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Research Article

An Investigation on Gas Lift Performance Curve in an Oil-Producing Well

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The main objective in oil production system using gas lift technique is to obtain the optimum gas injection rate which yields the maximum oil production rate. Relationship between gas injection rate and oil production rate is described by a continuous gas lift performance curve (GLPC). Obtaining the optimum gas injection rate is important because excessive gas injection will reduce production rate, and also increase the operation cost. In this paper, we discuss a mathematical model for gas lift technique and the characteristics of the GLPC for a production well, for which one phase (liquid) is flowing in the reservoir, and two phases (liquid and gas) in the tubing. It is shown that in certain physical condition the GLPC exists and is unique. Numerical computations indicate unimodal properties of the GLPC. It is also constructed here a numerical scheme based on genetic algorithm to compute the optimum oil production.

1. Introduction

Gas lift is one of the most common artificial lift methods used in oil production system. During the lift process, gas is injected at a selected point in tubing (see Figure 1.1), resulting in reduction of natural bottomhole pressure which then increases the pressure difference between the reservoir and the bottomhole. The goal of gas lift is to deliver the fluid to the top of the wellhead while keeping the bottomhole pressure low enough to provide high pressure drop between the reservoir and the bottomhole.

Reduction of bottomhole pressure due to gas injection will normally increase liquid (oil) production rate, because gas injection will lighten the fluid column, therefore larger amount of fluid will flow along the tubing. However, injecting too much amount of gas will increase the bottomhole pressure which may decrease the oil production rate. This

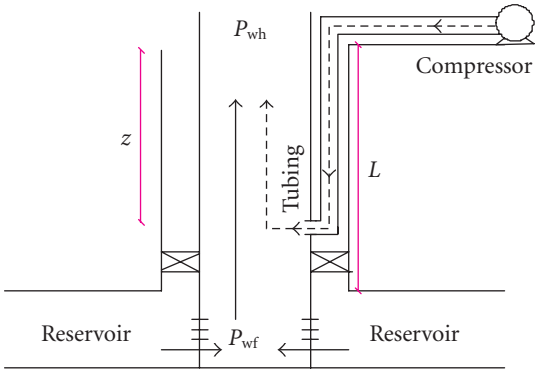

Figure 1.1. Simple gas lift scheme.

happened because too high gas injection rate causes slippage, where gas phase moves faster than liquid, leaving the liquid phase behind. In this condition, less amount of liquid will flow along the tubing. Hence, there must be an optimum gas injection rate that yields maximum oil production rate. Finding this optimum gas injection rate is the main objective of gas lift allocation optimization problem.

Oil production process in natural way can be considered as a combination of two fluid flows, first in reservoir and second along the tubing. Both fluid flows may be a one-phase (liquid) or a two-phase (liquid and gas). In this paper, both fluid flows are assumed naturally single phase (liquid), up to the point of gas injection in the tubing. Above gas injection point, two-phase flow takes place.

In mathematical formulation, gas lift performance problem (in normalized form) can be modelled as a two-parameter family of an ordinary differential equation (ODE) representing the steady flow equation along the tubing

$$\frac{dP}{dz} = F(z,P;q_g,q_l), \tag{1.1}$$

with boundary conditions

$$P(0) = P_{\text{wh}}, \tag{1.2}$$

$$P(1) = P_{\text{wf}}. \tag{1.3}$$

Here, $F(z,P;q_g,q_l)$ is a nonnegative real-valued function

$$F : [0,1] \times [P_{\text{wh}}, P_{\text{wf}}] \longrightarrow \mathbb{R}^+, \tag{1.4}$$

q_g and q_l, where $0 \le q_g < \infty$ and $0 \le q_l \le 1$ are gas injection rate and liquid production rate, respectively, P_{wh} and P_{wf} are wellhead pressure and bottomhole pressure, respectively. Here, P_{wf} is a function of parameter q_l. Note that we assume the fluid rate q_l is constant along the tubing. In reality, the fluid rate changes slightly with pressure (see comment about the effect of changes in q_l in Section 3). Solution of (1.1)-(1.2) satisfying

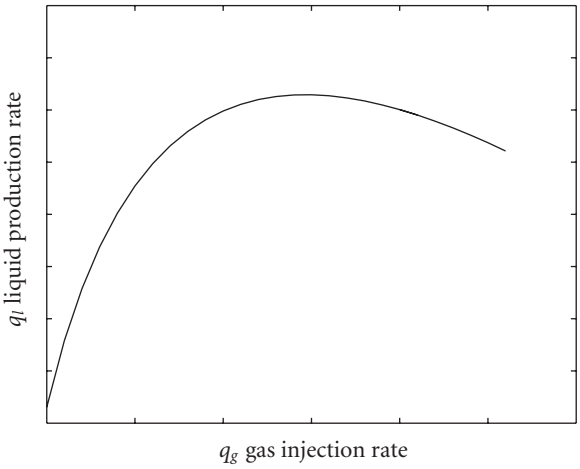

Figure 1.2. Sketch of experimental GLPC.

(1.3), if exists, will generate the gas lift performance curve (GLPC) in $q_g - q_l$ plane, in the form of an implicit function

$$\zeta(q_g, q_l) = 0. \tag{1.5}$$

Therefore, the problem is reduced to a constrained optimization problem in parameter space (q_g, q_l). In this paper, we discuss the existence and uniqueness of GLPC and construct a numerical scheme to compute the maximum value of q_l satisfying (1.1)–(1.3).

In practice, GLPC is constructed experimentally for specified wells [1]. By use of this curve, we can estimate the influence of gas injection rate to the liquid production rate, and determine gas injection rate required to obtain production rate as desired. Therefore, the characteristic of the GLPC is important to study, in order to find the optimum gas injection rate.

For practical purpose, GLPC can be obtained usually by conducting numerical simulation by an appropriate method such as nodal analysis (see [2]), and also can be obtained from field data by measuring gas injection rate and liquid production rate. Then, based on the data, curve fitting can be done to obtain the GLPC. As mentioned in [1, 3], a quadratic polynomial function is usually used for fitting GLPC from field data by least square method. In 2002, Alarcón et al. [3] proposed new function to improve the previous quadratic polynomial for fitting GLPC from field data by adding a logarithmic term to the quadratic polynomial. Sukarno et al. [4] proposed exponential function for fitting GLPC from field data. However, the application of exponential GLPC is limited only for gas injection between the lowest value up to the optimum gas injected. Another researcher [5] uses a piecewise linear function for fitting GLPC from field data.

The GLPC, resulted by curve fitting is usually found to be unimodal as illustrated in Figure 1.2. So, in case when a large amount of gas is available, the peak of the GLPC, which corresponds to the possible maximum production rate, can be obtained from gas

lift. However, none guarantees that GLPC is always unimodal, and there is a lack of theory for explaining the characteristic of GLPC. For this reason, in this paper, we will show that GLPC as a solution of boundary value problem (1.1), (1.2), and (1.3) exists and is unique. Further GLPC is shown to have a maximum point. The latter is important to ensure the existence of the optimum gas injection rate. So, when we construct a numerical method to compute the optimum gas injection rate, the numerical solution will converge to the maximum point.

2. Mathematical formulation of gas lift

Before we derive the mathematical formulation, we list below all symbols with corresponding units (in oil field and SI) used in this paper.

P_{wh}	Wellhead pressure (psi, Pa)
P_{wf}	Bottomhole pressure (psi, Pa)
P_r	Reservoir pressure (psi, Pa)
q_g	Gas injection rate (MSCF/d, m^3/s)
q_l	Liquid production rate (STB/d, m^3/s)
q_o	Oil production rate (STB/d, m^3/s)
J	Productivity index (STB/d/psi, $m^3/s/Pa$)
y_g	Void fraction (dimensionless)
ρ_m	In situ average density (lb/ft^3, kg/m^3)
ρ_l	Liquid density (lb/ft^3, kg/m^3)
ρ_g	Gas density (lb/ft^3, kg/m^3)
u_m	Mixture velocity (ft/s, m/s)
u_{sg}	Superficial gas velocity (ft/s, m/s)
u_{sl}	Superficial liquid velocity (ft/s, m/s)
U_d	Drift velocity (ft/s, m/s)
C_0	Parameter distribution (dimensionless)
WOR	Water-oil ratio (SCF/SCF, m^3/m^3)
GLR	Gas-liquid ratio (SCF/STB, m^3/m^3)
f	Friction factor (dimensionless)
D	Diameter of tubing (ft, m)
g	Acceleration of gravity (ft/s^2, m/s^2)
γ_g	Specific gravity of gas (dimensionless)
γ_o	Specific gravity of oil (dimensionless)
γ_w	Specific gravity of water (dimensionless)
R	Universal gas constant (psi ft^3/lb-mole-^{o}R)
Z	Gas compressibility factor (dimensionless)
T	Temperature (^{o}F, ^{o}C, ^{o}R).

For one-phase fluid flow (liquid) in the reservoir, in steady state, the model leads to Darcy's law

$$P_{\mathrm{wf}} = \frac{JP_r - q_l}{J}. \tag{2.1}$$

From (2.1), for fixed value of P_r, we see that lower bottomhole pressure P_{wf} will result in higher liquid production rate q_l. By assuming that fluid in the reservoir consists of oil and water only, we have

$$q_l = q_o(1 + \mathrm{WOR}). \tag{2.2}$$

Equation for two-phase fluid flow (liquid and gas) along the tubing can be derived from the *mechanical energy balance* equation (see [6, 7])

$$\frac{dP}{dz} = \frac{g}{g_c}\overline{\rho} + 2\frac{f\overline{\rho}u_m^2}{g_c D} + \frac{\overline{\rho}}{2g_c}\frac{d}{dz}u_m^2, \tag{2.3}$$

with an initial condition

$$P(0) = P_{\mathrm{wh}}. \tag{2.4}$$

The terms $(g/g_c)\overline{\rho}$, $2f\overline{\rho}u_m^2/g_c D$, and $(\overline{\rho}/2g_c)(d/dz)u_m^2$ in (2.3) correspond to the pressure drop due to gravity, friction, and acceleration, respectively. Since the pressure drop due to acceleration is quite small, here its contribution is neglected. The term $\overline{\rho}$ is in situ average density and can be expressed as

$$\overline{\rho} = y_g\rho_g + (1 - y_g)\rho_l, \tag{2.5}$$

where gas density ρ_g depends on the pressure P given by

$$\rho_g = \gamma P, \quad \text{with } \gamma = \frac{28.97\gamma_g}{ZRT}. \tag{2.6}$$

Average void fraction y_g is given by drift-flux model of Zuber and Findlay (see [8–10])

$$y_g = \frac{u_{\mathrm{sg}}}{C_0(u_{\mathrm{sg}} + u_{\mathrm{sl}}) + U_d}. \tag{2.7}$$

Parameter distribution C_0 accounts for the effects of the nonuniform distribution of both velocity and concentration profiles. Drift flux velocity U_d accounts for mean relative velocity between the two phases. There are several drift flux correlations offering procedures to compute C_0 and U_d (see [11]).

Mixture velocity u_m is given by the sum of liquid and gas superficial velocities

$$u_m = u_{\mathrm{sl}} + u_{\mathrm{sg}}, \tag{2.8}$$

where

$$u_{\mathrm{sl}} = \frac{q_l}{A}, \qquad u_{\mathrm{sg}} = \frac{ZP_{\mathrm{sc}}T}{T_{\mathrm{sc}}P}\frac{q_g}{A}, \qquad A = \pi\frac{D^2}{4} \tag{2.9}$$

is the cross section of the tubing.

Friction factor f is the Fanning friction factor. In laminar flow, the friction factor is a simple function of the Reynolds number N_{Re}, $f = 16/N_{\mathrm{Re}}$, whereas in turbulent flow, the

Table 2.1. Behavior of ρ and u with respect to P, q_g, and q_l.

	P	q_g	q_l
ρ	Increasing	Decreasing	Increasing
u	Decreasing	Increasing	Increasing

friction factor may depend on both Reynolds number and the relative pipe roughness ϵ. The Fanning friction factor is most commonly obtained from Colebrook equation (see [7]). So, there is a discontinuity of f in the transitional zone (between laminar and turbulent). In this study, we assume the friction factor is constant in any condition.

By scaling,

$$\begin{aligned} \widetilde{P} &= \frac{P}{P_r}, & \widetilde{z} &= \frac{z}{L}, \\ \widetilde{q}_l &= \frac{q_l}{JP_r}, & \widetilde{q}_g &= \frac{q_g}{LD\sqrt{gD}}, \\ \widetilde{\rho} &= \frac{\overline{\rho}Lg}{g_c P_r}, & \widetilde{u} &= \frac{u_m}{\sqrt{gD}}, \end{aligned} \tag{2.10}$$

(2.3), (2.4), and (2.1) can be written in dimensionless form

$$\frac{d\widetilde{P}}{d\widetilde{z}} = F(\widetilde{z},\widetilde{P};\widetilde{q}_g,\widetilde{q}_l) = \widetilde{\rho}(1+2f\widetilde{u}^2), \tag{2.11}$$

$$\widetilde{P}(0) = \widetilde{P}_{\text{wh}}, \tag{2.12}$$

$$\widetilde{P}_{\text{wf}} = 1 - \widetilde{q}_l, \tag{2.13}$$

respectively. Since $F(\widetilde{z},\widetilde{P};\widetilde{q}_g,\widetilde{q}_l)$ is continuously differentiable, the solution $P(z;q_g,q_l)$ of the initial value problem (IVP) (2.11)-(2.12) is continuously differentiable with respect to $\widetilde{q}_g$ and $\widetilde{q}_l$.

Next, for simplicity we drop tilde from (2.11), (2.12) and (2.13). It is clear that ρ and u depend on P, q_g, and q_l. Their dependence can be described in Table 2.1.

Let us consider the following dimensionless parameters:

$$\begin{aligned} \alpha_1 &= \frac{L\rho_l}{P_r}\frac{g}{g_c}, & \alpha_2 &= L\gamma\frac{g}{g_c}, \\ \alpha_3 &= D\gamma\frac{g}{g_c}, & \alpha_4 &= \frac{U_d}{\sqrt{gD}}, \\ \alpha_5 &= \frac{JP_r}{\pi\sqrt{gD}D^2}, & \alpha_6 &= \frac{ZTP_{\text{sc}}}{\pi T_{\text{sc}}P_r}. \end{aligned} \tag{2.14}$$

Note that $\alpha_1 > \alpha_2$ since $\rho_l > \rho_g$. Applying (2.14), (2.11) can be written in rational form

$$\frac{dP}{dz} = F(z,P;q_g,q_l) = \frac{a_0 + a_1P + a_2P^2 + a_3P^3}{b_2P^2 + b_3P^3}, \tag{2.15}$$

where

$$\begin{aligned}
a_0 &= 128\alpha_1\alpha_2^3\alpha_6^3(C_0-1)fq_g^3,\\
a_1 &= 32\alpha_2^2\alpha_6^2(4\alpha_2^2\alpha_6q_g+4\alpha_1\alpha_3\alpha_5(3C_0-2)q_l+\alpha_1\alpha_3\alpha_4)fq_g^2,\\
a_2 &= 4\alpha_3\alpha_6\alpha_2(32\alpha_1f\alpha_5^2\alpha_3(3C_0-1)q_l^2+16f\alpha_5(4\alpha_6\alpha_2^2q_g+\alpha_1\alpha_4\alpha_3)q_l+\alpha_1\alpha_3(C_0-1))q_g,\\
a_3 &= \alpha_3^2(4\alpha_6\alpha_2^2q_g+4\alpha_1C_0\alpha_5q_l\alpha_3+\alpha_1\alpha_4\alpha_3)(1+32f\alpha_5^2q_l^2),\\
b_2 &= 4\alpha_2\alpha_3^2\alpha_6C_0q_g,\\
b_3 &= (4C_0\alpha_3\alpha_5q_l+\alpha_3\alpha_4)\alpha_3^2
\end{aligned} \tag{2.16}$$

are nonnegative since $C_0 \geq 1$.

We see from (2.15), that when no gas flows in the tubing ($q_g = 0$), that is, before gas lift technique is applied, the flow is one phase (liquid), and

$$\frac{dP}{dz} = \alpha_1(1+32f\alpha_5^2q_l^2). \tag{2.17}$$

This means that pressure in the tubing P is a linear function of z since the right hand side of (2.17) is constant. On the other hand, when $q_l = 0$, that is, no liquid flows in the tubing (gas flow only), the flow is one-phase (gas). In this case, $C_0 = 1$, $U_d = 0$, and $y_g = 1$. Hence (2.15) reduces to

$$\frac{dP}{dz} = F(z,P;q_g,0) = \frac{(32fq_g^2\alpha_6^2\alpha_2^2+\alpha_3^2P^2)}{\alpha_3^2P}. \tag{2.18}$$

Next, we will examine some properties of (2.18).

Lemma 2.1. *Let $P(z;q_g,0)$ be the solution of (2.18) with initial condition (2.12), then $P(z;q_g,0)$ is increasing with respect to q_g. If $P_{wh} < e^{-\alpha_2}$ then there exists a unique positive q_g^* such that $P(1;q_g^*,0) = 1$.*

Proof. By solving (2.18) with initial condition $P(0) = P_{\text{wh}}$, we have

$$P(z;q_g,0) = \frac{\sqrt{32f\alpha_6^2\alpha_2^2(e^{2\alpha_2z}-1)q_g^2+e^{2\alpha_2z}P_{\text{wh}}^2\alpha_3^2}}{\alpha_3}. \tag{2.19}$$

Hence, $P(z;q_g,0)$ is increasing with respect to q_g. Particularly at $z = 1$, we have

$$P(1;q_g,0) = \frac{\sqrt{32f\alpha_6^2\alpha_2^2(e^{2\alpha_2}-1)q_g^2+e^{2\alpha_2}P_{\text{wh}}^2\alpha_3^2}}{\alpha_3}. \tag{2.20}$$

So, if $P_{\text{wh}} < e^{-\alpha_2}$, there exists a unique positive q_g^* such that $P(1;q_g^*,0) = 1$ where

$$q_g^* = \frac{\alpha_3\sqrt{2f(e^{2\alpha_2}-1)(1-e^{2\alpha_2}P_{\text{wh}}^2)}}{8\alpha_2\alpha_6f(e^{2\alpha_2}-1)}. \tag{2.21}$$

□

Rewrite (2.11) with initial condition (2.12) implicitly as

$$P(z;q_g,q_l) = \int_0^z F(t,P;q_g,q_l)\,dt + P_{\text{wh}}, \tag{2.22}$$

in the following lemma, we examine behavior of pressure function in (2.22) with respect to q_l.

Lemma 2.2. *Let $P(z;q_g,q_l)$ be a pressure function satisfying (2.22). Then, for any fixed z and q_g, $P(z;q_g,q_l)$ is increasing with respect to q_l.*

Proof. Since for any fixed z, P, and q_g, the terms ρ and u in (2.10) are increasing with respect to q_l (see Table 2.1), then for any fixed z, P, and q_g, the pressure gradient $F(z,P;q_g,q_l)$ in (2.11) is also increasing with respect to q_l. Now, we will show that $P(z;q_g,q_l)$ is also increasing with respect to q_l. Let $0 < q_{l1} < q_{l2}$, then

$$\frac{dP(0;q_g,q_{l1})}{dz} = F(0,P_{\text{wh}};q_g,q_{l1}) < F(0,P_{\text{wh}};q_g,q_{l2}) = \frac{dP(0;q_g,q_{l2})}{dz}. \tag{2.23}$$

This means that for some $\epsilon > 0$, $P(z;q_g,q_{l1}) \le P(z;q_g,q_{l2})$ for all $z \in (0,\epsilon)$. We will show that this result is also correct for $\epsilon \le z \le 1$. Suppose that the statement is not correct, that is, $P(z^*;q_g,q_{l1}) > P(z^*;q_g,q_{l2})$ for some $z^* \ge \epsilon$. Then, there exists $\overline{z} \in (\epsilon,z^*)$ such that $P(\overline{z};q_g,q_{l1}) = P(\overline{z};q_g,q_{l2})$ and $P(z;q_g,q_{l1}) > P(z;q_g,q_{l2})$ for all $z \in (\overline{z},z^*)$. Let $P_1^* = P(\overline{z};q_g,q_{l1})$ and $P_2^* = P(\overline{z};q_g,q_{l2})$, then

$$\frac{dP(\overline{z};q_g,q_{l1})}{dz} = F(\overline{z},P_1^*;q_g,q_{l1}) < F(\overline{z},P_2^*;q_g,q_{l2}) = \frac{dP(\overline{z};q_g,q_{l2})}{dz}. \tag{2.24}$$

This means that for some $\delta > 0$, $P(z;q_g,q_{l1}) \le P(z;q_g,q_{l2})$ for all $z \in (\overline{z},\overline{z}+\delta)$, which leads to a contradiction. □

3. Gas lift performance curve

Let $P(z;q_g,q_l)$ be a two-parameter family of continuous function satisfying the IVP (2.11)-(2.12). The production rate in gas lift technique can be stated as to find

$$q_l = \varphi(q_g) \tag{3.1}$$

on interval $0 \le q_g \le q_g^*$, such that

$$P(1;q_g,q_l) = P_{\text{wf}}, \tag{3.2}$$

where P_{wf} is given by (2.13) and q_g^* is given by (2.21). If such a function $\varphi(q_g)$ exists, then the graph of $q_l = \varphi(q_g)$ is called *gas lift performance curve* (GLPC).

In Theorem 3.1, we will show the existence and uniqueness of the GLPC on interval $[0,q_g^*]$.

Theorem 3.1 (existence and uniqueness of GLPC). *If $P_{wh} < e^{-\alpha_2}$, then GLPC exists and is unique on interval $0 \le q_g \le q_g^*$.*

Proof. Define a function

$$\zeta(q_g, q_l) = P(1; q_g, q_l) - (1 - q_l), \tag{3.3}$$

where $P(z; q_g, q_l)$ satisfies (2.22). We will show that there exists a unique curve $q_l = \varphi(q_g)$, $0 \le q_g \le q_g^*$, such that

$$\zeta(q_g, \varphi(q_g)) = 0. \tag{3.4}$$

Note that from Lemma 2.1, $P(1; q_g, 0)$ is increasing with respect to q_g and $P(1; q_g^*, 0) = 1$. Then, for any $q_g \in [0, q_g^*)$,

$$\begin{aligned} \zeta(q_g, 0) &= P(1; q_g, 0) - 1 < 0, \\ \zeta(q_g, 1) &= P(1; q_g, 1) > 0. \end{aligned} \tag{3.5}$$

By intermediate value theorem, for any $q_g \in [0, q_g^*)$, there exists a $q_l \in (0, 1)$ such that

$$\zeta(q_g, q_l) = 0. \tag{3.6}$$

Further, since $(\partial/\partial q_l)P(1; q_g, q_l) > 0$ (Lemma 2.2), then $(\partial/\partial q_l)\zeta(q_g, q_l) > 0$. Then, by implicit function theorem, there exists a unique curve $q_l = \varphi(q_g)$ on interval $[0, q_g^*]$ such that $\zeta(q_g, \varphi(q_g)) = 0$. □

Based on the assumptions used in Theorem 3.1, the GLPC starts at $q_g = 0$ (with the corresponding $\varphi(0) > 0$). This means that without gas injection, some amount of liquid can still be produced from the well. In this case, only liquid flows across the tubing. In case the liquid flow rate $q_l = 1$, the bottomhole pressure equals to 0. Physically, $q_l = 1$ is impossible to reach, since $P_{\text{wh}} > 0$. Hence $q_l = 1$ here is just an upper bound for q_l. In case of the liquid flow rate $q_l = 0$, the bottomhole pressure equals to reservoir pressure and only gas flows across the tubing. Physically, as gas in the tubing increases, the bottomhole pressure will increase. Hence, before $q_l = 0$ (no oil production) occurs, gas lift process must be stopped.

In the following theorem, we will show that in certain condition, the maximum value of GLPC occurs at an interior point of $(0, q_g^*)$.

Theorem 3.2 (maximum point of GLPC). *Under condition*

$$f < \frac{\alpha_1 - \alpha_2}{16\alpha_5(2\alpha_5(\alpha_2 + \alpha_1(2C_0 - 1)) + \alpha_1\alpha_4)}, \tag{3.7}$$

there exists $\overline{q}_g \in (0, q_g^*)$ such that the GLPC (3.1) reaches the maximum value at $q_g = \overline{q}_g$. Namely, constrained oil production maximization problem

$$\text{Max}\{q_l = \varphi(q_g) \mid P(1; q_g, q_l) = P_{wf},\ 0 < q_g < q_g^*\}, \tag{3.8}$$

where $P(z; q_g, q_l)$ satisfies (2.22), obtains its maximum point in the interior of $(0, q_g^*)$.

Proof. From Theorem 3.1, $\varphi(0) > 0$ and $\varphi(q_g^*) = 0$. Hence, it is enough to show that GLPC $q_l = \varphi(q_g)$ is increasing for small q_g. Let $F(z,P;q_g,q_l)$ be the pressure gradient (2.11). Then, at $q_g = 0$, its partial derivative with respect to q_g is given by

$$\begin{aligned}\frac{\partial F(z,P;0,q_l)}{\partial q_g} &= \frac{64\alpha_2\alpha_5\alpha_6 q_l(\alpha_1(2\alpha_5(2C_0-1)q_l+\alpha_4)+2\alpha_2\alpha_5 q_l P)f}{\alpha_3 P(4C_0\alpha_5 q_l+\alpha_4)} \\ &\quad - \frac{4\alpha_2\alpha_6(\alpha_1-\alpha_2 P)}{\alpha_3 P(4C_0\alpha_5 q_l+\alpha_4)}.\end{aligned} \tag{3.9}$$

Then, if (3.7) is satisfied, $\partial/\partial q_g F(z,P;q_g,q_l) < 0$ for small q_g, namely, for any fixed z, P, and q_l, $F(z,P;q_g,q_l)$ is decreasing with respect to q_g for small q_g. By the same way as in the proof of Lemma 2.2, it could be shown that $P(z;q_g,q_l)$, particularly $P(1;q_g,q_l)$, is also decreasing with respect to q_g for small q_g.

Along GLPC, $P(1;q_g,q_l) = 1 - q_l$, thus

$$\frac{dq_l(q_g)}{dq_g} = \frac{(-\partial/\partial q_g)P(1;q_g,q_l)}{1+(\partial/\partial q_l)P(1;q_g,q_l)}. \tag{3.10}$$

Since $(\partial/\partial q_l)P(1;q_g,q_l) > 0$ (by Lemma 2.2), then the numerator is always positive. Hence, for small q_g

$$\frac{dq_l(q_g)}{dq_g} > 0, \tag{3.11}$$

that is, GLPC $q_l = \varphi(q_g)$ is increasing for small q_g. □

Remark 3.3. In this paper, liquid production rate q_l is assumed to be constant along the tubing. In practical application, liquid production rate may change slightly with pressure. At low pressures, liquid may release gas which is dissolved at higher pressures. This assumption may fit the application on heavy oil cases.

4. Genetic algorithm for oil production maximization problem

Liquid production rate from a production well can be illustrated from combination of inflow performance relationship (IPR) and vertical lift performance (VLP), that is, a combination of ability of reservoir to deliver the fluid into the tubing and ability to deliver the fluid along the tubing from the bottom to the wellhead at a required wellhead pressure. Geometrically, for any fixed q_g, liquid production rate q_l can be obtained from the intersection between IPR curve and VLP curve. Here, IPR curve can be plotted from normalized Darcy's law (2.13) and VLP curve is plotted from bottomhole pressure $P(1;q_g,q_l)$, where $P(z;q_g,q_l)$ is given by (2.22). These curves are illustrated in Figure 4.1.

Each intersection point (q_g,q_l) of IPR and VLP satisfies $\zeta(q_g,q_l) = 0$, where $\zeta(q_g,q_l)$ is given by (3.3). GLPC can be constructed from these intersection points. So, the computation problem can be written as follows.

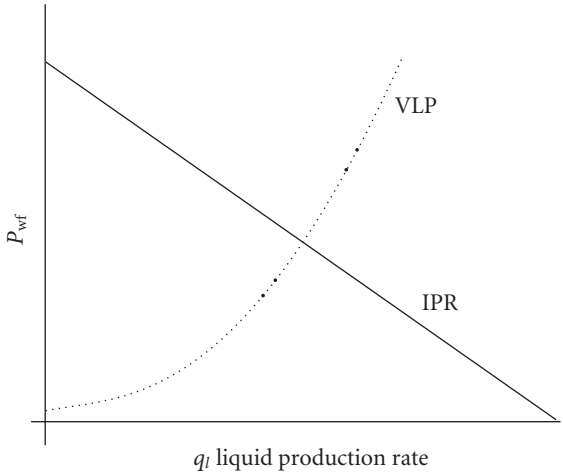

Figure 4.1. IPR and VLP curves for fixed q_g.

Determining $(q_g, q_l) \in R = \{(q_g, q_l) \mid 0 \le q_g \le q_g^*,\ 0 \le q_l \le 1\}$ such that

$$\zeta(q_g, q_l) = P(1; q_g, q_l) - (1 - q_l) = 0, \tag{4.1}$$

where q_g^* satisfies $P(1; q_g^*, 0) = 1$, $P(z; q_g, q_l)$ is solution of the IVP (2.11)-(2.12).

We plot some computational GLPCs obtained by shooting method in Figure 4.2. Constant values $C_0 = 1.08$ and $U_d = 0.45\,\text{m/s} = 1.4764\,\text{ft/s}$ are obtained from Toshiba correlation (see [11]). We see that for friction factor $f = 0.002$, GLPCs are unimodal for well data given by Table 4.2. Wells data is obtained from [4]. In order to fit with assumptions in this paper, we chose here constant wellhead pressure $P_{\text{wh}} = 100$ psi, constant temperature $T = 595^o$ R, injection depth equals to welldepth and natural gas-to-liquid-ratio GLR = 0.

In this section, we will construct a genetic algorithm to find the solution of constrained oil production maximization problem (3.8). First, we transform (3.8) into an unconstrained optimization problem using a penalty approach.

Let us consider a family of unconstrained minimization problem

$$\Omega(q_g, q_l) = \frac{1}{q_l + 1} + \lambda\left[P(1; q_g, q_l) - (1 - q_l)\right]^2, \tag{4.2}$$

where $P(z; q_g, q_l)$ is the solution of IVP (2.11)-(2.12) and λ is a positive constant normally called *penalty factor*. Then, the solution of unconstrained minimization problems

$$\min_{(q_g, q_l) \in D} \Omega(q_g, q_l), \tag{4.3}$$

where

$$D = \{(q_g, q_l) \mid 0 \le q_g \le q_g^*,\ 0 \le q_l \le 1\}, \tag{4.4}$$

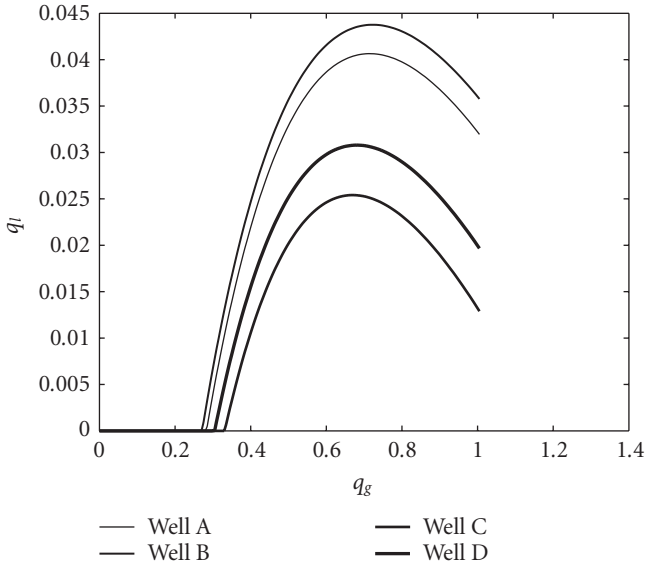

Figure 4.2. GLPCs by shooting method.

will converge to the optimal solution of the original problem, that is, the maximum point of the GLPC (3.1) as $\lambda \to \infty$ (see [12]). Since we do not have explicit formula for $P(z;q_g,q_l)$, then for given (q_g,q_l), $P(1;q_g,q_l)$ is approximated using fourth-order Runge-Kutta (RK-4). So, in the numerical scheme we compute the solution of

$$\min_{(q_g,q_l)\in\overline{D}} \overline{\Omega}(q_g,q_l) = \frac{1}{q_l+1} + \lambda\left[\overline{P}(1;q_g,q_l) - (1-q_l)\right]^2 \tag{4.5}$$

in the domain

$$\overline{D} = \left\{(q_g,q_l) \mid 0 \le q_g \le \overline{q}_g^*,\ 0 \le q_l \le 1\right\}, \tag{4.6}$$

where $\overline{P}(z;q_g,q_l)$ is the numerical solution of the IVP (2.11)-(2.12) using RK-4, and $\overline{q}_g^*$ is the solution of $\overline{P}(1;q_g,0) = 1$.

The computational procedure using genetic algorithm (see [13] for more details) can be written as follows.

(1) Initialize a population of chromosomes $v_1,v_2,\ldots,v_r$ which correspond to pairs $(q_{g1},q_{l1}),(q_{g2},q_{l2}),\ldots,(q_{gr},q_{lr})$.
(2) For each pair (q_{gk},q_{lk}), $k = 1,2,\ldots,r$, compute $\overline{P}(1;q_{gk},q_{lk})$ using RK-4.
(3) Evaluate the fitness values $\overline{\Omega}(q_{gk},q_{lk})$, $k = 1,2,\ldots,r$.
(4) Create new chromosomes by doing crossover and applying mutation.
(5) Apply a selection to get a new population.
(6) Return to step (2) until stopping criteria is satisfied.

The solution to the penalty problem can be made arbitrarily close to the optimal solution of the original problem by choosing penalty factor λ sufficiently large. However, if

Table 4.1. Numerical optimum points using genetic algorithm.

	Optimum q_g	Optimum q_l
Well A	0.71147	0.040501
Well B	0.72367	0.042929
Well C	0.65975	0.026463
Well D	0.67821	0.030506

Table 4.2. Well data.

	Well A	Well B	Well C	Well D	Unit
P_r	790	815	750	795	psi
J	11	11	11	11	STB/d/psi
L	6900	7000	7000	7217.8	feet
D	2.875	2.875	2.875	2.875	inch
WOR	1	1	1	1	—
γ_g	0.65	0.65	0.65	0.65	—
γ_o	0.876 (30°API)	0.876	0.876	0.876	—
γ_w	1.01	1.01	1.01	1.01	—

Table 4.3. Conversion of numerical optimum points to the field units.

	Optimum q_g	Optimum q_l
Well A	1.96006 MMSCF/d	351.95369 STB/d
Well B	2.02257 MMSCF/d	384.85848 STB/d
Well C	1.84392 MMSCF/d	218.31975 STB/d
Well D	1.95449 MMSCF/d	266.77497 STB/d

we choose a very large λ and attempt to solve the penalty problem, we may get into some computational difficulties. The population will move very quickly toward feasible points which may be still far from the optimum and difficult to move toward the optimum point. To overcome with this difficulty, we choose as the penalty factor, a sequence that increases with respect to generation. In this case, we choose $\lambda = n^2$, where n is generation number. Such a penalty, where the current generation number is involved in the computation of the corresponding penalty factor is known as *dynamic penalties*. Advantages and disadvantages of using dynamic penalties can be seen in [14].

The following are computational results using genetic algorithm. The computation process is conducted until second hundred generation with the numerical optimum points given in Table 4.1. Conversion of the optimum points into oil field units is shown in Table 4.3. In case where there is enough amount of gas available for injection, the optimum points in Table 4.3 correspond to the optimum gas injection and liquid production rate for each well obtained by gas lift technique. Comparing the results as in [4], the results require higher gas injection rate to obtain maximum oil production for each well (in [4], the corresponding optimum gas injection rates estimated from the GLPCs are about 1.3, 1.41, 0.7, and 1.1 MMSCF/d). This different may result from assumptions in

our model such as contribution of acceleration term, influence of temperature, friction factor, gas phase in reservoir, injection point, and also wellhead pressure.

5. Conclusion and discussion

Here we obtain existence and uniqueness of GLPC if $P_{\text{wh}} < e^{-\alpha_2}$. In addition to that, under condition

$$f < \frac{\alpha_1 - \alpha_2}{16\alpha_5\left(2\alpha_5\left(\alpha_2 + \alpha_1\left(2C_0 - 1\right)\right) + \alpha_1\alpha_4\right)}, \tag{5.1}$$

gas lift performance curve has a maximum point in the interval $(0, q_g^*)$.

In Section 5, we showed that the numerical scheme using genetic algorithm gives a good estimation for solution of minimization problem (4.3). The flexibility and robustness of the method are potential for handling more complicated gas lift problem such as Dual gas lift or Multiwells gas lift.

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