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Research Article

Operator Representation of Fermi-Dirac and Bose-Einstein Integral Functions with Applications

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Fermi-Dirac and Bose-Einstein functions arise as quantum statistical distributions. The Riemann zeta function and its extension, the polylogarithm function, arise in the theory of numbers. Though it might not have been expected, these two sets of functions belong to a wider class of functions whose members have operator representations. In particular, we show that the Fermi-Dirac and Bose-Einstein integral functions are expressible as operator representations in terms of themselves. Simpler derivations of previously known results of these functions are obtained by their operator representations.

1. Introduction

The study of analytic functions is very useful for the application of mathematics to various physical and engineering problems and for the development of a further understanding of mathematics itself. In particular, the Riemann zeta function [1, page 1]

$$\zeta(s) := \sum_{n=0}^{\infty} \frac{1}{n^s} \quad (s = \sigma + it,\ \sigma > 1), \tag{1.1}$$

has played an important role in number theory. There have been several generalizations of the zeta function. Of special interest for our purposes is the polylogarithm function

$$\phi(x,s) = Li_s(x) := F(x,s) := \sum_{n=0}^{\infty} \frac{x^n}{n^s}, \tag{1.2}$$

which extends the zeta function as

$$\phi(1,s) = Li_s(1) = F(x,1) = \zeta(s). \tag{1.3}$$

It has been studied extensively by several authors including Lambert, Legendre, Abel, Kummer, Appell, Lerch, Lindelöf, Wirtinger, Jonquière, Truesdell, and others. It is related to the Fermi-Dirac and Bose-Einstein integral functions which in turn come from the Fermi-Dirac and Bose-Einstein statistics for the quantum description of collections of particles of spins $(n+1/2)\hbar$ and $n\hbar$, respectively. For the asymptotic expansions and other properties of these functions, we refer to the works in [2–10].

We present a series representation of a class of functions and deduce the well-known series representation and operator forms of the Fermi-Dirac (and Bose-Einstein) integral and other related functions. The present formulation helps us to find an alternate proof of the Euler formula for the closed-form representation of the zeta function at even integral values. Lindelöf proved the expansion [11, equation (15), page 30]

$$\phi(x,s) = Li_s(x) = \Gamma(1-s)(-\log x)^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n) \frac{(\log(x))^n}{n!} \quad (|\log x| < 2\pi), \tag{1.4}$$

which is useful for numerical evaluation of the function. The function (1.2) is related to the Fermi-Dirac integral function [4, page 30]

$$F_p(x) := \frac{1}{\Gamma(p+1)} \int_0^{\infty} \frac{t^p}{e^{t-x}+1} dt \quad (p > -1), \tag{1.5}$$

and the Bose-Einstein integral function [4, page 53]

$$B_p(x) := \frac{1}{\Gamma(p+1)} \int_0^{\infty} \frac{t^p}{e^{t-x}-1} dt \quad (p > 0), \tag{1.6}$$

as we have

$$F_{p-1}(x) = -\phi(-e^x, p) = -Li_p(-e^x), \tag{1.7}$$

$$B_{p-1}(x) = \phi(e^x, p) = Li_p(e^x). \tag{1.8}$$

Putting $p = 0$ in (1.5), we find (see also [4, page 20])

$$F_0(x) = x + \ln(1+e^{-x}). \tag{1.9}$$

Note that the Fermi-Dirac and Bose-Einstein integral functions are also related by the duplication formula

$$F_p(x) = B_p(x) - 2^{-p} B_p(2x), \tag{1.10}$$

which is useful in translating the properties of these functions.

2. The Mellin and Weyl transform representations

The Mellin transform of a function $\varphi(t)$ $(0 \le t < \infty)$, if it exists, is defined by (see [12, page 79])

$$\Phi_M(s) := M[\varphi;s] := \int_0^\infty t^{s-1}\varphi(t)dt \quad (s = \sigma + it). \tag{2.1}$$

The inversion formula for the Mellin transform is given by [12, page 80]

$$\varphi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi_M(z)t^{-z}dz. \tag{2.2}$$

If $\varphi \in L^1_{\text{loc}}[0,\infty)$ is such that $\varphi(t) = O(t^{-\sigma_1})$, $t \to 0^+$ and $\varphi(t) = O(t^{-\sigma_2})$, $t \to \infty$, the integral $\Phi_M(s)$ in (2.1) defines a function in the strip $\sigma_1 < \sigma < \sigma_2$. Moreover, if the function $\varphi(t)$ is continuous in $[0,\infty)$ and has rapid decay at infinity, the Mellin transform (2.1) will converge absolutely for $\sigma > 0$. In particular, if the integral (2.1) converges uniformly and absolutely in the strip $\sigma_1 < \sigma < \sigma_2$, the function $\Phi_M(s)$ is analytic in the interior of the strip $\sigma_1 < \sigma < \sigma_2$ [12, page 80].

The Weyl transform of a function $\varphi(t)$ $(0 \le t < \infty)$, if it exists, is defined by [6, page 201]

$$\Phi(s;x) := W^{-s}[\varphi(t)](x) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\varphi(t+x)dt \quad (\sigma > 0,\ x > 0). \tag{2.3}$$

We define $\Phi(0;x) \doteq \varphi(x)$ and

$$\Phi(-s;x) := (-1)^n \frac{d^n}{dx^n}[\Phi(s;x)], \tag{2.4}$$

where n is the smallest integer greater than σ. Then, we have the representation

$$\Phi(-n;x) := (-1)^n \frac{d^n}{dx^n}[\Phi(0;x)] = (-1)^n \frac{d^n}{dx^n}[\varphi(x)] \quad (n = 0,1,2,\dots). \tag{2.5}$$

Since [6, page 243] $W^{-\alpha}W^{-\beta} = W^{-\alpha-\beta} = W^{-\beta}W^{-\alpha}$, we have

$$\Phi(\alpha+\beta;x) = W^{-\alpha}[\Phi(\beta;t)](x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\Phi(\beta;t+x)dt. \tag{2.6}$$

3. A class of good functions and applications

To prove our main result, we define a class χ of functions that we call "good." A function $\varphi \in L^1_{\text{loc}}[0,\infty)$ is said to be a member of the class χ if

(P.1) $\Phi(0;t)$ has a power series representation at $t = 0$;

(P.2) the integral in (2.1) is absolutely and uniformly convergent in the strip $0 < \sigma_1 \le \sigma \le \sigma_2 < 1$.

The class of good functions is nonempty, as e^{-t} and $(1/(e^t - 1) - 1/t)$ belong to the class. We prove our representation formulae here and discuss their applications in the next sections.

Theorem 3.1. *The Weyl transform of a good function, φ, can be represented by*

$$\Phi(s;x) = \sum_{n=0}^{\infty} \Phi(s-n;0)\frac{(-x)^n}{n!} \quad (0 \le \sigma < 1,\ x > 0). \tag{3.1}$$

Proof. Since $\varphi \in \chi$, the corresponding function $\Phi(s;x) \in \chi$ must have the Taylor series expansion about $x = 0$:

$$\Phi(s;x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\left[\Phi(s;x)\right]_{x=0}\frac{x^n}{n!}. \tag{3.2}$$

However, we have

$$\frac{d^n}{dx^n}\left[\Phi(s;x)\right]_{x=0} = (-1)^n\Phi(s-n;0). \tag{3.3}$$

The proof follows directly from (3.2) and (3.3). □

Theorem 3.2. *For a good function, φ, the Weyl transform of*

$$\psi(t) := t^{-\mu} + \varphi(t) \tag{3.4}$$

is

$$\Psi(s;x) = \frac{\Gamma(\mu-s)}{\Gamma(\mu)}x^{s-\mu} + \sum_{n=0}^{\infty}\Phi(s-n;0)\frac{(-x)^n}{n!} \quad (0 < \sigma < \mu,\ x > 0). \tag{3.5}$$

Proof. Taking the Weyl transform of both sides in (3.4) and using (see [6, equation (7.7), page 249])

$$W^{-s}\left[t^{-\mu}\right](x) = \frac{\Gamma(\mu-s)}{\Gamma(\mu)}x^{s-\mu} \quad (0 < \sigma < \mu,\ x > 0), \tag{3.6}$$

we arrive at (3.5). □

4. Applications to Fermi-Dirac and Bose-Einstein integral functions

We show that the Fermi-Dirac and Bose-Einstein functions are expressible as the Weyl transform of a good function and recover their classical series representations by using the result (3.1) in a simple way. It is to be remarked that the result (3.1) is applicable to a wider class of functions. For example, $\cos(t)$ and $\sin(t)$ are good functions having Weyl transforms $\cos\,(x+\pi s/2)$ and $\sin\,(x+\pi s/2)$ $(0 < \sigma < 1)$. An application of (3.1) leads to the representations

$$\begin{aligned}\cos\left(x+\frac{\pi}{2}s\right) &= \sum_{n=0}^{\infty}\cos\left(\frac{\pi}{2}(s-n)\right)\frac{(-x)^n}{n!},\\ \sin\left(x+\frac{\pi}{2}s\right) &= \sum_{n=0}^{\infty}\sin\left(\frac{\pi}{2}(s-n)\right)\frac{(-x)^n}{n!}.\end{aligned} \tag{4.1}$$

Theorem 4.1. *The Fermi-Dirac integral function has the Taylor series representation*

$$F_{s-1}(x) = \sum_{n=0}^{\infty} (1 - 2^{n-s+1})\zeta(s-n)\frac{x^n}{n!} \quad (0 < \sigma < 1,\ x > 0), \tag{4.2}$$

with coefficients involving the zeta values.

Proof. Replacing x by $-x$ in (1.5) and putting $p = s - 1$, we obtain the operator form of the Fermi-Dirac integral

$$F_{s-1}(-x) = W^{-s}\left[\frac{1}{e^t + 1}\right](x). \tag{4.3}$$

Putting

$$\varphi(t) := \frac{1}{e^t + 1}, \tag{4.4}$$

we find that

$$F_{s-1}(-x) = W^{-s}\left[\frac{1}{e^t + 1}\right](x) = \Phi(s;x). \tag{4.5}$$

However, we have (see [1, equation (2.7.1), page 23])

$$F_{s-1}(0) = \Phi(s;0) = (1 - 2^{1-s})\zeta(s). \tag{4.6}$$

Putting these values in the representation Theorem 3.1 with x in place of $-x$, we arrive at (4.2). □

Theorem 4.2. *The Bose-Einstein integral function has the series representation*

$$B_{s-1}(-x) = \Gamma(1-s)x^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n)\frac{(-x)^n}{n!} \quad (0 < \sigma < 1,\ x > 0), \tag{4.7}$$

with coefficients involving the values of the zeta function.

Proof. Replacing x by $-x$ in (1.6) and putting $p = s - 1$, we obtain the operator form of the integral function

$$B_{s-1}(-x) = W^{-s}\left[\frac{1}{e^t - 1}\right](x). \tag{4.8}$$

Putting

$$\theta(t) := \frac{1}{e^t - 1} - \frac{1}{t}, \tag{4.9}$$

we find that

$$\frac{1}{e^t - 1} = \frac{1}{t} + \theta(t) \tag{4.10}$$

and (see [1, equation (2.7.1), page 23])

$$W^{-s}[\theta(t)](0) = W^{-s}\left[\frac{1}{e^t - 1} - \frac{1}{t}\right](0) = \zeta(s). \tag{4.11}$$

Since the function $\theta(t) \in \chi$, it follows from the representation Theorem 3.1 that

$$\Theta(s;x) = \sum_{n=0}^{\infty} \zeta(s-n)\frac{(-x)^n}{n!}. \tag{4.12}$$

However, from (3.6), we have

$$W^{-s}\left[\frac{1}{t}\right](x) = \Gamma(1-s)x^{s-1}. \tag{4.13}$$

Taking the Weyl transform of both sides in (4.10) and using (4.11)–(4.13), we get

$$B_{s-1}(-x) = \Gamma(1-s)x^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n)\frac{(-x)^n}{n!}. \tag{4.14}$$

□

Remark 4.3. The present formulation of the Weyl transform representation of the Fermi-Dirac integral functions leads to the representation (see (2.6))

$$F_{\alpha+\beta}(-x) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}F_{\beta}(-t-x)dt \quad (\sigma > 0). \tag{4.15}$$

Putting $\beta = \alpha - 1$ in (4.15), we get

$$F_{2\alpha-1}(-x) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}F_{\alpha-1}(-t-x)dt \quad (\sigma > 0). \tag{4.16}$$

Similarly, it follows from the operational formulation (4.8) that

$$B_{\alpha+\beta}(-x) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}B_{\beta}(-t-x)dt \quad (\sigma > 0). \tag{4.17}$$

The operator representations (4.15) and (4.17) provide a useful relation between the functions and their transforms.

5. Alternate derivation of Euler's formula

Euler's formula relating the Riemann zeta function to the Bernoulli numbers, B_n, is one of the important results in the theory of the zeta function. The usual derivation is long and complicated. From the results obtained above for the Fermi-Dirac and Bose-Einstein integral functions, we obtain the Euler formula more simply. The formula is

$$\zeta(2n) = (-1)^{n+1}\frac{2(2\pi)^{2n}}{(2n)!}B_{2n} \quad (n = 1,2,3,\ldots), \tag{5.1}$$

where the Bernoulli numbers are defined by [2, page 804]

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{(-x)^n}{n!}. \tag{5.2}$$

The Euler numbers are defined by

$$\frac{2e^x}{e^x+1}=E_0+\sum_{n=1}^{\infty}E_n\frac{(-x)^n}{n!}. \tag{5.3}$$

Putting $s=0$ in (4.2) and using (1.7), we find that

$$F_{-1}(x)=\frac{e^x}{e^x+1}=\sum_{n=0}^{\infty}\zeta(-n)\left(1-2^{n+1}\right)\frac{(x)^n}{n!}. \tag{5.4}$$

However, it follows from the Riemann functional equation that

$$\zeta(-n)=-\frac{1}{2}(2\pi)^{-n-1}\sin\left(\frac{\pi n}{2}\right)\Gamma(n+1)\zeta(n+1). \tag{5.5}$$

From (5.4) and (5.5), we obtain

$$2F_{-1}(x)=\frac{2e^x}{e^x+1}=1+\sum_{n=1}^{\infty}\left[\left(2^{n+1}-1\right)\sin\left(\frac{n\pi}{2}\right)\Gamma(n+1)\zeta(n+1)\right]\frac{(x)^n}{n!}, \tag{5.6}$$

where we have taken

$$\lim_{s\to 0}\left[\sin\left(\frac{s\pi}{2}\right)\zeta(s+1)\right]=1. \tag{5.7}$$

Comparing the coefficients of equal powers of x in (5.3) and (5.6), we get

$$\begin{aligned} E_0&=1,\\ E_{2n}&=0\quad(n=1,2,3,\ldots),\\ E_{2n-1}&=(-1)^n\frac{4\left(2^{2n}-1\right)(2n-1)!}{(2\pi)^{2n}}\zeta(2n), \end{aligned} \tag{5.8}$$

which can be rewritten to give

$$\zeta(2n)=(-1)^n\frac{(2\pi)^{2n}E_{2n-1}}{4\left(2^{2n}-1\right)(2n-1)!}\quad(n=1,2,3,\ldots). \tag{5.9}$$

Now inserting the relation between the Euler and Bernoulli numbers (see [2, page 805])

$$E_n=\frac{2-2^{n+2}}{n+1}B_{n+1}\quad(n=1,2,3,\ldots), \tag{5.10}$$

we obtain (5.1) as desired. The representation of the zeta values at odd integers remains a challenging task. We hope that the present formulation of the operator representation of the Fermi integrals may lead to the desired formula.

6. Concluding remarks

Transform techniques are extremely powerful tools for dealing with functions and constructing solutions of equations. In particular, the Weyl transform, which is at the heart of the "fractional calculus," has been extensively used for various purposes. In this paper, we have used it to construct the Fermi-Dirac and Bose-Einstein integral functions from elementary functions. These functions are related to probabilities arising from the Fermi-Dirac and Bose-Einstein distribution functions which give the quantum description of collections of identical particles of half-odd integer and integral intrinsic spin, respectively. Due to their physical significance, these distribution functions have been extensively studied. Bosons and fermions were regarded as being mutually exclusive with no fundamental physical quantity described by some function "between" these two in any sense. However, it was later realized that there can be "effective particles," called *anyons*, that are neither *fermions* nor *bosons* but something between the two. The process of obtaining the integral functions by the Weyl transform can be used to develop a candidate for an anyon integral function.

Our procedure has significant "spinoffs." We recover the well-known connections between the Fermi-Dirac and Bose-Einstein integral functions and with the zeta and polylogarithm function. Of special interest is an alternative, and extremely elegant, derivation of the Euler formula relating the Riemann zeta function to the even-integer argument and the Bernoulli numbers. This demonstrates the significance and power of the Weyl transform method, as applied here. It also leads us to hope that the Fermi-Dirac integral function may provide a way of constructing a formula for the odd-integer argument. Zeta function has remained an open problem for the last 300 years.

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