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Research Article

On a Class of Integrals Involving a Bessel Function Times Gegenbauer Polynomials

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We provide information and explicit formulae for a class of integrals involving Bessel functions and Gegenbauer polynomials. We present a simple proof of an old formula of Gegenbauer. Some interesting special cases and applications of this result are obtained. In particular, we give a short proof of a recent result of A. A. R. Neves et al. regarding the analytical evaluation of an integral of a Bessel function times associated Legendre functions. These integrals arise in problems of vector diffraction in electromagnetic theory.

1. Introduction

In the recent article [1], Neves et al. gave an analytical evaluation of the integral

$$I_n^m := \int_0^\pi \sin\theta \exp(iR\cos\alpha\cos\theta) P_n^m(\cos\theta) J_m(R\sin\alpha\sin\theta)\, d\theta, \tag{1.1}$$

where $P_n^m(\cos\theta)$ is the associated Legendre function and $J_m(z)$ is the Bessel function of the first kind and order m.

The authors of [1] encountered this integral in their work [2] dealing with the calculation of the optical force of the optical tweezers in a complete electromagnetic treatment for any beam shape focused on an arbitrary position. The integral (1.1) appears also in fields related to vector diffraction theory where computationally intensive methods or approximations are employed.

As the authors of [1] pointed out, an explicit formula for the integral (1.1) would be a useful result, making unnecessary any numerical approximations of it. More specifically,

Neves et al. showed in [1] that for all integers n, m such that $n \geq 0$ and $-n \leq m \leq n$, one has

$$I_n^m = 2i^{n-m} P_n^m(\cos\alpha) j_n(R), \tag{1.2}$$

where $j_n(R)$ is the spherical Bessel function of order n, that is,

$$j_n(R) = \sqrt{\frac{\pi}{2R}} J_{n+1/2}(R). \tag{1.3}$$

The authors mention in [1] that the integral (1.1) has not been reported in a closed form and it is not shown in any integral tables. In [3, page 379, formula (1)], however, the following closely related formula is given:

$$\begin{aligned} &\int_0^\pi \exp(iR\cos\alpha\cos\theta) J_{\nu-1/2}(R\sin\alpha\sin\theta) C_n^\nu(\cos\theta)\sin^{\nu+1/2}\theta\, d\theta \\ &\quad = \left(\frac{2\pi}{R}\right)^{1/2} i^n \sin^{\nu-1/2}\alpha C_n^\nu(\cos\alpha) J_{\nu+n}(R), \end{aligned} \tag{1.4}$$

where $C_n^\nu(x)$ is the Gegenbauer (or ultraspherical) polynomial of degree n and order ν defined by the generating function

$$\frac{1}{(1-2xt+t^2)^\nu} = \sum_{k=0}^{\infty} C_k^\nu(x) t^k. \tag{1.5}$$

The formula (1.4) is due to Gegenbauer and holds for all real numbers ν such that $\nu > -1/2$.

An equivalent form of (1.4) can also be found in the integral tables [4, pages 838-839, formula 7.333 (1) and (2)].

In this note we show that (1.2) follows easily from (1.4) using properties of the associated Legendre functions.

A proof of (1.4) is given in [3, pages 378-379], using a method which is based on integration over a unit sphere. Since formula (1.4) is important in applications in many different fields in physics, especially for those that require partial wave decomposition, we give in the next section a new proof of (1.4) which is simpler than the one given in [3]. We will also present some other consequences of (1.4).

2. Proofs and additional comments

We first show how to obtain (1.2) from (1.4). Suppose first that m, n are nonnegative integers such that $m \leq n$. It is well known that the relation between associated Legendre functions $P_n^m(\cos\theta)$ and Gegenbauer polynomials is given by the formula

$$P_n^m(\cos\theta) = (-1)^m \frac{(2m)!}{2^m m!} \sin^m\theta C_{n-m}^{m+1/2}(\cos\theta), \tag{2.1}$$

see for example [4, page 1052, formula 8.936 (2)]. Using the relation (2.1) in (1.1) and applying (1.4) for $\nu = m + 1/2 \geq 1/2$, we immediately obtain (1.2). When the integer m

is negative such that $-n \le m < 0$, we cannot apply (1.4) as above, because the condition $\nu = m + 1/2 > -1/2$ is not fulfilled. This case can be handled using the formulae

$$J_m(z) = (-1)^m J_{-m}(z), \tag{2.2}$$

see [3, page 15, formula (2)], and

$$P_n^m(x) = (-1)^m \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} P_n^{-m}(x), \quad -1 \le x \le 1, \tag{2.3}$$

see [4, page 1025, formula 8.752 (2)]. It follows from (2.2) and (2.3) that

$$I_n^m = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} I_n^{-m}. \tag{2.4}$$

Since $n \ge -m > 0$, applying (1.2) established in the previous case and using (2.3) and (2.4), we get

$$I_n^m = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} 2i^{n+m} P_n^{-m}(\cos\alpha) j_n(R) = 2i^{n-m} P_n^m(\cos\alpha) j_n(R), \tag{2.5}$$

which completes the proof of (1.2).

We next give a simple proof of (1.4). Let $\nu > -1/2$. The Gegenbauer polynomials $C_n^\nu(\cos\theta)$ satisfy the following orthogonality property:

$$\begin{aligned} &\int_0^\pi C_n^\nu(\cos\theta) C_m^\nu(\cos\theta) \sin^{2\nu}\theta \, d\theta \\ &\quad = \begin{cases} 0, & n \ne m, \\ \dfrac{\sqrt{\pi}\Gamma(2\nu+n)\Gamma(\nu+1/2)}{(\nu+n)n!\Gamma(2\nu)\Gamma(\nu)} = \dfrac{\sqrt{\pi}\Gamma(\nu+1/2)}{(\nu+n)\Gamma(\nu)} C_n^\nu(1), & n = m. \end{cases} \end{aligned} \tag{2.6}$$

We will also use the formula

$$\begin{aligned} &\frac{J_{\nu-1/2}(R\sin\alpha\sin\theta)}{(R\sin\alpha\sin\theta)^{\nu-1/2}} \exp(iR\cos\alpha\cos\theta) \\ &\quad = \sqrt{2}\frac{\Gamma(\nu)}{\Gamma(\nu+1/2)} \sum_{k=0}^\infty (k+\nu) i^k \frac{J_{k+\nu}(R) C_k^\nu(\cos\theta) C_k^\nu(\cos\alpha)}{R^\nu C_k^\nu(1)}, \end{aligned} \tag{2.7}$$

which holds for all ν, see [4, page 1053, formula 8.936 (4)]. A proof of formula (2.7) is given in [3, pages 369-370]. It should be noted that the series above is absolutely convergent. This follows easily using the standard estimate

$$|J_\nu(R)| \le \frac{R^\nu}{\Gamma(\nu+1)2^\nu}, \quad R > 0, \ \nu > -\frac{1}{2}, \tag{2.8}$$

(cf. [3, page 49, formula (1)]), and [5, Theorem 7.33.1] concerning the maximum of Gegenbauer polynomials on $[-1,1]$. Therefore, the series in (2.7) can be termwise integrated. Then applying the dominated convergence theorem and the orthogonality relation (2.6), we obtain

$$\begin{aligned}
&\int_0^\pi \frac{J_{\nu-1/2}(R\sin\alpha\sin\theta)}{(R\sin\alpha\sin\theta)^{\nu-1/2}}\exp(iR\cos\alpha\cos\theta)C_n^\nu(\cos\theta)\sin^{2\nu}\theta\,d\theta \\
&\quad = \sqrt{2}\frac{\Gamma(\nu)}{\Gamma(\nu+1/2)}\int_0^\pi\left[\sum_{k=0}^{\infty}(k+\nu)i^k\frac{J_{k+\nu}(R)C_k^\nu(\cos\theta)C_k^\nu(\cos\alpha)}{R^\nu C_k^\nu(1)}\right]C_n^\nu(\cos\theta)\sin^{2\nu}\theta\,d\theta \\
&\quad = \sqrt{2\pi}i^n\frac{J_{n+\nu}(R)C_n^\nu(\cos\alpha)}{R^\nu},
\end{aligned} \tag{2.9}$$

whence (1.4) follows at once.

It is a further confirmation of the importance of formula (1.4) that some other results are obtained by it as special cases. Indeed, taking the limiting case of (1.4) when $\alpha \to 0$ using the dominated convergence theorem and the fact that

$$\lim_{x\to 0} x^{-\lambda}J_\lambda(x) = \frac{1}{2^\lambda\Gamma(\lambda+1)}, \quad \text{for } \lambda > -1, \tag{2.10}$$

we obtain

$$J_{\nu+n}(R) = \frac{(-i)^n}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{R}{2}\right)^\nu\int_0^\pi \exp(iR\cos\theta)\frac{C_n^\nu(\cos\theta)}{C_n^\nu(1)}\sin^{2\nu}\theta\,d\theta. \tag{2.11}$$

Formula (2.11) is Gegenbauer's generalization of the Poisson integral representation of Bessel functions. For $n = 0$, it reduces to Poisson's formula

$$J_\nu(R) = \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{R}{2}\right)^\nu\int_0^\pi \exp(iR\cos\theta)\sin^{2\nu}\theta\,d\theta, \quad \text{when } \nu > -\frac{1}{2}, \tag{2.12}$$

see [4, page 48, formula (6)].

Since

$$\lim_{\nu\to 0}\frac{C_n^\nu(\cos\theta)}{C_n^\nu(1)} = \cos n\theta, \tag{2.13}$$

taking the limit in (2.11) as $\nu \to 0$ applying once more the dominated convergence theorem, we deduce that for all nonnegative integers n, we have

$$\begin{aligned}
i^nJ_n(R) &= \frac{1}{\pi}\int_0^\pi \exp(iR\cos\theta)\cos n\theta\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\exp(iR\cos\theta)\exp(in\theta)d\theta.
\end{aligned} \tag{2.14}$$

In view of (2.2), these equalities hold also for all negative integers n, hence (2.14) gives all the Fourier coefficients of the function $\exp(iR\cos\theta)$.

Replacing θ with $\pi/2-\theta$ in (2.14) and using the 2π-periodicity of the integrand, we get Bessel's formula

$$J_n(R)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-R\sin\theta)d\theta, \tag{2.15}$$

see [3, page 20, formula (2)].

Finally, we note that the Gouesbet and Lock result [6] for the integral

$$\int_0^{\pi}\sin^{|m|+1}\theta\exp(\pm iR\cos\theta)P_n^{|m|}(\cos\theta)d\theta=2(\pm i)^{n+|m|}\frac{(n+|m|)!}{(n-|m|)!}\frac{j_n(R)}{R^{|m|}}, \tag{2.16}$$

where n, m are integers such that $|m|\le n$, can be derived from (2.11) in exactly the same way as (1.2) is obtained from (1.4).

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