ &\leq 2(\|q(\omega) - \xi_n(\omega)\| + \|\xi_n(\omega) - p(\omega)\|) \\ &\leq \varepsilon. \end{aligned} \tag{3.35}$$

Therefore, $T_i(\omega, p(\omega)) = p(\omega)$, for all $i = 1, 2, 3, \ldots, N$. This completes the proof. □

Remark 3.6. By using the similar method given in the above proof with different approximation obtained from many results, we can conclude that Theorems 3.1, 3.3, and 3.5 are also valid for the errors considered, in the sense of Xu [15], under the appropriately controllable conditions on the parameters $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$.

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