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Research Article

On T-Fuzzy Ideals in Nearrings

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We introduce the notion of fuzzy ideals in nearrings with respect to a t-norm T and investigate some of their properties. Using T-fuzzy ideals, characterizations of Artinian and Noetherian nearrings are established. Some properties of T-fuzzy ideals of the quotient nearrings are also considered.

1. Introduction

Nearrings are one of the generalized structures of rings. Substantial work on nearrings related to group theory and ring theory was studied by Zassenhaus and Wielandt in 1930. World War II interrupted the study of nearrings, but in 1950s, the research of nearring redeveloped by Wielandt, Frohlich, and Blackett. Since then, work in this area has grown and was diversified to include applications to projective geometry, groups with nearring operators, automata theory, formal language theory, nonlinear interpolation theory, optimization theory [1, 2].

The theory of fuzzy sets was first inspired by Zadeh [3]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. There have been wide-ranging applications of the theory of fuzzy sets, from the design of robots and computer simulation to engineering and water resources planning. Rosenfeld [4] introduced the fuzzy sets in the realm of group theory. Since then, many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader framework of the fuzzy setting (e.g., [4–9]). Triangular norms were introduced by Schweizer and Sklar [10, 11] to model the distances in probabilistic metric spaces. In fuzzy sets theory, triangular norm (t-norm) is extensively used to model the logical connective: conjunction (AND). There

are many applications of triangular norms in several fields of mathematics and artificial intelligence [12].

Abou-Zaid [13] introduced the notion of a fuzzy subnearring and studied fuzzy left (right) ideals of a nearring, and gave some properties of fuzzy prime ideals of a nearring. In this paper, we introduce the notion of fuzzy ideals in nearrings with respect to a t-norm T and investigate some of their properties. Using T-fuzzy ideals, characterizations of Artinian and Noetherian nearrings are established. Some properties of T-fuzzy ideals of the quotient nearrings are also considered.

2. Preliminaries

In this section, we review some elementary aspects that are necessary for this paper.

Definition 2.1. An algebra $(R,+,\cdot)$ is said to be a *nearring* if it satisfies the following conditions:

(1) $(R,+)$ is a (not necessarily abelian) group,
(2) $(R,\cdot)$ is a semigroup,
(3) for all $x,y,z \in R$, $x\cdot(y+z) = x\cdot y + x\cdot z$.

Definition 2.2. A subset I of a nearring R is said to subnearring if $(I,+,\cdot)$ is also a nearring.

Proposition 2.3. *A subset I of a nearring R is a subnearring of R if and only if $x-y$, $xy \in I$ for all $x,y \in I$.*

Definition 2.4. A mapping $f : R_1 \to R_2$ is called a (nearring) *homomorphism* if $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$ for all $x,y \in R_1$.

Definition 2.5. An ideal I of nearring $(R,+,\cdot)$ is a subset of R such that

(a) $(I,+)$ is a normal subgroup of $(R,+)$,
(b) $RI \subset I$,
(c) $(r+i)s - rs \in I$ for all $i \in I$ and $r,s \in R$.

Note that I is a left ideal of R if I satisfies (a) and (b), and I is a right ideal of R if I satisfies (a) and (c). If I is both left and right ideal, I is called an ideal of R.

Definition 2.6. A quotient nearring (also called a residue-class nearring) is a nearring that is the quotient of a nearring and one of its ideals I, denoted R/I. If I is an ideal of a nearring R and $a \in R$, then a coset of I is a set of the form $a+I = \{a+s \mid s \in I\}$. The set of all cosets is denoted by R/I.

Theorem 2.7. *If I is an ideal of a nearring R, the set R/I is a nearring under the operations $(a+I)+(b+I) = (a+b)+I$ and $(a+I)\cdot(b+I) = (a\cdot b)+I$.*

Definition 2.8 [14]. A nearring R is said to be left (right) *Artinian* if it satisfies the descending chain condition on left (right) ideals of R. R is said to be *Artinian* if R is both left and right Artinian.

Definition 2.9 [14]. A nearring R is said to be left (right) *Noetherian* if it satisfies the ascending chain condition on left (right) ideals of R. R is said to be *Noetherian* if R is both left and right Noetherian.

Lemma 2.10. *If a nearring R is Artinian, then R is Noetherian.*

Definition 2.11 [3]. A mapping $\mu : X \to [0,1]$, where X is an arbitrary nonempty set and is called a *fuzzy set* in X.

Definition 2.12 [13]. A fuzzy subset μ in a nearring R is said to be a *fuzzy subnearring* of R if it satisfies the following conditions:
(F1) for all $x, y \in R$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
(F2) for all $x, y \in R$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.13 [13]. A fuzzy subnearring μ of R is said to be *fuzzy ideal* if it satisfies the following conditions:
(F3) for all $x, y \in R$, $\mu(y+x-y) \geq \mu(x)$,
(F4) for all $x, y \in R$, $\mu(xy) \geq \mu(y)$,
(F5) for all $x, y, z \in R$, $\mu((x+z)y - xy) \geq \mu(z)$.

Lemma 2.14. *If μ is a fuzzy ideal of R, then $\mu(0) \geq \mu(x)$ for all $x \in R$.*

Definition 2.15 [10]. A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions for all $(x, y, z \in [0,1])$:
(T1) $T(x,1) = x$,
(T2) $T(x,y) = T(y,x)$,
(T3) $T(x, T(y,z)) = T(T(x,y), z)$,
(T4) $T(x,y) \leq T(x,z)$ whenever $y \leq z$.

A simple example of such defined t-norm is a function $T(x,y) = \min(x,y)$. In general case, $T(x,y) \leq \min(x,y)$ and $T(x,0) = 0$ for all $x, y \in [0,1]$.

3. T-fuzzy ideals in nearrings

Definition 3.1. A fuzzy set μ in R is called fuzzy subnearring with respect to a t-norm (shortly, T-fuzzy subnearring) of R if
(TF1) for all $x, y \in R$, $\mu(x-y) \geq T(\mu(x), \mu(y))$,
(TF2) for all $x, y \in R$, $\mu(xy) \geq T(\mu(x), \mu(y))$.

Definition 3.2. A T-fuzzy subnearring μ in R is called *T-fuzzy ideal* of R if:
(TF3) for all $x, y \in R$, $\mu(y+x-y) \geq \mu(x)$,
(TF4) for all $x, y \in R$, $\mu(xy) \geq \mu(y)$,
(TF5) for all $x, y, z \in R$, $\mu((x+z)y - xy) \geq \mu(z)$.

Note that μ is a T-fuzzy left ideal of R if it satisfies (TF1), (TF2), (TF3), and (TF4), and μ is a T-fuzzy right ideal of R if it satisfies (TF1), (TF2), (TF3), and (TF5). μ is called T-fuzzy ideal of R if μ is both left and right T-fuzzy ideal of R.

Example 3.3. Consider a nearring $R = \{a, b, c, d\}$ with the following Cayley's tables:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

We define a fuzzy subset $\mu : R \to [0,1]$ by $\mu(a) > \mu(b) > \mu(d) = \mu(c)$. Let $T_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_m(x, y) = \max(x + y - 1, 0)$ which is a t-norm for all $x, y \in [0,1]$. By routine calculations, it is easy to check that μ is a T_m-fuzzy ideal of R.

The following propositions are obvious.

Proposition 3.4. *A fuzzy set μ in a nearring R is a T-fuzzy ideal of R if and only if the level set*

$$U(\mu; \alpha) = \{x \in R \mid \mu(x) \geq \alpha\} \tag{3.1}$$

is an ideal of R when it is nonempty.

Proposition 3.5. *Every T-fuzzy ideal of a nearring R is a T-fuzzy subnearring of R.*

Converse of Proposition 3.5 may not be true in general as seen in the following example.

Example 3.6. Let $R := \{a, b, c, d\}$ be a set with binary operations as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R, +, \cdot)$ is a nearring. We define a fuzzy subset $\mu : R \to [0,1]$ by $\mu(a) > \mu(b) > \mu(d) = \mu(c)$. Let $T_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_m(x, y) = \max(x + y - 1, 0)$ which is a t-norm for all $x, y \in [0,1]$. By routine computations, it is easy to see that μ is a T_m-fuzzy subnearring of R. It is clear that μ is also left T_m-fuzzy ideal of R, but μ is not T_m-fuzzy right ideal of R since $\mu((c + d)d - cd) = \mu(d) < \mu(b)$.

Definition 3.7. Let R_1 and R_2 be two nearrings and f a function of R_1 into R_2. If ν is a fuzzy set in R_2, then the *image* of μ under f is the fuzzy set in R_1 defined by

$$f(\mu)(x) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

for each $y \in R_2$.

Theorem 3.8. *Let $f : R_1 \to R_2$ be an onto homomorphism of nearrings. If μ is a T-fuzzy ideal in R_1, then $f(\mu)$ is a T-fuzzy ideal in R_2.*

Proof. Let $y_1, y_2 \in R_2$. Then

$$\{x \mid x \in f^{-1}(y_1 - y_2)\} \supseteq \{x_1 - x_2 \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\}, \tag{3.3}$$

and hence

$$\begin{aligned} f(\mu)(y_1 - y_2) &= \sup\{\mu(x) \mid f^{-1}(y_1 - y_2)\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\mu(x_1 - x_2) \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\} \\ &= T(\sup\{\mu(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{\mu(x_2) \mid x_2 \in f^{-1}(y_2)\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)), \end{aligned} \tag{3.4}$$

and since

$$\begin{aligned} &\{x \mid x \in f^{-1}(y_1 y_2)\} \supseteq \{x_1 x_2 \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\}, \\ f(\mu)(y_1 y_2) &= \sup\{\mu(x) \mid f^{-1}(y_1 y_2)\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\mu(x_1 x_2) \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\} \\ &= T(\sup\{\mu(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{\mu(x_2) \mid x_2 \in f^{-1}(y_2)\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned} \tag{3.5}$$

This shows that $f(\mu)$ is a T-fuzzy subnearring in R_2.

Let $y_1, y_2, y_3 \in R_2$. Then

$$\begin{aligned} f(\mu)(y_1 + y_2 - y_1) &= \sup\{\mu(x) \mid f^{-1}(y_1 + y_2 - y_1)\} \\ &\geq \sup\{\mu(x_1 + x_2 - x_1) \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\mu(x_1) \mid x_1 \in f^{-1}(y_1)\} = f(\mu)(y_1), \\ f(\mu)(y_1 y_2) &= \sup\{\mu(x) \mid f^{-1}(y_1 y_2)\} \\ &\geq \sup\{\mu(x_1 x_2) \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\mu(x_2) \mid x_2 \in f^{-1}(y_2)\} = f(\mu)(y_2), \end{aligned}$$

$$\begin{aligned} f(\mu)((y_1+y_2)y_3-y_1y_3) &= \sup\{\mu(x) \mid f^{-1}((y_1+y_2)y_3-y_1y_3)\} \\ &\geq \sup\{\mu((x_1+x_2)x_3-x_1x_3) \mid x_1 \in f^{-1}(y_1), \\ &\qquad x_2 \in f^{-1}(y_2),\ x_3 \in f^{-1}(y_3)\} \\ &\geq \sup\{\mu(x_3) \mid x_3 \in f^{-1}(y_3)\} = f(\mu)(y_3). \end{aligned} \tag{3.6}$$

Hence $f(\mu)$ is a T-fuzzy ideal of nearring in R_2. □

The following proposition is trivial.

Proposition 3.9. *If μ and λ are T-fuzzy ideals of a nearring R, then the function $\mu \wedge \lambda : R \to [0,1]$ defined by*

$$(\mu \wedge \lambda)(x) = T(\mu(x), \lambda(x)) \tag{3.7}$$

for all $x \in R$ is a T-fuzzy ideal of R.

Definition 3.10. A fuzzy ideal μ of a nearring R is said to be *normal* if $\mu(0) = 1$.

Theorem 3.11. *Let μ be a T-fuzzy ideal of a nearring R and let μ^* be a fuzzy set in R defined by $\mu^*(x) = \mu(x) + 1 - \mu(0)$ for all $x \in R$. Then μ^* is a normal T-fuzzy ideal of R containing μ.*

Proof. For any $x, y \in R$,

$$\begin{aligned} \mu^*(x-y) &= \mu(x-y)+1-\mu(0) \geq T(\mu(x),\mu(y))+1-\mu(0) \\ &= T(\mu(x)+1-\mu(0),\mu(y)+1-\mu(0)) = T(\mu^*(x),\mu^*(y)), \\ \mu^*(xy) &= \mu(xy)+1-\mu(0) \geq T(\mu(x),\mu(y))+1-\mu(0) \\ &= T(\mu(x)+1-\mu(0),\mu(y)+1-\mu(0)) = T(\mu^*(x),\mu^*(y)). \end{aligned} \tag{3.8}$$

This shows that μ^* is a T-fuzzy subnearring of R. For any $x, y, z \in R$,

$$\begin{aligned} \mu^*(y+x-y) &= \mu(y+x-y)+1-\mu(0) \geq \mu(x)+1-\mu(0) = \mu^*(x), \\ \mu^*(xy) &= \mu(xy)+1-\mu(0) \geq \mu(y)+1-\mu(0) = \mu^*(y), \\ \mu^*((x+y)z-xz) &= \mu((x+y)z-xz)+1-\mu(0) \geq \mu(z)+1-\mu(0) = \mu^*(z). \end{aligned} \tag{3.9}$$

Hence μ^* is a T-fuzzy ideal of nearring of R. Clearly, $\mu^*(0) = 1$ and $\mu \subset \mu^*$. This ends the proof. □

Theorem 3.12. *If μ is a T-fuzzy ideal of a nearring R, then for all $x \in R$,*

$$\mu(x) = \sup\{t \in [0,1] \mid x \in U(\mu;t)\}. \tag{3.10}$$

Proof. Let $s := \sup\{t \in [0,1] \mid x \in U(\mu;t)\}$, and let $\epsilon > 0$. Then $s - \epsilon < t$ for some $t \in [0,1]$ such that $x \in U(\mu;t)$, and so $s - \epsilon < \mu(x)$. Since ϵ is an arbitrary, it follows that $s \leq \mu(x)$. Now let $\mu(x) = v$, then $x \in U(\mu;v)$ and so $v \in \{t \in [0,1] \mid x \in U(\mu;t)\}$. Thus $\mu(x) = v \leq \sup\{t \in [0,1] \mid x \in U(\mu;t)\} = s$. Hence $\mu(x) = s$. This completes the proof. □

We now consider the converse of Theorem 3.12.

Theorem 3.13. *Let $\{R_w \mid w \in \wedge\}$, where $\wedge \subseteq [0,1]$, be a collection of ideals of a nearring R such that*

(i) $R = \bigcup_{w\in\wedge} R_w$,

(ii) $\alpha > \beta$ *if and only if* $R_\alpha \subset R_\beta$ *for all* $\alpha,\beta \in \wedge$.

Then a fuzzy set μ in R defined by

$$\mu(x) = \sup\{w \in \wedge \mid x \in R_w\} \tag{3.11}$$

is a T-fuzzy ideal of R.

Proof. In view of Proposition 3.4, it is sufficient to show that every nonempty level set $U(\mu;\alpha)$ is an ideal of R. Assume $U(\mu;\alpha) \neq \alpha$ for some $\alpha \in [0;1]$. Then the following cases arise:

(1)

$$\alpha = \sup\{\beta \in \wedge \mid \beta < \alpha\} = \sup\{\beta \in \wedge \mid R_\alpha \subset R_\beta\}, \tag{3.12a}$$

(2)

$$\alpha \neq \sup\{\beta \in \wedge \mid \beta < \alpha\} = \sup\{\beta \in \wedge \mid R_\alpha \subset R_\beta\}. \tag{3.12b}$$

Case (1) implies that

$$x \in U(\mu;\alpha) \Longleftrightarrow x \in R_w \quad \forall w < \alpha \Longleftrightarrow x \in \bigcap_{w<\alpha} R_w. \tag{3.13}$$

Hence $U(\mu;\alpha) = \bigcap_{w<\alpha} R_w$, which is an ideal of R.

For case (2), there exists $\epsilon > 0$ such that $(\alpha - \epsilon, \alpha) \cap \wedge = \varnothing$. We claim that in this case $U(\mu;\alpha) = \bigcup_{\beta\geq\alpha} R_\beta$. Indeed, if $x \in \bigcup_{\beta\geq\alpha} R_\beta$, then $x \in R_\beta$ for some $\beta \geq \alpha$, which gives $\mu(x) \geq \beta \geq \alpha$. Thus $x \in U(\mu;\alpha)$, that is, $\bigcup_{\beta\geq\alpha} R_\beta \subseteq U(\mu;\alpha)$. Conversely, $x \notin \bigcup_{\beta\geq\alpha} R_\beta$, then $x \notin R_\beta$ for all $\beta \geq \alpha$, which implies that $x \notin R_\beta$ for all $\beta > \alpha - \epsilon$, that is, if $x \in R_\beta$, then $\beta \leq \alpha - \epsilon$. Thus $\mu(x) \leq \alpha - \epsilon$. So $x \notin U(\mu;\alpha)$. Thus $U(\mu;\alpha) \subseteq \bigcup_{\beta\geq\alpha} R_\beta$. Hence $U(\mu;\alpha) = \bigcup_{\beta\geq\alpha} R_\beta$, which is an ideal of R. This completes the proof. □

4. Characterizations of Artinian and Noetherian nearrings

Lemma 4.1. *Let μ be a T-fuzzy ideal of a nearring R and let $s,t \in \mathrm{Im}(\mu)$. Then $U(\mu;s) = U(\mu;t) \Leftrightarrow s = t$.*

Proof. Routine. □

Theorem 4.2. *Every T-fuzzy ideal of a nearring R has a finite number of values if and only if a nearring R is Artinian.*

Proof. Suppose that every T-fuzzy ideal of a nearring R has a finite number of values and R is not Artinian. Then there exists strictly descending chain

$$R = U_0 \supset U_1 \supset U_2 \supset \cdots \tag{4.1}$$

of ideals of R. Define a fuzzy set μ in R by μ being a fuzzy set in R defined by

$$\mu(x) := \begin{cases} \dfrac{n}{n+1} & \text{if } x \in U_n \setminus U_{n+1},\ n = 0,1,2,\ldots, \\ 1 & \text{if } x \in \displaystyle\bigcap_{n=0}^{\infty} U_n. \end{cases} \tag{4.2}$$

Let $x, y \in R$, then $x - y$, $xy \in U_n \setminus U_{n+1}$ for some $n = 0,1,2,\ldots$, and either $x \notin U_{n+1}$ or $y \notin U_{n+1}$. So for definiteness, let $y \in U_n \setminus U_{n+1}$ for $k \leq n$. It follows that

$$\begin{aligned} \mu(x-y) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq T(\mu(x),\mu(y)), \\ \mu(xy) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq T(\mu(x),\mu(y)). \end{aligned} \tag{4.3}$$

In (TF3)–(TF5) the process of verification is analogous. Thus μ is T-fuzzy ideal of R and μ has infinite number of different values. This contradiction proves that R is Artinian nearring.

Conversely, let a nearring R be an Artinian and let μ be a T-fuzzy ideal of R. Suppose that $\mathrm{Im}(\mu)$ is an infinite. Note that every subset of $[0,1]$ contains either a strictly increasing or strictly decreasing sequence.

Let $t_1 < t_2 < t_3 < \cdots$ be a strictly increasing sequence in $\mathrm{Im}(\mu)$. Then

$$U(\mu;t_1) \supset U(\mu;t_2) \supset U(\mu;t_3) \supset \cdots \tag{4.4}$$

is strictly descending chain of ideals of R. Since R is Artinian, there exists a natural number i such that $U(\mu;t_i) = U(\mu;t_{i+n})$ for all $n \geq 1$. Since $t_i \in \mathrm{Im}(\mu)$ for all i, it allow that from Lemma 4.1 that $t_i = t_{i+n}$ for all $n \geq 1$. This is a contradiction since t_i are different.

On the other hand, if $t_1 > t_2 > t_3 > \cdots$ is a strictly decreasing sequence in $\mathrm{Im}(\mu)$, then

$$U(\mu;t_1) \subset U(\mu;t_2) \subset U(\mu;t_3) \subset \cdots \tag{4.5}$$

is an ascending chain of ideals of R. Since R is Noetherian by Lemma 2.10, there exists a natural number j such that $U(\mu;t_j) = U(\mu;t_{j+n})$ for all $n \geq 1$. Since $t_j \in \mathrm{Im}(\mu)$ for all j, by Lemma 4.1 $t_j = t_{j+n}$ for all $n \geq 1$, which is again contradiction since t_j are distinct. Hence $\mathrm{Im}(\mu)$ is finite. □

Theorem 4.3. *Let a nearring R be Artinian and let μ be a T-fuzzy ideal of R. Then $|U_\mu| = |\mathrm{Im}(\mu)|$, where U_μ denote a family of all level ideals of R with respect to μ.*

Proof. Since R is Artinian, it follows from Theorem 4.2 that $\mathrm{Im}(\mu)$ is finite. Let $\mathrm{Im}(\mu) = \{t_1, t_2, \ldots, t_n\}$ where $t_1 < t_2 < \cdots < t_n$. It is sufficient to show that U_μ consists of level ideals of R with respect to μ for all $t_i \in \mathrm{Im}(\mu)$, that is, $U_\mu = \{U(\mu; t_i) \mid 1 \le i \le n\}$. Clearly, $U(\mu; t_i) \in U_\mu$ for all $t_i \in \mathrm{Im}(\mu)$. Let $0 \le t \le \mu(0)$ and let $U(\mu; t)$ be a level ideal of R with respect to μ. Assume that $t \notin \mathrm{Im}(\mu)$. If $t < t_1$, then clearly $U(\mu; t) = U(\mu; t_1)$, and so let $t_i < t < t_{i+1}$ for some i. Then $U(\mu; t_{i+1}) \subseteq U(\mu; t)$. Let $x \in U(\mu; t)$. Then $\mu(x) > t$ since $t \notin \mathrm{Im}(\mu)$, and so $\mu(x) \ge U(\mu; t_{i+1})$. Thus $U(\mu; t) = U(\mu; t_{i+1})$, which shows that U_μ consists of level ideals of R with respect to μ for all $t_i \in \mathrm{Im}(\mu)$. Hence $|U_\mu| = |\mathrm{Im}(\mu)|$. □

Theorem 4.4. *Let a nearring R be Artinian and let μ and ν be a T-fuzzy ideals of R. Then $|U_\mu| = |U_\nu|$ and $\mathrm{Im}(\mu)$=$\mathrm{Im}(\nu)$ if and only if μ= ν.*

Proof. If $\mu = \nu$, then clearly $U_\mu = U_\nu$ and $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$. Now suppose that $U_\mu = U_\nu$ and $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$. By Theorems 4.2 and 4.3, $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$ are finite and $|U_\mu| = |\mathrm{Im}(\mu)|$ and $|U_\nu| = |\mathrm{Im}(\nu)|$. Let

$$\mathrm{Im}(\mu) = \{t_1, t_2, \ldots, t_n\}, \qquad \mathrm{Im}(\nu) = \{s_1, s_2, \ldots, s_n\}, \tag{4.6}$$

where $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$. Clearly, $t_i = s_i$ for all i. We now prove that $U(\mu; t_i) = U(\nu; t_i)$ for all i. Note that $U(\mu; t_1) = R = U(\nu; t_1)$. Consider $U(\mu; t_2)$, $U(\nu; t_2)$, suppose that $U(\mu; t_2) \neq U(\nu; t_2)$. Then $U(\mu; t_2) = U(\nu; t_k)$ for some $k > 2$ and $U(\nu; t_2) = U(\mu; t_j)$ for some $j > 2$. If there exist $x \in R$ such that $\mu(x) = t_2$, then

$$\mu(x) < t_j \quad \forall j > 2. \tag{4.7}$$

Since $U(\mu; t_2) = U(\nu; t_k)$, $x \in U(\nu; t_k)$. Then $\nu(x) \ge t_k > t_2$, $k > 2$. Thus $x \in U(\nu; t_2)$. Since $U(\nu; t_2) = U(\mu; t_j)$, $x \in U(\mu; t_j)$. Thus

$$\mu(x) \ge t_j \quad \text{for some } j > 2. \tag{4.8}$$

Clearly, (4.7) and (4.8) contradict each other. Hence $U(\mu; t_2) = U(\nu; t_2)$. Continuing in this way, we get $U(\mu; t_i) = U(\nu; t_i)$ for all i.

Now let $x \in R$. Suppose that $\mu(x) = t_i$ for some i. Then $x \notin U(\mu; t_j)$ for all $i+1 \le j \le n$. This implies that $x \notin U(\nu; t_j)$ for all $i+1 \le j \le n$. But then $\nu(x) < t_j$ for all $i+1 \le j \le n$. Suppose that $\nu(x) = t_m$ for some $i \le m \le i$. If $i \neq m$, then $x \in U(\nu; t_i)$. On the other hand, since $\mu(x) = t_i$, $x \in U(\mu; t_i) = U(\nu; t_i)$. Thus we have a contradiction. Hence $i = m$ and $\mu(x) = t_i = \nu(x)$. Consequently, $\mu = \nu$. □

Theorem 4.5. *A nearring R is Noetherian if and only if the set of values of any T-fuzzy ideal of R is a well-ordered subset of* $[0, 1]$.

Proof. Suppose that μ is a T-fuzzy ideal of R whose set of values is not a well-ordered subset of $[0,1]$. Then there exists a strictly decreasing sequence $\{\lambda_n\}$ such that $\mu(x_n) = \lambda_n$. Denote by U_n the set $\{x \in R \mid \mu(x) \geq \lambda_n\}$. Then

$$U_1 \subset U_2 \subset U_3 \cdots \tag{4.9}$$

is a strictly ascending chain of ideals of R, which contradicts that R is Noetherian.

Conversely, assume that the set of values of any T-fuzzy ideal of R is a well-ordered subset of $[0,1]$ and R is not Noetherian nearring. Then there exists a strictly ascending chain

$$U_1 \subset U_2 \subset U_3 \cdots \tag{$*$}$$

of ideals of R. Define a fuzzy set μ in R by

$$\mu(x) := \begin{cases} \dfrac{1}{k} & \text{for } x \in U_k \backslash U_{k-1}, \\ 0 & \text{for } x \notin \displaystyle\bigcup_{k=1}^{\infty} U_k. \end{cases} \tag{4.10}$$

It can be easily seen that μ is a T-fuzzy ideal of R. Since the chain $(*)$ is not terminating, μ has a strictly descending sequence of values, contradicting that the value set of any T-fuzzy ideal is well ordered. Consequently, R is Noetherian. □

Proposition 4.6. *Let $R = \{\lambda_n \in (0,1) \mid n \in \mathbb{N}\} \cup \{0\}$, where $\lambda_i > \lambda_j$ whenever $i < j$. Let $\{U_n \mid n \in \mathbb{N}\}$ be a family of ideals of nearring R such that $U_1 \subset U_2 \subset U_3 \subset \cdots$. Then a fuzzy set μ in R defined by*

$$\mu(x) := \begin{cases} \lambda_1 & \text{if } x \in U_1, \\ \lambda_n & \text{if } x \in U_n \setminus U_{n-1},\ n = 2,3,\ldots, \\ 0 & \text{if } x \in R \setminus \displaystyle\bigcup_{n=1}^{\infty} U_n \end{cases} \tag{4.11}$$

is a T-fuzzy ideal of R.

Proof. Straightforward. □

Theorem 4.7. *Let $R = \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\} \cup \{0\}$ where $\{\lambda_n\}$ is a fixed sequence, strictly decreasing to 0 and $0 < \lambda_n < 1$. Then a nearring R is Noetherian if and only if for each T-fuzzy ideal μ of R, $\mathrm{Im}(\mu) \subset R \Rightarrow \exists n_0 \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subset \{\lambda_1, \lambda_2, \ldots, \lambda_{n_0}\} \cup \{0\}$.*

Proof. If R is Noetherian, then $\mathrm{Im}(\mu)$ is a well-ordered subset of $[0,1]$ by Theorem 4.5 and so the condition is necessary by noticing that a set is well ordered if and only if it does not contain any infinite descending sequence.

Conversely, let R be not Noetherian. Then there exists a strictly ascending chain of ideals of R:

$$U_1 \subset U_2 \subset U_3 \cdots. \tag{4.12}$$

Define a fuzzy set μ in R by

$$\mu(x) := \begin{cases} \lambda_1 & \text{if } x \in U_1, \\ \lambda_n & \text{if } x \in U_n \setminus U_{n-1},\ n = 2,3,\ldots, \\ 0 & \text{if } x \in R \setminus \bigcup_{n=1}^{\infty} U_n. \end{cases} \tag{4.13}$$

Then, by Proposition 4.6, μ is a T-fuzzy ideal of R. This contradicts our assumption. Hence R is Noetherian. □

Theorem 4.8. *If R is a Noetherian nearring, then every T-fuzzy ideal of R is finite valued.*

Proof. Let $\mu : R \to [0,1]$ be a T-fuzzy ideal of R which is not finite valued. Then there exists an infinite sequence of distinct numbers $\mu(0) = t_1 > t_2 > \cdots > t_n > \cdots$, where $t_i = \mu(x_i)$ for some $x_i \in R$. This sequence induces an infinite sequence of distinct ideals of R:

$$U(\mu;t_1) \subset U(\mu;t_2) \subset \cdots \subset U(\mu;t_n) \subset \cdots, \tag{4.14}$$

which is a contradiction. This completes the proof. □

5. The quotient nearrings via fuzzy ideals

Theorem 5.1. *Let I be an ideal of a nearring R. If μ is a T-fuzzy ideal of R, then the fuzzy set $\overline{\mu}$ of R/I defined by*

$$\overline{\mu}(a+I) = \sup_{x\in I} \mu(a+x) \tag{5.1}$$

is a T-fuzzy ideal of the quotient nearring R/I of R with respect to I.

Proof. Let $a,b \in R$ be such that $a+I = b+I$. Then $b = a+y$ for some $y \in I$. Thus

$$\overline{\mu}(b+I) = \sup_{x\in I} \mu(b+x) = \sup_{x\in I} \mu(a+y+x) = \sup_{x+y=z\in I} (a+z) = \overline{\mu}(a+I). \tag{5.2}$$

This shows that $\overline{\mu}$ is well-defined. Let $x+I$, $y+I \in R/I$, then

$$\begin{aligned}
\overline{\mu}((x+I)-(y+I)) &= \overline{\mu}((x-y)+I) = \sup_{z\in I}\mu((x-y)+z) \\
&= \sup_{z=u-v\in I}\mu((x-y)+(u-v)) \\
&\geq \sup_{u,v\in I} T\{\mu(x+u),\mu(y+v)\} \\
&= T\left\{\sup_{u\in I}\mu(x+u),\sup_{v\in I}\mu(y+v)\right\} \\
&= T\{\overline{\mu}(x+I),\overline{\mu}(y+I)\}, \\
\overline{\mu}((x+I)(y+I)) &= \overline{\mu}(xy+I) = \sup_{z\in I}\mu(xy+z) \\
&= \sup_{z=uv\in I}\mu(xy+uv) \\
&\geq \sup_{u,v\in I} T\{\mu(x+u),\mu(y+v)\} \\
&= T\left\{\sup_{u\in I}\mu(x+u),\sup_{v\in I}\mu(y+v)\right\} \\
&= T\{\overline{\mu}(x+I),\overline{\mu}(y+I)\}.
\end{aligned} \tag{5.3}$$

In (TF3)–(TF5) the process of verification is analogous. Thus $\overline{\mu}$ is a T-fuzzy ideal of R/I. □

Theorem 5.2. *Let I be an ideal of a nearring R. Then there is a one-to-one correspondence between the set of T-fuzzy ideals μ of R such that $\mu(0)=\mu(s)$ for all $s\in I$ and the set of all T-fuzzy ideals $\overline{\mu}$ of R/I.*

Proof. Let μ be a T-fuzzy ideal of R. Using Theorem 5.1, we prove that $\overline{\mu}$ defined by

$$\overline{\mu}(a+I) = \sup_{x\in I}\mu(a+x) \tag{5.4}$$

is a T-fuzzy ideal of R/I. Since $\mu(0)=\mu(s)$ for all $s\in I$,

$$\mu(a+s) \geq T(\mu(a),\mu(s)) = \mu(a). \tag{5.5}$$

Again,

$$\mu(a) = \mu(a+s-s) \geq T(\mu(a+s),\mu(s)) = \mu(a+s). \tag{5.6}$$

Thus $\mu(a+s)=\mu(a)$ for all $s\in I$, that is, $\overline{\mu}(a+I)=\mu(a)$. Hence the correspondence $\mu\mapsto\overline{\mu}$ is one-to-one. Let $\overline{\mu}$ be a T-fuzzy ideal of R/I and define fuzzy set μ in R by $\mu(a)=\overline{\mu}(a+I)$ for all $a\in I$.

For $x, y \in R$, we have

$$\begin{aligned}
\mu(x-y) &= \overline{\mu}((x-y)+I) = \overline{\mu}((x+I)-(y+I)) \\
&\geq T\{\overline{\mu}(x+I), \overline{\mu}(y+I)\} = T\{\mu(x), \mu(y)\}, \\
\mu(xy) &= \overline{\mu}((xy)+I) = \overline{\mu}((x+I)\cdot(y+I)) \\
&\geq T\{\overline{\mu}(x+I), \overline{\mu}(y+I)\} = T\{\mu(x), \mu(y)\}.
\end{aligned} \tag{5.7}$$

In (TF3)–(TF5) the process of verification is analogous. Thus μ is a T-fuzzy ideal of R. Note that $\mu(z) = \overline{\mu}(z+I) = \overline{\mu}(I)$ for all $z \in I$, which shows that $\mu(z) = \mu(0)$ for all $z \in I$. This ends the proof. □

Theorem 5.3. *Let T be a t-norm and I an ideal of a nearring R. Then for all $\lambda \in [0,1]$, there exists a T-fuzzy ideal μ of R such that $\mu(0) = \lambda$ and $U(\mu; I) = I$.*

Proof. Let $\mu : R \to [0,1]$ be a fuzzy subset of R defined by

$$\mu(x) := \begin{cases} \lambda & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \tag{5.8}$$

where λ is fixed number in $[0,1]$. Then clearly, $U(\mu;\lambda) = I$. Let $x, y \in R$, then a routine calculation shows that μ is a T-fuzzy ideal of R. □

Theorem 5.4. *Let μ be a T-fuzzy ideal of a nearring R and let $\mu(0) = \lambda$. Then the fuzzy subset μ^* of the quotient nearring $R/U(\mu;\lambda)$ defined by $\mu^*(x + U(\mu;\lambda)) = \mu(x)$ for all $x \in R$ is a T-fuzzy ideal of $R/U(\mu;\lambda)$.*

Proof. Straightforward. □

Theorem 5.5. *Let I be an ideal of a nearring R and ϕ T-fuzzy ideal of R/I such that $\phi(x+I) = \phi(x)$ only if $x \in I$. Then there exists a T -fuzzy ideal of R such that $U(\mu;\lambda) = I$, $\lambda = \mu(0)$, and $\phi = \mu^*$.*

Proof. Define a T-fuzzy ideal μ of R by $\mu(x) = \phi(x+I)$ for all $x \in R$. It is easy to see that μ is T-fuzzy ideal of R. Next, we prove that $U(\mu;\lambda) = I$. Let $x \in U(\mu;\lambda)$,

$$\Longleftrightarrow \mu(x) = \mu(0) \Longleftrightarrow \phi(x+I) = \phi(I) \Longleftrightarrow x \in I. \tag{5.9}$$

Hence $U(\mu;\lambda) = I$. Finally, we prove that $\mu^* = \phi$,

$$\text{Since } \mu^*(x+I) = \mu^*(x + U(\mu;\lambda)) = \mu(x) = \phi(x+I). \tag{5.10}$$

Hence $\mu^* = \phi$. This completes the proof. □

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