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Research Article

Construction of Planar Harmonic Functions

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Dedicated to the memory of Evelyn Marie Silvia (1948–2006)
Evelyn was a friend and colleague of Jay for 21 years and Herb for 37 years

Recommended by Teodor Bulboaca

Complex-valued harmonic functions that are univalent and sense-preserving in the open unit disk can be written in the form $f = h + \overline{g}$, where h and g are analytic in the open unit disk. The functions h and g are called the analytic and coanalytic parts of f, respectively. In this paper, we construct certain planar harmonic maps either by varying the coanalytic parts of harmonic functions that are known to be harmonic starlike or by adjoining analytic univalent functions with coanalytic parts that are related or derived from the analytic parts.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset C$ if both u and v are real harmonic in D. In any simply connected domain, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic and g the coanalytic part of f. Clunie and Sheil-Small [1] pointed out that a necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D.

Denote by H the class of functions f that are harmonic univalent and sense-preserving in the open unit disk $\Delta = \{z : |z| < 1\}$ with $f(0) = f_z(0) - 1 = 0$. Thus the analytic and coanalytic parts of the function $f = h + \overline{g}$ may be written, respectively, as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Note that H reduces to S, the class of normalized univalent analytic functions whenever the coanalytic part of f is zero, that is, $g \equiv 0$. For $f = h + \overline{g} \in H$, the condition $h'(0) = 1 > |g'(0)| = |b_1|$ implies that the function $(f - \overline{b_1 f})/(1 - |b_1|^2)$ is also in H. Consequently, we may sometimes restrict ourselves to $\widehat{H}$, the subclass of H for which $b_1 = f_{\overline{z}}(0) = 0$. Clunie and Sheil-Small [1] showed that H and $\widehat{H}$ are normal families but only $\widehat{H}$ is compact. For more references on harmonic mappings, see Duren [2].

In this paper, we construct certain planar harmonic maps either by varying the coanalytic parts of functions $f = h + \overline{g}$ or by adjoining functions in S with coanalytic parts that are related to or derived from the analytic parts. One form of the latter process, known as shearing, is due to Clunie and Sheil-Small [1]. An application of this method has recently appeared in [3] and a detailed discussion of shear construction can be found in [4].

2. Functions harmonic starlike by proximity

In this section, some harmonic functions are identified by virtue of neighborhood proximity to a given function. When discussing combinations of functions involving coefficients, it is convenient to indicate the kth coefficient of a function f as $a_k(f)$. Using this convention, a normalized harmonic function $f = h + \overline{g}$ of the form (1.1) can be written in the form

$$f(z) = z + a_{-1}(f)\overline{z} + \sum_{|k|=2}^{\infty} a_k(f)\phi_k(z), \tag{2.1}$$

where $\phi_k(z) = z^k$ for $k \geq 2$ and $\phi_k(z) = (\overline{z})^{-k}$ for $k \leq -2$.

It is proved in [5] (see also [6]) that for harmonic functions $f \in H$ of the form (2.1) if

$$(1+\alpha)\left|a_{-1}(f)\right| + \sum_{|k|=2}^{\infty} \left|(k-\alpha)\right|\left|a_k(f)\right| \leq 1-\alpha, \quad 0 \leq \alpha < 1, \tag{2.2}$$

then $(\partial/\partial\theta)(\arg f(re^{i\theta})) \geq \alpha$, $z = re^{i\theta} \in \Delta$, that is, f is harmonic sense-preserving and starlike of order α in Δ. The condition (2.2) is also necessary if $f \in \widehat{H}$. For the special cases of (2.2) when $a_{-1}(f) = \alpha = 0$, see Silverman [7] and when only $\alpha = 0$, see Silverman and Silvia [8]. For the sake of simplicity, the class of sense-preserving harmonic functions starlike of order zero in Δ is called sense-preserving harmonic starlike.

For f of the form (2.1) and $\delta > 0$, the δ-neighborhood of f, denoted by $N_\delta(f)$, is defined as the class of all functions F of the form (2.1) so that

$$\left|a_{-1}(f) - a_{-1}(F)\right| + \sum_{|k|=2}^{\infty} |k|\left|a_k(f) - a_k(F)\right| \leq \delta. \tag{2.3}$$

We let $\widehat{N}_\delta(f) \equiv N_\delta(f) \cup \widehat{H}$.

Ruscheweyh [9] introduced the notion of δ-neighborhoods for subclasses of analytic univalent functions and proved that $N_{1/4}(f)$ is analytic starlike for all analytic convex functions f. The natural extension of Ruscheweyh's δ-neighborhood to the harmonic case was introduced by Avci and Zlotkiewicz [10].

From the coefficient condition (2.2) for $\alpha = 0$, we conclude that, over the class H of sense-preserving harmonic functions, the 1-neighborhood $N_1(z)$ is a subset of sense-preserving harmonic starlike functions. In other words, coefficient proximity of planar harmonic functions to the identity function can yield starlikeness. This readily supports the construction of starlike planar harmonic functions. For example, the harmonic function

$$f(z) = h(z) + \overline{g}(z) = z + \frac{i}{3}\overline{z} + \sum_{|k|=2}^{\infty} a_k(f)\phi_k(z), \tag{2.4}$$

where $a_k(f) = a_k = 1/(k2^k)$ for $k \geq 2$ and $a_{-k}(f) = b_k = i/(k3^k)$ for $k \geq 2$ is sense-preserving harmonic starlike in Δ.

For $0 < \alpha < 1$ and for f and F of the form (2.1), let $F \in N_\alpha(f)$. Using the triangle inequality, we can write

$$\begin{aligned}
&\left|a_{-1}(F)\right| + \sum_{|k|=2}^{\infty} |k|\left|a_k(F)\right| \\
&\quad = \left|a_{-1}(f) - a_{-1}(F)\right| + \sum_{|k|=2}^{\infty} |k|\left|a_k(f) - a_k(F)\right| + \left|a_{-1}(f)\right| + \sum_{|k|=2}^{\infty} |k|\left|a_k(f)\right| \\
&\quad < \left|a_{-1}(f) - a_{-1}(F)\right| + \sum_{|k|=2}^{\infty} |k|\left|a_k(f) - a_k(F)\right| + (1+\alpha)\left|a_{-1}(f)\right| \\
&\qquad + \sum_{|k|=2}^{\infty} |k|\left|a_k(f)\right| \leq \alpha + (1-\alpha) = 1.
\end{aligned} \tag{2.5}$$

Therefore, by (2.2) and according to the definition (2.3), we have proved the following.

Theorem 2.1. *If f of the form (2.1) satisfies the coefficient condition (2.2), then $N_\alpha(f)$, $0 < \alpha < 1$, consists of sense-preserving harmonic starlike functions.*

Since the condition (2.2) is also necessary if $f \in \widehat{H}$, the above theorem yields the following.

Corollary 2.2. *If $f \in \widehat{H}$ is starlike of order α, $0 < \alpha < 1$, then $\widehat{N}_\alpha(f)$, $0 < \alpha < 1$, consists of sense-preserving harmonic starlike functions.*

For the special case $\alpha = 1/2$, Theorem 2.1 yields the following corollary which was also obtained by Avci and Zlotkiewicz [10].

Corollary 2.3. *If $f \in \widehat{H}$ is of the form (2.1) and if $\sum_{|k|=2}^{\infty} k^2|a_k(f)| \leq 1$, then $\widehat{N}_{1/2}$ consists of sense-preserving harmonic starlike functions.*

The more general form of the above results is given in the following theorem.

THEOREM 2.4. *For* $0 < \alpha < 1$, *let* $f = h + \overline{g}$ *satisfy the condition (2.2), where* f *is of the form (2.1) and* h *and* g *are given by (1.1). If the complex sequences* $\{c_k\}_{k=2}^{\infty}$ *and* $\{d_k\}_{k=1}^{\infty}$ *satisfy the inequality*

$$|d_1| + \sum_{k=2}^{\infty} k(|c_k| + |d_k|) \leq \frac{\alpha}{2-\alpha}, \tag{2.6}$$

then the function $F = H + \overline{G}$ *is sense-preserving harmonic starlike in* Δ, *where*

$$H(z) = h(z) + \sum_{k=2}^{\infty} c_k z^k, \qquad G(z) = g(z) + \sum_{k=1}^{\infty} d_k z^k. \tag{2.7}$$

Proof. For fixed α, $0 < \alpha < 1$, set $|a_k| = \lambda_k((1-\alpha)/(k-\alpha))$ and $|b_k| = \mu_k((1-\alpha)/(k+\alpha))$. In view of (2.2), $\mu_1 + \sum_{k=2}^{\infty}(\lambda_k + \mu_k) \leq 1$. Since $k((1-\alpha)/(k-\alpha))$ is decreasing in k and $k((1-\alpha)/(k+\alpha))$ is increasing in k and observing that $\max\{2((1-\alpha)/(2-\alpha)), (1-\alpha)\} = 2((1-\alpha)/(2-\alpha))$, it follows that $|b_1| + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 2((1-\alpha)/(2-\alpha))$, with equality when $\lambda_2 = 1$. Then under the hypotheses of the theorem, we conclude that $F = H + \overline{G}$ is sense-preserving harmonic starlike in Δ because

$$\begin{aligned}
&\sum_{k=2}^{\infty} k|a_k + c_k| + \sum_{k=1}^{\infty} k|b_k + d_k| \\
&\quad \leq |b_1| + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) + |d_1| + \sum_{k=2}^{\infty} k(|c_k| + |d_k|) \\
&\quad \leq 2\left(\frac{1-\alpha}{2-\alpha}\right) + \frac{\alpha}{2-\alpha} = 1.
\end{aligned} \tag{2.8}$$

□

We note that the result of Theorem 2.4 is sharp when $a_2 = (1-\alpha)/(2-\alpha)$ and $c_2 = \alpha/2(2-\alpha)$.

The approach used to prove Theorem 2.4 can be used to prove the following more general result.

THEOREM 2.5. *Under the hypotheses of Theorem 2.4,* $N_\delta(f)$ *consists of functions that are sense-preserving harmonic starlike of order* β, $\beta < \alpha$, *when* $\delta = 2(\alpha-\beta)/(2-\alpha)(2-\beta)$.

Proof. Since $((k-\beta)/(1-\beta))((1-\alpha)/(k-\alpha))$ is decreasing in k and $((k+\beta)/(1-\beta))((1-\alpha)/(k+\alpha))$ is increasing in k, we can write

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta}{1-\beta}\right)|a_k| + \sum_{k=1}^{\infty}\left(\frac{k+\beta}{1-\beta}\right)|b_k| \leq \left(\frac{2-\beta}{1-\beta}\right)\left(\frac{1-\alpha}{2-\alpha}\right) \tag{2.9}$$

with equality only when $|a_2| = (1-\alpha)/(2-\alpha)$. Thus,

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta}{1-\beta}\right)|c_k| + \sum_{k=1}^{\infty}\left(\frac{k+\beta}{1-\beta}\right)|d_k| \leq 1 - \left(\frac{2-\beta}{1-\beta}\right)\left(\frac{1-\alpha}{2-\alpha}\right). \tag{2.10}$$

Using a similar argument, we see that $|d_1|+\sum_{k=2}^{\infty}k(|c_k|+|d_k|)$ is maximized when

$$\frac{2-\beta}{1-\beta}|c_2| = 1-\left(\frac{2-\beta}{1-\beta}\right)\left(\frac{1-\alpha}{2-\alpha}\right) \quad \text{or} \quad |c_2| = \frac{\alpha-\beta}{(2-\beta)(2-\alpha)}. \tag{2.11}$$

Therefore, the theorem follows since

$$|d_1|+\sum_{k=2}^{\infty}k(|c_k|+|d_k|) \le \frac{2(\alpha-\beta)}{(2-\beta)(2-\alpha)}. \tag{2.12}$$

□

Note that the result of Theorem 2.5 is sharp for $|a_2| = (1-\alpha)/(2-\alpha)$ and $|c_2| = (\alpha-\beta)/(2-\beta)(2-\alpha)$.

3. Planar harmonic functions with directional convexity

A domain Ω is said to be convex in the direction $e^{i\phi}$ if, for every fixed complex number τ and real number t, the set $\Omega\cap\{\tau+te^{i\phi}\}$ is either connected or empty. In [1], the following characterization of harmonic functions with directional convexity was proved.

Lemma 3.1. *A harmonic function $f = h+\overline{g}$ locally univalent in Δ is a univalent mapping of Δ onto a domain that is convex in the direction of the real axis if and only if $h-g$ is a conformal mapping of Δ onto a domain that is convex in the direction of the real axis.*

By imposing a simple rotation and forming $f_\phi = h+e^{i\phi}\overline{g}$, the function f_ϕ obtained from $f = h+\overline{g}$ is convex in the direction $e^{i\phi}$. For the sake of brevity, in this section we only deal with functions convex in the direction of the real axis. The process discussed in the above lemma has recently been referred to as shear construction (see [4]) and provides a useful tool for constructing some planar harmonic functions with directional convexity. In the following, we use Lemma 3.1 to construct harmonic functions that are convex in the direction of the real axis.

Example 3.2. For $k = ie^{-it/2}[(e^{-it}-ie^{2it})/8\cos^3(t/2-\pi/4)]$, $c = -ie^{2it}$, $d = -ie^{it}$, and $0 < t < \pi$ set

$$\begin{aligned} h(z) &= k\,\mathrm{Log}\left(\frac{1+dz}{1-z}\right)+\frac{1}{(1+d)^2}\left(\frac{(d-2c+d^2)z+(2d^2+cd^2)z^2}{(1+dz)^2}\right),\\ g(z) &= k\,\mathrm{Log}\left(\frac{1+dz}{1-z}\right)-\frac{1}{(1+d)^2}\left(\frac{(1+2c+d)z+(c+2cd-2d^2)z^2}{(1+dz)^2}\right). \end{aligned} \tag{3.1}$$

Setting $G = h-g$, we obtain $G(z) = (z+cz^2)/(1+dz)^2$ which is analytic and convex in the direction of the real axis. This is because the function K defined by $G(z)=e^{-i(\pi/2)}K(e^{i(\pi/2)}z)$ is proved by Goodman and Saff [11] to be convex in the direction of the imaginary axis. Therefore, from Lemma 3.1, we conclude that $f = h+\overline{g}$ is a planar harmonic function that is convex in the direction of the real axis.

Another set of examples follows nicely from binomials in the form $z-az^n$ for selected n. To set things up, we have the following.

LEMMA 3.3. *Let $f_n(z) = z - az^n$, where a is real and n is a natural number greater than 1. Then*

(i) *f_2 is convex in the direction of the real axis whenever $|a| \le 1/2$;*
(ii) *f_3 is convex in the direction of the real axis whenever $-1/9 \le a \le 1/3$;*
(iii) *f_4 is convex in the direction of the real axis whenever $|a| \le 1/8$;*
(iv) *f_5 is convex in the direction of the real axis whenever $-1/5 \le a \le 1/25$.*

Proof. Since f_n is univalent in Δ if and only if $|a| \le 1/n$ and convex in Δ if and only if $|a| \le 1/n^2$, we need to consider only the intervals $-1/n \le a < -1/n^2$ and $1/n^2 < a \le 1/n$. □

According to the definition of directional convexity, f_n is convex in the direction of the real axis if the intersection of any line parallel to the real axis with $f_n(\Delta)$ is a connected set or empty. Consequently, f_n is convex in the direction of the real axis if and only if $\Im f_n(e^{i\theta})$ for $-\pi < \theta \le \pi$ has at most one relative extremum in the upper half-plane and at most one relative extremum in the lower half-plane.

(i) For $n = 2$, we have $\Im\{f_2(e^{i\theta})\} = \sin(\theta) - a\sin(2\theta)$. Let

$$g_2(\theta) = \frac{\partial(\Im\{f_2(e^{i\theta})\})}{\partial\theta} = \cos(\theta) - 2a\cos(2\theta) = 2a + \cos(\theta) - 4a\cos^2(\theta). \tag{3.2}$$

For $a \ne 0$, we observe that $g_2(\theta) = 0$ whenever $\cos(\theta) = (1 \pm \sqrt{1+32a^2})/(8a)$ are valid roots. We let $\gamma_1(a) = (1+\sqrt{1+32a^2})/(8a)$ and $\gamma_2(a) = (1-\sqrt{1+32a^2})/(8a)$ for $a \ne 0$.

First, consider $\gamma_1(a)$. For $1/4 < |a| < 1/2$, we obtain $8|a| - 1 > 0$ and $2|a| - 1 < 0$. Consequently, $16|a|(2|a|-1) = 32a^2 - 16|a| < 0$ from which it follows that $1 + 32a^2 > 64a^2 - 16|a| + 1 = (8|a|-1)^2$. Therefore, $|\gamma_1(a)| > 1$ and we conclude that $\cos(\theta) \ne \gamma_1(a)$ whenever $1/4 < |a| < 1/2$ and $-\pi < \theta \le \pi$. This means that there are no relative extrema when $1/4 < |a| < 1/2$ and $\cos(\theta) = \gamma_1(a)$. Similarly, this is the case for $|a| = 1/2$. This is because $\cos(\theta) = \gamma_1(1/2) = 1 = -\gamma_1(-1/2)$ which yields $\Im\{f_2(e^{i\theta})\} = g_2(\theta) = g_2'(\theta) = 0$.

Next, consider $\gamma_2(a)$. From

$$\frac{d\gamma_2}{da} = \frac{(1-\sqrt{1+32a^2})}{8a^2\sqrt{1+32a^2}}, \quad a \ne 0, \tag{3.3}$$

we see that γ_2 is decreasing for both $-1/2 < a < -1/4$ and $1/4 < a < 1/2$. Consequently, for $-1/2 < a < -1/4$, we have $0.366 \approx (\sqrt{3}-1)/2 < \gamma_2(a) \le 1/2$ and for $1/4 < a < 1/2$, we have $-1/2 \le \gamma_2(a) < (1-\sqrt{3})/2 \approx -0.366$.

Thus, corresponding to each coefficient a satisfying either $-1/2 < a < -1/4$ or $1/4 < a < 1/2$, there exists $\theta_a \in (-\pi, \pi)$ that is a critical number for $\Im\{f_2(e^{i\theta})\}$. On the other hand, the identity $\sin^2(\theta_a) = 1 - \cos^2(\theta_a)$ insures that any pair of relative extrema will not both occur in either the upper or lower half-plane. Hence, we conclude that $\Im\{f_2(e^{i\theta})\}$ has at most one relative extremum in the upper half-plane and at most one relative extremum in the lower half-plane when either $-1/2 < a < -1/4$ or $1/4 < a < 1/2$.

Combined with the condition $|a| \leq 1/4$ required for convexity, we conclude that f_2 is convex in the direction of the real axis whenever $|a| \leq 1/2$.

(ii) For $n = 3$, we have $\Im f_3(e^{i\theta}) = \Im\{e^{i\theta} - ae^{3i\theta}\} = \sin(\theta) - a\sin(3\theta)$ and

$$g_3(\theta) = \frac{\partial(\Im\{f_3(e^{i\theta})\})}{\partial\theta} = \cos(\theta) - 3a\cos(3\theta) = \big(\cos(\theta)\big)\big(1+9a-12a\cos^2(\theta)\big). \tag{3.4}$$

Since $(1+9a)/(12a) \leq 0$ for $-1/9 \leq a < 0$ and $(1+9a)/(12a) > 1$ for $0 < a < 1/3$, it follows that, if $-1/9 \leq a < 1/3$, then the only critical numbers for $\Im\{f_3(e^{i\theta})\}$ occur when $\cos(\theta) = 0$. Now,

$$g_3'(\theta) = \frac{dg_3}{d\theta} = \frac{\partial^2(\Im\{f_3(e^{i\theta})\})}{\partial\theta^2} = -\sin(\theta) + 9a\sin(3\theta) \tag{3.5}$$

and $-1/9 < a < 1/3$ yield that $g_3'(\pi/2) = -1-9a < 0$ and $g_3'(-\pi/2) = 1+9a > 0$. Hence, $\Im f_3(e^{i\theta})$ has a relative maximum when $\theta = \pi/2$ and a relative minimum when $\theta = -\pi/2$, while $g_3(\theta) = 4\cos(\theta)\sin^2(\theta)$ when $a = 1/3$ and $g_3'(0) = g_3'(\pi) = 0$ yield that there is no additional relative extremum. Therefore, f_3 is convex in the direction of the real axis for a satisfying $-1/9 \leq a \leq 1/3$.

(iii) When $n = 4$, we have $\Im f_4(e^{i\theta}) = \Im\{e^{i\theta} - ae^{4i\theta}\} = \sin(\theta) - a\sin(4\theta)$ and

$$g_4(\theta) = \frac{\partial(\Im\{f_4(e^{i\theta})\})}{\partial\theta} = -32a\cos^4(\theta) + 32a\cos^2(\theta) + \cos(\theta) - 4a. \tag{3.6}$$

Let

$$H_a(x) = -32ax^4 + 32ax^2 + x - 4a, \quad -1 \leq x \leq 1. \tag{3.7}$$

We will show that, for a satisfying $1/16 < |a| \leq 1/8$, H_a has exactly one zero in $[-1,1]$.

For $1/16 < a \leq 1/8$, note that $0 \leq -32ax^4 + 32ax^2 \leq 8a$ and $32ax^2 + x - 4a$ is nonpositive for

$$u_1(a) = \frac{-1-\sqrt{1+2^9a^2}}{64a} \leq x \leq \frac{-1+\sqrt{1+2^9a^2}}{64a} = u_2(a). \tag{3.8}$$

Hence, $H_a(x) \geq 0 + x - 4a \geq x - 1/2 \geq 0$ for $1/2 \leq x \leq 1$ and $H_a(x) \leq 8a + x - 4a = 4a + x < 0$ for $-1 \leq x < -1/2$. Since $H_a(x) = -32ax^4 + (32ax^2 + x - 4a) < 0$ for $u_1(a) \leq x \leq u_2(a)$ and $u_1(a) \in (-1,-1/2]$ when $1/16 < a \leq 1/8$, we have that $H_a(x) < 0$ for $-1 \leq x \leq u_2(a)$.

For $-1/8 \leq a < -1/16$, we have $8a \leq -32ax^4 + 32ax^2 \leq 0$ for $-1 \leq x \leq 1$ and that $32ax^2 + x - 4a$ is nonnegative for $u_2(a) \leq x \leq u_1(a)$. Therefore, it follows that $H_a(x) < 0$ for $-1 \leq x < -1/2$ and $H_a(x) > 0$ for $u_2(a) \leq x \leq 1$.

Combining these yields that if $1/16 < a \leq 1/8$, then $H_a \neq 0$ for $x \in [-1, u_2(a)) \cup [1/2, 1]$ and has at least one zero for $x \in [u_2(a), 1/2)$ and if $-1/8 \leq a < -1/16$, then $H_a \neq 0$ for $x \in [-1,-1/2] \cup (u_2(a), 1]$ and has at least one zero for $x \in (-1/2, u_2(a))$.

Since $u_2(a)$ is decreasing from $(-1+\sqrt{3})/4$ to $1/4$ for $1/16 < a \leq 1/8$ and increasing from $-1/4$ to $(1-\sqrt{3})/4$ for $-1/8 \leq a < -1/16$, it follows that the intervals $[u_2(a), 1/2)$

for $1/16 < a \leq 1/8$ and $(-1/2, u_2(a)]$ for $-1/8 \leq a < -1/16$ are contained in $(0, \sqrt{2}/2]$ and $[-\sqrt{2}/2, 0)$, respectively.

Finally, $H_a'(x) = 2^6 ax(1-2x^2)+1 > 0$ for $x \in [-\sqrt{2}/2, 0)$ when $-1/8 \leq a < -1/16$ and for $x \in (0, \sqrt{2}/2]$ when $1/16 < a \leq 1/8$. Consequently, H_a has exactly one zero in $[u_2(a), 1/2)$ for $1/16 < a \leq 1/8$ and exactly one zero in $(-1/2, u_2(a)]$ for $-1/8 \leq a < -1/16$. Therefore, for each a satisfying $1/16 < |a| \leq 1/8$, H_a has exactly one zero in $[-1,1]$. It follows that there is one value of $\cos(\theta)$ for which $g_4(\theta) = 0$.

We conclude that, for each a satisfying $1/16 < |a| \leq 1/8$, g_4 has exactly one zero in each of the upper and lower half-planes. Hence, $\Im f_4(e^{i\theta})$ has at most one relative extremum in each of the upper and lower half-planes, that is, f_4 is convex in the direction of the real axis for $1/16 < |a| \leq 1/8$. This combined with the convexity for $|a| \leq 1/16$ yields the claimed result.

(iv) For $n = 5$, we have $\Im f_5(e^{i\theta}) = \Im\{e^{i\theta} - ae^{5i\theta}\} = \sin(\theta) - a\sin(5\theta)$ and

$$\begin{aligned} g_5(\theta) &= \frac{\partial(\Im f_5(e^{i\theta}))}{\partial\theta} = \cos(\theta) - 5a\cos(5\theta) \\ &= \big(\cos(\theta)\big)\big[(1-25a) + 100a\cos^2(\theta) - 80a\cos^4(\theta)\big]. \end{aligned} \tag{3.9}$$

We observe that $g_5(\theta) = 0$ when $\cos(\theta) = 0$ and for $a \in (-\infty, -4/25] \cup (0, \infty)$ whenever $0 \leq (25a \pm \sqrt{125a^2 + 20a})/(20a) \leq 1$.

Because f_5 is not univalent for $|a| > 1/5$, we restrict ourselves to the consideration of $\gamma_1(a) = (25a + \sqrt{125a^2+20a})/(20a)$ and $\gamma_2(a) = (25a - \sqrt{125a^2+20a})/(20a)$ for $a \in \cup(0, 1/5]$.

Observe that $\gamma_1(-4/25) = 5/4$ and $\gamma_2(a) > 5/4$ for $-1/5 \leq a \leq -4/25$. Also $\gamma_1(a) > \gamma_1(-1/5) = 1$ for $-1/5 < a < -4/25$ because $\gamma_1'(a) = [-a/(2a^2\sqrt{125a^2+20a})] > 0$. Since $\gamma_1(-1/5) = 1$, we have that $g_5(\theta) = 0$ when $\sin^2(\theta) = \cos^2(\theta) - 1 = 0$. This in conjunction with the fact that $g_5'(\theta) = (dg_5)/(d\theta) = (\partial^2(\Im\{f_5(e^{i\theta})\}))/(\partial\theta^2) = 0$ at the critical numbers lead to the conclusion that there are no relative extrema.

For $-1/5 < a < -1/25$, the only critical numbers for $\Im\{f_5(e^{i\theta})\}$ are when $\cos(\theta) = 0$. Furthermore,

$$g_5'(\theta) = \frac{dg_5}{d\theta} = \frac{\partial^2(\Im\{f_5(e^{i\theta})\})}{\partial\theta^2} = -\sin(\theta) + 25a\sin(5\theta) \tag{3.10}$$

yields that $g_5'(\pi/2) = -1 + 25a < 0$ and $g_5'(-\pi/2) = 1 - 25a > 0$. Hence, $\Im\{f_5(e^{i\theta})\}$ has a relative maximum when $\theta = \pi/2$ and a relative minimum when $\theta = -\pi/2$. This combined with the convexity of f_5 when $|a| \leq 1/25$ yields that f_5 is convex in the direction of the real axis for a satisfying $-1/5 \leq a \leq 1/25$.

In view of Lemma 3.1, examples like those found in Lemma 3.3 lead naturally to the construction of examples of planar harmonic functions that are convex in the direction of the real axis. This is illustrated in the following.

THEOREM 3.4. *The following functions are convex in the direction of the real axis.*

(i) $F(z) = z + ((1-2a)/2)z^2 + (1/2)\overline{z}^2 + \sum_{|k|=3}^{\infty}((1-2a)/|k|)\phi_k(z)$; $|a| \leq 1/2$.

(ii) $G(z) = \sum_{k=1}^{2}(1/k)z^k + ((1-3a)/3)z^3 + \sum_{k=2}^{3}(1/k)\overline{z}^k + \sum_{|k|=4}^{\infty}((1-3a)/|k|)\phi_k(z)$; $-1/9 \leq a \leq 1/3$.

(iii) $H(z) = \sum_{k=1}^{3}(1/k)z^k + ((1-4a)/4)z^4 + \sum_{k=2}^{4}(1/k)\overline{z}^k + \sum_{|k|=5}^{\infty}((1-4a)/|k|)\phi_k(z)$; $|a| \leq 1/8$.

(iv) $J(z) = \sum_{k=1}^{4}(1/k)z^k + ((1-5a)/5)z^5 + \sum_{k=2}^{5}(1/k)\overline{z}^k + \sum_{|k|=6}^{\infty}((1-5a)/|k|)\phi_k(z)$; $-1/5 \leq a \leq 1/25$.

Proof. (i) Since $f_2(z) = z - az^2 = h_2 - g_2$ given by Lemma 3.3 is convex in the direction of the real axis for $-1/2 \leq a \leq 1/2$, we want to solve the system of equations consisting of $h_2'(z) - g_2'(z) = 1 - 2az$ and $zh_2'(z) - g_2'(z) = 0$. This yields that

$$g_2'(z) = \frac{z - 2az^2}{1 - z}. \tag{3.11}$$

A little more elementary calculus leads to

$$\begin{aligned} h_2(z) &= z - (1-2a)[\operatorname{Log}(1-z) + z] = z + (1-2a)\sum_{n=2}^{\infty}\frac{z^n}{n}, \\ g_2(z) &= az^2 - (1-2a)z - (1-2a)\operatorname{Log}(1-z) = \frac{1}{2}z^2 + (1-2a)\sum_{n=3}^{\infty}\frac{z^n}{n}. \end{aligned} \tag{3.12}$$

In view of Lemma 3.1, we conclude that, for each a satisfying $|a| \leq 1/2$, the function $F = h_2 + \overline{g_2}$ is harmonic and convex in the direction of the real axis.

(ii) For a satisfying $-1/9 \leq a \leq 1/3$, setting $f_3 = h_3 - g_3$ as given by Lemma 3.3 and $zh_3'(z) - g_2'(z) = 0$ yields that $G = h_3 + \overline{g_3}$ is harmonic and convex in the direction of the real axis where

$$\begin{aligned} h_3(z) &= z + \frac{3}{2}az^2 - (1-3a)[\operatorname{Log}(1-z) + z], \\ g_3(z) &= \frac{3}{2}az^2 + az^3 - (1-3a)[\operatorname{Log}(1-z) + z]. \end{aligned} \tag{3.13}$$

(iii) For a such that $|a| \leq 1/8$, setting $f_4 = h_4 - g_4$ and $zh_4'(z) - g_4'(z) = 0$ yields that $H = h_4 + \overline{g_4}$ is harmonic and convex in the direction of the real axis where

$$\begin{aligned} h_4(z) &= z + 2az^2 + \frac{4}{3}az^3 - (1-4a)[\operatorname{Log}(1-z) + z], \\ g_4(z) &= 2az^2 + \frac{4}{3}az^3 + az^4 - (1-4a)[\operatorname{Log}(1-z) + z]. \end{aligned} \tag{3.14}$$

(iv) For $-1/5 \leq a \leq 1/25$, set $f_5 = h_5 - g_5$, where

$$\begin{aligned} h_4(z) &= z + \frac{5}{2}az^2 + \frac{5}{3}az^3 + \frac{5}{4}az^4 - (1-5a)[\operatorname{Log}(1-z) + z], \\ g_4(z) &= \frac{5}{2}az^2 + \frac{5}{3}az^3 + \frac{5}{4}az^4 + az^5 - (1-5a)[\operatorname{Log}(1-z) + z]. \end{aligned} \tag{3.15}$$

Then $J = h_4 + \overline{g_4}$ is harmonic and convex in the direction of the real axis. □

We conclude with an example of harmonic function construction where the coanalytic part of the function is a modification of its analytic part.

Theorem 3.5. *Let the function h given by (1.1) be analytic univalent in Δ and set $g(z) = h(z) - z$. Then the function $F(z) = h(z) + \overline{G(z)} = h(z) + \overline{g(z) + b_b z^p}$ is convex in the direction of the real axis if $0 < b_p \leq 1/p^2$ and $\sum_{k=2}^{\infty} k|a_k| \leq (p-1)/2p$.*

Proof. Note that

$$\begin{aligned} |G'(z)| &= \left| \sum_{k=2}^{\infty} k a_k z^{k-1} + p b_p z^{p-1} \right| \\ &< \sum_{k=2}^{\infty} k|a_k| + \frac{1}{p} < 1 - \sum_{k=2}^{\infty} k|a_k| < |h'(z)| \end{aligned} \tag{3.16}$$

whenever $\sum_{k=2}^{\infty} k|a_k| \leq (p-1)/2p$. Consequently, F is locally univalent and sense-preserving in Δ. Since the image of $|z| = r < 1$ under F is a closed curve, the function F is convex in the direction of the real axis if the imaginary part of F has at most one maximum and one minimum. □

For $z = re^{i\theta}$, we obtain $\Im\{F(z)\} = r\sin\theta - b_p r^p \sin(p\theta) = \Im\{Q(z)\}$, where $Q(z) = z - b_p z^p$. The result now follows because $|b_p| \leq 1/p^2$ implies that Q is convex in Δ.

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