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Research Article

On Some Analytic Functions Defined by a Multiplier Transformation

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We introduce and study a new class of analytic functions defined in the unit disc using a certain multiplier transformation. Some inclusion results and other interesting properties of this class are investigated.

1. Introduction

Let $P_k(\eta)$ be the class of functions $p(z)$ analytic in the unit disc $E = \{z : |z| < 1\}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \eta}{1-\eta} \right| d\theta \le k\pi, \tag{1.1}$$

where $z = re^{i\theta}$, $k \ge 2$, $0 \le \eta < 1$. For $\eta = 0$, we obtain the class P_k defined by Pinchuk [1], and for $\eta = 0$, $k = 2$, we have the class P of functions with positive real part, whereas $P_2(\eta) = P(\eta)$ is the class of functions with positive real part greater than η. We can write (1.1) as

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + (1-2\eta)ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.2}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2, \qquad \int_0^{2\pi} |d\mu(t)| \le k. \tag{1.3}$$

We can also write (1.1), for $p \in P_k(\eta)$ in E, if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad p_1, p_2 \in P(\eta). \tag{1.4}$$

It is known [2] that the class $P_k(\eta)$ is a convex set. Let A be the class of functions f, defined by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1.5}$$

which are analytic in E. By S, K, S^*, and C, we denote the subclasses of A which are univalent, close-to-convex, starlike, and convex in E, respectively. The class A is closed under the Hadamard product or convolution:

$$(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^{m+1}, \tag{1.6}$$

where

$$f(z) = \sum_{m=0}^{\infty} a_m z^{m+1}, \qquad g(z) = \sum_{m=0}^{\infty} b_m z^{m+1}. \tag{1.7}$$

We define the following.

Definition 1.1. Let $f \in$ A. Then, for $\alpha, \beta \geq 0$, $0 \leq \eta < \alpha + \beta \leq 1$, $k \geq 2$, and $z \in E$, $f \in Q_k(\alpha, \beta, \eta)$ if and only if

$$\{\alpha f'(z) + \beta (z f'(z))'\} \in P_k(\eta). \tag{1.8}$$

We note that, for $\beta = 0$ and $k = 2$, $f' \in P(\eta) \subset P$ for $z \in E$ and this implies that f is univalent in E, see [3]. For any real number s, the multiplier transformations I_λ^s of functions $f \in A$ are defined by

$$f_\lambda^s(z) = I_\lambda^s f(z) = z + \sum_{m=2}^{\infty} \left(\frac{m+\lambda}{1+\lambda}\right)^s a_m z^m \quad (\lambda > -1). \tag{1.9}$$

It is obvious that $I_\lambda^s(I_\lambda^t f(z)) = I_\lambda^{s+t} f(z)$ for all real numbers s and t. The operator I_λ^s has been studied by several authors for different choices of s and λ, see [4–7]. It is worth noting that, for s any nonnegative integer and $\lambda = 0$, the operator I_λ^s is the differential operator defined by Sălăgeam [8]. Also the operator I_λ^s is related rather closely to the multiplier transformation discussed by Flett [9]. Using (1.9) and convolution, function $f_{\lambda,\mu}^s$ is defined as follows:

$$f_\lambda^s(z) * f_\lambda^s(z) = \frac{z}{(1-z)^\mu}, \quad z \in E, \ \mu > 0. \tag{1.10}$$

Motivated essentially by Choi et al. operator [10] and Noor integral operator [11–14], Cho and Kim [15] defined the operator $I^s_{\lambda,\mu} : A \to A$ as

$$I^s_{\lambda,\mu} f(z) = f^s_{\lambda,\mu}(z) * f(z), \tag{1.11}$$

where s is real, $\lambda > -1$, $\mu > 0$, and $f \in A$. In particular, $I^0_{0,2} f(z) = z f'(z)$, $I^1_{0,2} f(z) = f(z)$. From (1.10) and (1.11), we have

$$z\left(I^{s+1}_{\lambda,\mu} f(z)\right)' = (\lambda+1) I^s_{\lambda,\mu} f(z) - \lambda I^{s+1}_{\lambda,\mu} f(z), \tag{1.12}$$

$$z\left(I^s_{\lambda,\mu} f(z)\right)' = \mu I^s_{\lambda,\mu+1} f(z) - (\mu-1) I^s_{\lambda,\mu} f(z). \tag{1.13}$$

We now define the following.

Definition 1.2. Let $f \in A$. Then, for s real, $\lambda > 1$, $\mu > 0$,

$$f \in Q^s_k(\lambda,\mu,\alpha,\beta,\eta) \quad \text{iff } I^s_{\lambda,\mu} f(z) \in Q_k(\alpha,\beta,\eta) \text{ for } z \in E. \tag{1.14}$$

2. Preliminary results

Lemma 2.1. *If $h(z)$ is analytic in E with $h(0) = 1$ and if λ_1 is a complex number satisfying* $\operatorname{Re}\lambda_1 \geq 0$ $(\lambda_1 \neq 0)$, *then* $\{h(z) + \lambda_1 z h'(z)\} \in P_k(\delta)$, $0 \leq \delta < 1$, *implies* $h(z) \in P_k(\delta + (1-\delta)(2\gamma - 1)$ *and*

$$\gamma = \int_0^1 \left(1 + t^{\operatorname{Re}\lambda_1}\right)^{-1} dt, \tag{2.1}$$

where γ is an increasing function of $\operatorname{Re}\lambda_1$ *and* $1/2 \leq \gamma < 1$. *The estimate is sharp.*

Proof. Let $h(z) = (k/4 + 1/2)h_1(z) - (k/4 - 1/2)h_2(z)$, $h(z)$ is analytic in E with $h(0) = 1$. Then, $h(z) + \lambda_1 z h'(z) = (k/4 + 1/2)[h_1(z) + \lambda_1 z h_1'(z)] - (k/4 - 1/2)[h_2(z) + \lambda_1 z h_2'(z)]$. Since $[h(z) + \lambda_1 z h'(z)] \in P_k(\delta)$, we use (1.4) to have $[h_i(z) + \lambda_1 z h_i'(z)] \in P(\delta)$, $i = 1,2$. We now apply a lemma in [16] to conclude that $h_i \in P(\delta_1)$, $i = 1,2$, and $\delta_1 = \delta + (1-\delta)(2\gamma - 1)$, where γ is given by (2.1) and it is an increasing function of $\operatorname{Re}\lambda_1$ with $1/2 \leq \gamma < 1$. Consequently $h \in P_k(\delta_1)$ in E. □

Lemma 2.2 [17]. *If $p(z)$ is analytic in E with $p(0) = 1$, then, for any function F, analytic in E, the function $p * F$ takes values in the convex hull of image of E under F.*

Lemma 2.3. *Let $\beta_1 < 1$. If the function p is analytic in E, with $p(0) = 1$, then $p \in P_k(\beta_2)$, $\beta_2 = (2\beta_1 - 1) + 2(1-\beta_1)\ln 2$, $z \in E$. This result is sharp.*

Proof. The proof is immediate when we use (1.4) and a similar result for the class $P(\beta_2)$ in [18]. □

Lemma 2.4. *For $\eta_1 \le 1$ and $\eta_2 \le 1, P_k(\eta_1) * P_k(\eta_2) \subset P_k(1-2(1-\eta_1)(1-\eta_2))$. This result is sharp.*

Proof. Let $H \in P_k(\eta_1)$, $p \in P_k(\eta_2)$. Then, using (1.4), we can write

$$(H*p)(z) = \left(\frac{k}{4}+\frac{1}{2}\right)[(H_1*p_1)(z)] - \left(\frac{k}{4}-\frac{1}{2}\right)[(H_2*p_2)(z)],$$
$$H_i \in P(\eta_1), \quad p_i \in P(\eta_2), \quad i = 1,2. \tag{2.2}$$

Now using a result from [19], we have, for $i = 1,2$,

$$(H_i * p_i) \in P(\eta), \quad \eta = 1-2(1-\eta_1)(1-\eta_2). \tag{2.3}$$

This result is shown to be sharp in [19] and consequently $(H*p) \in P_k(\eta)$. □

3. Main results

Theorem 3.1. *$Q_k^s(\lambda,\mu,\alpha,\beta,\eta) \subset Q_k^s(\lambda,\mu,1,0,\sigma)$ for*

$$\sigma = \sigma_1 + (1-\sigma_1)(2\sigma_2 - 1), \quad \sigma_1 = \frac{\eta}{\alpha+\beta},$$
$$\sigma_2 = \int_0^1 \left(1+t^{\beta/(\alpha+\beta)}\right)^{-1} dt, \quad \text{with } \frac{1}{2} \le \sigma_2 \le 1. \tag{3.1}$$

Proof. Let $f \in Q_k^s(\lambda,\mu,\alpha,\beta,\eta)$. Then, by definition it follows that

$$\{\alpha(I_{\lambda,\mu}^s f)' + \beta(z(I_{\lambda,\mu}^s f)')'\} \in P_k(\eta), \quad z \in E. \tag{3.2}$$

Set $(I_{\lambda,\mu}^s f(z))' = p(z)$. Then p is analytic in E with $p(0) = 1$ and for $z \in E$,

$$\left\{\frac{\alpha(I_{\lambda,\mu}^s f(z))' + \beta(z(I_{\lambda,\mu}^s f(z))')' - \eta}{\alpha+\beta-\eta}\right\}$$
$$= \left\{\frac{\alpha+\beta}{\alpha+\beta-\eta}p(z) + \frac{\beta}{\alpha+\beta-\eta}zp'(z) - \frac{\eta}{\alpha+\beta-\eta}\right\} \in P_k. \tag{3.3}$$

From (1.4) and (3.4), we have, for $i = 1,2$,

$$\left[\frac{\alpha+\beta}{\alpha+\beta-\eta}p_i(z) + \frac{\beta}{\alpha+\beta-\eta}zp_i'(z) - \frac{\eta}{\alpha+\beta-\eta}\right] = h_i(z) \in P. \tag{3.4}$$

By putting $\sigma_1 = \eta/(\alpha+\beta)$, we see that

$$p_i(z) + \frac{\beta}{\alpha+\beta}zp_i'(z) = (1-\sigma_1)h_i(z) + \sigma_1 = H_i(z) \in P(\sigma_1). \tag{3.5}$$

Now using Lemma 2.1, we obtain $p_i \in P(\sigma)$, where σ is given by (3.1). Therefore, $(I_{\lambda,\mu}^s f)' \in P_k(\sigma)$ and consequently $f \in Q_k^s(\lambda,\mu,1,0,\sigma)$ in E. □

Remark 3.2. By writing $\sigma_1 = \eta/(\alpha+\beta)$, $\alpha_1 = \alpha/(\alpha+\beta)$, we can deduce from Definition 1.2 that $f \in Q_k^s(\lambda,\mu,\alpha,\beta,\eta)$, if and only if, for $0 \le \alpha_1 \le 1$,

$$\left[\alpha_1 (I_{\lambda,\mu}^s f)' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f)')'\right] \in P_k(\sigma_1), \quad z \in E. \tag{3.6}$$

In this case, we say that $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ in E.

Theorem 3.3. *Let s be real, $\lambda > -1$, $\mu > 0$. Then,*

$$Q_k^s(\lambda,\mu+1,\alpha_1,\sigma_1) \subset Q_k^s(\lambda,\mu,\alpha_1,\delta_1) \subset Q_k^{s+1}(\lambda,\mu,\alpha_1,\delta_2), \tag{3.7}$$

where α_1 and σ_1 are as defined in Remark 3.2 and

$$\delta_1 = \sigma_1 + (1-\sigma_1)(2\eta_1 - 1), \quad \eta_1 = \int_0^1 (1+t^{1/\mu})^{-1} dt, \tag{3.8}$$

$$\delta_2 = \delta_1 + (1-\delta_1)(2\eta_2 - 1), \quad \eta_2 = \int_0^1 (1+t^{1/(\lambda+1)})^{-1} dt. \tag{3.9}$$

Proof. We first show that $Q_k^s(\lambda,\mu+1,\alpha_1,\sigma_1) \subset Q_k^s(\lambda,\mu,\alpha_1,\delta_1)$.

Let $f \in Q_k^s(\lambda,\mu+1,\alpha_1,\sigma_1)$ and set

$$p(z) = \alpha_1\left[(I_{\lambda,\mu}^s f(z))'\right] + (1-\alpha_1)\left[(zI_{\lambda,\mu}^s f(z)')'\right]. \tag{3.10}$$

From (1.13) and (3.10), we have, for $z \in E$,

$$\left\{\alpha_1 (I_{\lambda,\mu+1}^s f(z))' + (1-\alpha_1)(z(I_{\lambda+\mu+1} f(z))')'\right\} = \left\{p(z) + \frac{1}{\mu} z p'(z)\right\} \in P_k(\sigma_1) \tag{3.11}$$

and, on using (1.4), it follows that $\operatorname{Re}\{p_i(z) + (1/\mu) z p_i'(z)\} > \sigma_1$, $z \in E$, $i = 1,2$.

Now, applying Lemma 2.1, we have $\operatorname{Re} p_i(z) > \delta_1$, $i = 1,2$, where δ_1 is given by (3.8). This implies $p \in P_k(\delta_1)$ for $z \in E$ and hence $f \in Q_k^s(\lambda,\mu,\alpha_1,\delta_1)$ in E. To prove $Q_k^s(\lambda,\mu,\alpha_1,\delta_1) \subset Q_k^{s+1}(\lambda,\mu,\alpha_1,\delta_2)$, we proceed as follows. Set

$$\left\{\alpha_1 (I_{\lambda,\mu}^{s+1} f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^{s+1} f(z))')'\right\} = h(z). \tag{3.12}$$

Then, using (1.12), we have

$$\left\{\alpha_1 (I_{\lambda,\mu}^s f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f(z))')'\right\} = \left\{h(z) + \frac{1}{\lambda+1} z h'(z)\right\} \in P_k(\delta_1). \tag{3.13}$$

With similar argument as detailed above, we obtain the required result. □

Theorem 3.4. *The class $Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ is closed under the convolution with a convex function. That is, if $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ and $\phi \in C$ for $z \in E$, then $(\phi * f) \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$.*

Proof. Let $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$. Consider

$$\begin{aligned}
&\alpha_1\left(I_{\lambda,\mu}^s(\phi*f)(z)\right)' + (1-\alpha_1)\left(z\left(I_{\lambda,\mu}^s(\phi*f)(z)\right)'\right)' \\
&\quad = \alpha_1\left(f_{\lambda,\mu}^s(z)*(\phi*f)(z)\right)' + (1-\alpha_1)\left(z\left(f_{\lambda,\mu}^s(z)*(\phi*f)(z)\right)'\right)' \\
&\quad = \alpha_1\left(\phi(z)*f_{\lambda,\mu}^s(z)*f(z)\right)' + (1-\alpha_1)\left(z\left(\phi(z)*f_{\lambda,\mu}^s(z)*f(z)\right)'\right)' \\
&\quad = \frac{\phi(z)}{z}*\left\{\alpha_1\left(I_{\lambda,\mu}^s f(z)\right)' + (1-\alpha_1)\left(z\left(I_{\lambda,\mu}^s f(z)\right)'\right)'\right\} \\
&\quad = \left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\phi(z)}{z}*h_1(z)\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\phi(z)}{z}*h_2(z)\right],
\end{aligned} \tag{3.14}$$

where $\phi(z)/z \in P(1/2)$ and $h_i \in P(\sigma_1)$. Using Lemma 2.2, we see that $[(\phi(z)/z)*h_i(z)] \in P(\sigma_1)$ and consequently $h \in P_k(\sigma_1)$, which implies that $\phi*f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$; the proof is complete. □

Corollary 3.5. *The class $Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ is invariant under the following integral operators:*

(i) $f_1(z) = \int_0^z (f(t)/t)dt$,

(ii) $f_2(z) = (2/z)\int_0^z f(t)dt$ *(Libera's operator [20])*,

(iii) $f_3(z) = \int_0^z (f(t) - f(xt)/(t-xt))dt, |x| \le 1, x \ne 1$,

(iv) $f_4(z) = ((1+c)/z^c)\int_0^z t^{c-1} f(t)dt, \operatorname{Re} c > 0$.

One may write (see [21, 22])

$$\begin{aligned}
f_1(z) &= f(z)*\phi_1(z), & f_2(z) &= f(z)*\phi_2(z), \\
f_3(z) &= f(z)*\phi_3(z), & f_4(z) &= f(z)*\phi_4(z),
\end{aligned} \tag{3.15}$$

where ϕ_i, $i = 1,2,3,4$, are convex and

$$\begin{aligned}
\phi_1(z) &= -\log(1-z) = \sum_{n=1}^{\infty}\frac{1}{n}z^n, \\
\phi_2(z) &= \frac{-2[z+\log(1-z)]}{z} = \sum_{n=1}^{\infty}\frac{2}{n+1}z^n, \\
\phi_3(z) &= \frac{1}{1-x}\log\left[\frac{1-xz}{1-z}\right] = \sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)^n}z^n, \quad |x| \le 1,\ x \ne 1, \\
\phi_4(z) &= \sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n, \quad \operatorname{Re} c > 0.
\end{aligned} \tag{3.16}$$

Now, the result follows by applying Theorem 3.4. Let μ_1 and μ_2 be linear operators defined on the class S as follows:

$$\mu_1(f(z)) = zf'(z), \qquad \mu_2(f(z)) = \frac{[f(z)+zf'(z)]}{2} \quad \text{(Livingston's operator [23])}. \tag{3.17}$$

Then, both of these operators can be written as a convolution operator [21], given by $\mu_i(f) = h_i * f$, $i = 1,2$, where

$$h_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}, \qquad h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2}. \tag{3.18}$$

It can easily be verified that the radius of convexity $r_c(h_1) = 2 - \sqrt{3}$ and $r_c(h_2) = 1/2$. These facts together with Theorem 3.4 yield the following.

Theorem 3.6. *Let $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$. Then,*

$$\begin{aligned} \mu_1(f) &= (f * h_1) \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1), \quad \text{for } |z| < 2 - \sqrt{3}, \\ \mu_2(f) &= (f * h_2) \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1), \quad \text{for } |z| < \frac{1}{2}. \end{aligned} \tag{3.19}$$

Theorem 3.7. *Let $0 \le \alpha_1 < \alpha_2$. Then, $Q_k^s(\lambda,\mu,\alpha_1,\sigma_1) \subset Q_k^s(\lambda,\mu,\alpha_2,\sigma_1)$.*

Proof. If $\alpha_1 = 0$, the result is obvious. Therefore, we assume that $\alpha_1 > 0$ and $f \in Q_k^s(\lambda,\mu,\alpha_2,\sigma_1)$. Let $(I_{\lambda,\mu}^s f(z))' = H_1(z)$. Then, by Theorem 3.1, $H_1 \in P_k(\sigma_1)$. Also, let

$$\left\{\alpha_1 (I_{\lambda,\omega}^s f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f(z))')'\right\} = H_2(z), \quad H_2 \in P_k(\sigma_1) \text{ in } E. \tag{3.20}$$

Now,

$$\begin{aligned} \alpha_2 (I_{\lambda,\mu}^s f(z))' + (1-\alpha_2)(z(I_{\lambda,\mu}^s f(z))')' &= \frac{\alpha_2 - \alpha_1}{(1-\alpha_1)} H_1(z) + \frac{(1-\alpha_2)}{(1-\alpha_1)} H_2(z) \\ &= \frac{(\alpha_2 - \alpha_1)}{(1-\alpha_1)} H_1(z) + \left(1 - \frac{\alpha_2 - \alpha_1}{(1-\alpha_1)}\right) H_2(z). \end{aligned} \tag{3.21}$$

Since $H_1, H_2 \in P_k(\sigma_1)$ and $P_k(\sigma_1)$ is a convex set, see [2], we obtain the required result. □

Theorem 3.8. *Let $f_i \in Q_k^s(\lambda,\mu,\alpha_1,\zeta_i)$, $i = 1,2$, and let $\Psi = f_1 * f_2$. Then, $\Psi(z)/z \in Q_k^s(\lambda,\mu,1,\zeta)$ for $z \in E$, where $\zeta = 1 - \delta(1-\delta_1)(1-\delta_2)(\ln 2 - 1)^2$ and*

$$\delta_i = \zeta_i + (1-\zeta_i)(2m-1). \tag{3.22}$$

Proof. Since $f_i \in Q_k^s(\lambda,\mu,\alpha_1,\zeta_i)$, it follows from Theorem 3.1 that $f_i \in Q_k^s(\lambda,\mu,1,\delta_i)$, $\delta_i = \zeta_i + (1-\zeta_i)(2m-1)$, and

$$m = \int_0^1 (1 + t^{(1-\alpha)})^{-1} dt. \tag{3.23}$$

Now,

$$\begin{aligned} \left(z(I_{\lambda,\mu}^s \Psi(z))'\right)' &= I_{\lambda,\mu}^s\left[(\Psi'(z) + z\psi''(z)\right] = \left(z(I_{\lambda,\mu}^s(f_1' * f_2)(z))\right)' \\ &= I_{\lambda,\mu}^s\left[(f_1'(z) * f_2'(z))\right] = (I_{\lambda,\mu}^s f_1(z))' * (I_{\lambda,\mu}^s f_2(z))'. \end{aligned} \tag{3.24}$$

Since $f_i \in Q_k^s(\lambda,\mu,1,\delta_i)$, it follows, by Lemma 2.4, that $\{\Psi'(z)+z\Psi''(z)\} \in Q_k^s(\lambda,\mu,1,\delta)$, where

$$\delta = 1-2(1-\delta_1)(1-\delta_2). \tag{3.25}$$

From (3.25) and Lemma 2.3, we have

$$\Psi'(z) \in Q_k^s(\lambda,\mu,1,\{1+4(1-\delta_1)(1-\delta_2)(\ln 2-1)\}). \tag{3.26}$$

From (3.26) and Lemma 2.3, again, we have

$$\frac{\Psi(z)}{z} \in Q_k^s(\lambda,\mu,1,\{1-\delta(1-\delta_1)(1-\delta_2)(\ln 2-1)^2\}), \quad z \in E. \tag{3.27}$$

We now consider the converse case of Theorem 3.1 as follows. □

Theorem 3.9. *Let $f \in Q_k^s(\lambda,\mu,1,\sigma)$. Then, $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma)$, $0<\alpha_1 \le 1$, for $|z|<r_{\alpha_1}$ $(\alpha_1 \ne 1/2)$, where*

$$r_{\alpha_1} = \frac{1}{\{2(1-\alpha_1)+\sqrt{4\alpha_1^2-6\alpha_1+3}\}}. \tag{3.28}$$

This result is sharp.

Proof. Let $\phi_{\alpha_1}(z) = \alpha_1(I_{\lambda,\mu}^s f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f(z))')'$. Then,

$$\phi_{\alpha_1}(z) = \frac{k_{\alpha_1}(z)}{z} * (I_{\lambda,\mu}^s f(z))', \quad \text{where } k_{\alpha_1}(z) = \alpha_1 \frac{z}{1-z} + (1-\alpha_1)\frac{z}{(1-z)^2}. \tag{3.29}$$

It is known [23] that the function k_{α_1} is convex for $|z|<r_{\alpha_1}$, where r_{α_1} is given by (3.28) and this radius is sharp and consequently, for $|z|<r_{\alpha_1}$, by a well-known result, $k_{\alpha_1} \in P(1/2)$. Thus, using Lemma 2.2, and the given fact that $f \in Q_k^s(\lambda,\mu,1,\sigma)$, we obtain the required result. □

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