Research Article

Integral Transforms of Fourier Cosine and Sine Generalized Convolution Type

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Integral transforms of the form $f(x) \mapsto g(x) = (1 - d^2/dx^2)\{\int_0^\infty k_1(y)[f(|x+y-1|) + f(|x-y+1|) - f(x+y+1) - f(|x-y-1|)]dy + \int_0^\infty k_2(y)[f(x+y) + f(|x-y|)]dy\}$ from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$, $(1 \leq p \leq 2, p^{-1} + q^{-1} = 1)$ are studied. Watson's and Plancherel's theorems are obtained.

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1. Introduction

Let F_c be the Fourier cosine transform [1]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y) dy, \quad (1.1)$$

and let F_s be the Fourier sine transform [1]

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y) dy. \quad (1.2)$$

In 1941, Churchill introduced the convolution of two functions f and g for the Fourier cosine transform

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(x+y) + g(|x-y|)]dy, \quad x > 0, \quad (1.3)$$

and proved the following factorization equality [2]:

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y). \quad (1.4)$$

Using the factorization property (1.4), one can easily solve the integral equation with the Toeplitz-plus-Hankel kernel

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(|x-y|)]f(y)dy = g(x) \quad (1.5)$$

in case the Toeplitz kernel $k_2(x)$ and the Hankel kernel $k_1(x)$ are the same [3, 4]. The general case is still open.

The convolution of two functions f and g with the weight function $\gamma(y) = \sin y$ for the Fourier sine transform was introduced by Kakichev in [5]

$$\begin{aligned} (f \overset{\gamma}{\underset{F_s}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u) [\operatorname{sign}(x+u-1)g(|x+u-1|) + \operatorname{sign}(x-u+1)g(|x-u+1|) \\ - g(x+u+1) - \operatorname{sign}(x-u-1)g(|x-u-1|)]du, \quad x > 0, \end{aligned} \quad (1.6)$$

where the following factorization property has been established:

$$F_s(f \overset{\gamma}{\underset{F_s}{*}} g)(y) = \sin y (F_s f)(y) (F_s g)(y). \quad (1.7)$$

Further properties of this convolution have been studied in [6].

Churchill was also the first author who introduced the generalized convolution for two different integral transforms. Namely, in 1941, he defined the generalized convolution of two functions f and g for the Fourier sine and cosine transforms

$$(f \overset{*}{\underset{1}{*}} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u) [g(|x-u|) - g(x+u)]du, \quad x > 0, \quad (1.8)$$

and proved the following factorization identity [7]:

$$F_s(f \overset{*}{\underset{1}{*}} g)(y) = (F_s f)(y) \cdot (F_c g)(y). \quad (1.9)$$

It is easy to see that the integral equation with the Toeplitz-plus-Hankel kernel (1.5) can be written in the form

$$f(x) + \sqrt{2\pi} (f \overset{*}{\underset{F_c}{*}} h_1)(x) + \sqrt{2\pi} (f \overset{*}{\underset{1}{*}} h_2)(x) = g(x), \quad (1.10)$$

where $h_1 = (1/2)(k_1 + k_2)$ and $h_2 = (1/2)(k_2 - k_1)$. So studying generalized convolutions may shed light on how to solve the integral equation with the Toeplitz-plus-Hankel kernel (1.5) in closed form.

In 1998, Kakichev and Thao proposed a constructive method for defining a generalized convolution for three arbitrary integral transforms (see [8]). For example, for the Fourier cosine and Fourier sine transforms, the following convolution has been introduced in [9]:

$$(f \overset{*}{\underset{2}{*}} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u) [\operatorname{sign}(u-x)g(|u-x|) + g(u+x)]du, \quad x > 0. \quad (1.11)$$

For this convolution, the following factorization equality holds [9]:

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y). \quad (1.12)$$

Another generalized convolution with a weight function $\gamma(y) = \sin y$ for the Fourier cosine and sine transforms has been studied in [10]

$$\begin{aligned} (f \underset{2}{*}^\gamma g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u) [g(|x+u-1|) + g(|x-u+1|) \\ - g(x+u+1) - g(|x-u-1|)] du, \quad x > 0. \end{aligned} \quad (1.13)$$

It satisfies the factorization property [10]

$$F_c(f \underset{2}{*}^\gamma g)(y) = \sin y (F_s f)(y)(F_c g)(y). \quad (1.14)$$

In any convolution of two functions f and g , if we fix one function, say g , as the kernel, and allow the other function f vary in a certain function space, we will get an integral transform $f \mapsto f * g$. The most famous integral transforms constructed by that way are the Watson convolution and the Mellin convolution and the Mellin transform [11]

$$f(x) \mapsto g(x) = \int_0^\infty k(xy) f(y) dy. \quad (1.15)$$

Recently, a class of integral transforms that is related to the generalized convolution (1.11) has been introduced and investigated in [12]. In this paper, we will consider a class of integral transform which has a connection with the generalized convolution (1.13), namely, the transforms of the form

$$\begin{aligned} f(x) \mapsto g(x) = \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty k_1(y) [f(|x+y-1|) + f(|x-y+1|) \right. \\ \left. - f(x+y+1) - f(|x-y-1|)] dy \right. \\ \left. + \int_0^\infty k_2(y) [f(x+y) + f(|x-y|)] dy \right\}, \quad x > 0. \end{aligned} \quad (1.16)$$

We show that under certain conditions on the kernels k_1 and k_2 , transform (1.16) admits an inverse of similar form. We find conditions on the kernels k_1 and k_2 when transform (1.16) defines a bounded operator from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$ ($1 \leq p \leq 2$, $p^{-1} + q^{-1} = 1$). Moreover, Watson- and Plancherel-type theorems for transforms (1.16) in $L_2(\mathbb{R}_+)$ are also obtained.

2. A Watson-type theorem

LEMMA 2.1. *Let $f, g \in L_2(\mathbb{R}_+)$. Then for any $x > 0$, the following identity holds:*

$$\begin{aligned} \int_0^\infty f(u) [g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)] du \\ = 2\sqrt{2\pi} F_c(\sin y (F_s f)(y)(F_c g)(y))(x). \end{aligned} \quad (2.1)$$

Proof. Let f_1 be the odd extension of f from $\mathbb{R}_+$ to $\mathbb{R}$ and g_1 the even extension of g from $\mathbb{R}_+$ to $\mathbb{R}$. Then let Ff_1 is an odd function while Fg_1 is an even function, where F is the Fourier integral transform

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y) dy. \quad (2.2)$$

On $\mathbb{R}_+$, we have $Ff_1 = -iF_s f$ and $Fg_1 = F_c g$.

The Parseval identity for the Fourier integral transform yields

$$\begin{aligned} & \int_0^\infty f(u) [g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)] du \\ &= \int_0^\infty f(u) g_1(x-u+1) du - \int_0^\infty f(u) g(x+u+1) du \\ & \quad - \int_0^\infty f(u) g(x-u-1) du + \int_0^\infty f(u) g_1(x+u-1) du \\ &= \int_{-\infty}^\infty f_1(u) g_1(x-u+1) du - \int_{-\infty}^\infty f_1(u) g_1(x-u-1) du \\ &= \int_{-\infty}^\infty (Ff_1)(u) (Fg_1)(u) e^{i(x+1)u} du - \int_{-\infty}^\infty (Ff_1)(u) (Fg_1)(u) e^{i(x-1)u} du \\ &= \int_{-\infty}^\infty (Ff_1)(y) (Fg_1)(y) (\cos(x+1)y + i \sin(x+1)y) dy \\ & \quad - \int_{-\infty}^\infty (Ff_1)(y) (Fg_1)(y) (\cos(x-1)y + i \sin(x-1)y) dy. \end{aligned} \quad (2.3)$$

On the other hand, note that $(Ff_1)(y)(Fg_1)(y) \cos(x+1)y$, $(Ff_1)(y)(Fg_1)(y) \cos(x-1)y$ are odd functions in y . Hence, their integrals over $\mathbb{R}$ vanish, and therefore,

$$\begin{aligned} & \int_0^\infty f(u) [g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)] du \\ &= \int_{-\infty}^\infty (Ff_1)(y) (Fg_1)(y) i \sin(x+1)y dy - \int_{-\infty}^\infty (Ff_1)(y) (Fg_1)(y) i \sin(x-1)y dy \\ &= 2i \int_{-\infty}^\infty (Ff_1)(y) (Fg_1)(y) \sin y \cos(xy) dy \\ &= 2\sqrt{2\pi} F_c(\sin y (F_s f)(y) (F_c g)(y))(x). \end{aligned} \quad (2.4)$$

This completes the proof. We assumed that all the integrals over $\mathbb{R}$ are interpreted as Cauchy principal value integrals, if necessary. $\square$

THEOREM 2.2. *Let $k_1, k_2 \in L_2(\mathbb{R}_+)$. Then*

$$|2 \sin y(F_s k_1)(y) + (F_c k_2)(y)| = \frac{1}{\sqrt{2\pi}(1+y^2)}, \quad (2.5)$$

is a necessary and sufficient condition to ensure that the integral transform $f \mapsto g$

$$g(x) := \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty k_1(y) [f(|x+y-1|) + f(|x-y+1|) - f(x+y+1) - f(|x-y-1|)] dy + \int_0^\infty k_2(y) [f(x+y) + f(|x-y|)] dy \right\} \quad (2.6)$$

is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation has the form

$$f(x) = \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty k_1(y) [g(|x+y-1|) + g(|x-y+1|) - g(x+y+1) - g(|x-y-1|)] dy + \int_0^\infty k_2(y) [g(x+y) + g(|x-y|)] dy \right\}. \quad (2.7)$$

Proof

Necessity. Suppose that k_1 and k_2 satisfy condition (2.5). It is well known that $(1+y^2)h(y) \in L_2(\mathbb{R})$, if and only if $(Fh)(x)$, $(d/dx)(Fh)(x)$ and $(d^2/dx^2)(Fh)(x) \in L_2(\mathbb{R})$ ([11, Theorem 68, page 92]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = -F(y^2 h(y))(x). \quad (2.8)$$

In particular, if h is an even or odd function such that $(1+y^2)h(y) \in L_2(\mathbb{R}_+)$, then the following equalities hold:

$$\begin{aligned} \left(1 - \frac{d^2}{dx^2}\right)(F_c h)(x) &= F_c((1+y^2)h(y))(x), \\ \left(1 - \frac{d^2}{dx^2}\right)(F_s h)(x) &= F_s((1+y^2)h(y))(x). \end{aligned} \quad (2.9)$$

Using the factorization equalities for convolutions (1.3), (1.6), we have

$$\begin{aligned} g(x) &= \left(1 - \frac{d^2}{dx^2}\right) F_c(2\sqrt{2\pi} \sin y(F_s k_1)(y)(F_c f)(y) + \sqrt{2\pi}(F_c k_2)(y)(F_c f)(y))(x) \\ &= F_c(\sqrt{2\pi}(1+y^2)(2 \sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y))(x). \end{aligned} \quad (2.10)$$

By virtue of the Parseval equalities for the Fourier cosine and sine transforms $\|f\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)}$ and noting that k_1 and k_2 satisfy condition (2.5), we have

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} &= \left\| \sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y) \right\|_{L_2(\mathbb{R}_+)} \\ &= \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}\quad (2.11)$$

It follows that the transformation (2.6) is unitary.

On the other hand, in view of condition (2.5), $\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))$ is bounded, hence $\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c g)(y) \in L_2(\mathbb{R}_+)$. We have

$$\begin{aligned}g(x) &= F_c(\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y))(x) \\ &\Leftrightarrow (F_c g)(y) = \sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y) \\ &\Leftrightarrow (F_c f)(y) = \sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c g)(y).\end{aligned}\quad (2.12)$$

Using formula (2.9), we obtain

$$\begin{aligned}f(x) &= F_c(\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c g)(y))(x) \\ &= \left(1 - \frac{d^2}{dx^2}\right) F_c(2\sqrt{2\pi} \sin y F_s k_1(y)(F_c g)(y) + \sqrt{2\pi}(F_c k_2)(y)(F_c g)(y)) \\ &= \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty k_1(y) [g(|x+y-1|) + g(|x-y+1|) - g(x+y+1) \right. \\ &\quad \left. - g(|x-y-1|)] dy + \int_0^\infty k_2(y) [g(x+y) + g(|x-y|)] dy \right\}.\end{aligned}\quad (2.13)$$

Therefore, the transformation (2.6) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation has the form (2.7).

Sufficiency. If transform (2.6) is unitary, then the Parseval identities for the Fourier cosine and sine transforms yield

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} &= \left\| \sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y) \right\|_{L_2(\mathbb{R}_+)} \\ &= \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}\quad (2.14)$$

The middle equality is possible if and only if k_1 and k_2 satisfy condition (2.5). This completes the proof of the theorem. $\square$

Let $h_1, h_2 \in L_2(\mathbb{R}_+)$ satisfy

$$|(F_s h_1)(y)(F_s h_2)(y)| = \frac{1}{(1+y^2)(1+\sin^2 y)}, \quad (2.15)$$

and let k_1, k_2 be defined by

$$k_1(x) = \frac{1}{2\sqrt{2\pi}} (h_1 \underset{F_s}{*} h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}} (h_1 \underset{2}{*} h_2)(x). \quad (2.16)$$

Then $k_1, k_2 \in L_2(\mathbb{R}_+)$ and from (1.7) and (1.12), we have

$$\begin{aligned}
 & |2 \sin y (F_s k_1)(y) + (F_c k_2)(y)| \\
 &= \left| \frac{1}{\sqrt{2\pi}} \sin^2 y (F_s h_1)(y) (F_s h_2)(y) + \frac{1}{\sqrt{2\pi}} (F_s h_1)(y) (F_s h_2)(y) \right| \\
 &= \left| \frac{1}{\sqrt{2\pi}} (1 + \sin^2 y) (F_s h_1)(y) (F_s h_2)(y) \right| = \frac{1}{\sqrt{2\pi} (1 + y^2)}.
 \end{aligned} \tag{2.17}$$

Thus k_1 and k_2 satisfy condition (2.5).

3. A Plancherel-type theorem

In order to examine the Plancherel-type theorem, we will need the following lemma.

LEMMA 3.1. *Let f and g be $L_2(\mathbb{R}_+)$ functions, then*

$$\begin{aligned}
 & \int_0^\infty f(y) [g(|x+y-1|) + g(|x-y+1|) - g(x+y+1) - g(|x-y-1|)] dy \\
 &= \int_0^\infty g(y) [f(x+y+1) + \text{sign}(x-y+1)f(|x-y+1|) \\
 &\quad - \text{sign}(x-y-1)f(|x-y-1|) - \text{sign}(x+y-1)f(|x+y-1|)] dy,
 \end{aligned} \tag{3.1}$$

$$\int_0^\infty f(y) [g(x+y) + g(|x-y|)] dy = \int_0^\infty g(y) [f(x+y) + f(|x-y|)] dy. \tag{3.2}$$

Proof. Again, let f_1 be the odd extension of f from $\mathbb{R}_+$ to $\mathbb{R}$ and $g_1(x) = g(|x|)$ the even extension of g from $\mathbb{R}_+$ to $\mathbb{R}$. By the Parseval equality, we have

$$\begin{aligned}
 & \int_0^\infty f(y) [g(|x+y-1|) + g(|x-y+1|) - g(x+y+1) - g(|x-y-1|)] dy \\
 &= \int_0^\infty f(y) g(|x+y-1|) dy + \int_0^\infty f(y) g(|x-y+1|) dy \\
 &\quad - \int_0^\infty f(y) g(x+y+1) dy - \int_0^\infty f(y) g(|x-y-1|) dy \\
 &= - \int_{-\infty}^0 f_1(y) g_1(x-y-1) dy + \int_0^\infty f_1(y) g_1(x-y+1) dy \\
 &\quad + \int_{-\infty}^0 f_1(y) g_1(x-y+1) dy - \int_0^\infty f_1(y) g_1(x-y-1) dy \\
 &= \int_{-\infty}^\infty (F f_1)(u) (F g_1)(u) e^{i(x+1)u} du - \int_{-\infty}^\infty (F f_1)(u) (F g_1)(u) e^{i(x-1)u} du
 \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} g_1(y) f_1(x-y+1) dy - \int_{-\infty}^{\infty} g_1(y) f_1(x-y-1) dy \\
&= \int_0^{\infty} g_1(y) f_1(x-y+1) dy + \int_0^{\infty} g_1(y) f_1(x+y+1) dy \\
&\quad - \int_0^{\infty} g_1(y) f_1(x-y-1) dy - \int_0^{\infty} g_1(y) f_1(x+y-1) dy \\
&= \int_0^{\infty} g(y) [f(x+y+1) + \text{sign}(x-y+1)f(|x-y+1|) \\
&\quad - \text{sign}(x-y-1)f(|x-y-1|) - \text{sign}(x+y-1)f(|x+y-1|)] dy.
\end{aligned} \tag{3.3}$$

Then formula (3.1) holds. Formula (3.2) follows easily from formula (1.4)

$$\begin{aligned}
\int_0^{\infty} f(y) [g(x+y) + g(|x-y|)] dy &= \sqrt{2\pi} F_c[(F_c f)(y)(F_c g)(y)](x) \\
&= \sqrt{2\pi} F_c[(F_c g)(y)(F_c f)(y)](x) \\
&= \int_0^{\infty} g(y) [f(x+y) + f(|x-y|)] dy.
\end{aligned} \tag{3.4}$$

The lemma has been proved. $\square$

THEOREM 3.2. *Let k_1, k_2 be functions satisfying condition (2.5) and suppose that $K_1(x) = (1 - d^2/dx^2)k_1(x)$ and $K_2(x) = (1 - d^2/dx^2)k_2(x)$ are locally bounded. Let $f \in L_2(\mathbb{R}_+)$ and for each positive integer N , put*

$$\begin{aligned}
g_N(x) &= \int_0^{\infty} K_1(y) [f^N(|x+y-1|) + f^N(|x-y+1|) - f^N(x+y+1) \\
&\quad - f^N(|x-y-1|)] dy + \int_0^{\infty} K_2(y) [f^N(x+y) + f^N(|x-y|)] dy,
\end{aligned} \tag{3.5}$$

where $f^N = f \cdot \chi_{(0,N)}$, the restriction of f over $(0,N)$. Then

- (1) $g_N \in L_2(\mathbb{R}_+)$ and as $N \rightarrow \infty$, g_N converges in $L_2(\mathbb{R}_+)$ norm to a function $g \in L_2(\mathbb{R}_+)$ with $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$;
- (2) put $g^N = g \cdot \chi_{(0,N)}$, then

$$\begin{aligned}
f_N(x) &= \int_0^{\infty} K_1(y) [g^N(|x+y-1|) + g^N(|x-y+1|) - g^N(x+y+1) \\
&\quad - g^N(|x-y-1|)] dy + \int_0^{\infty} K_2(y) [g^N(x+y) + g^N(|x-y|)] dy
\end{aligned} \tag{3.6}$$

belongs to $L_2(\mathbb{R}_+)$ and converges in $L_2(\mathbb{R}_+)$ norm to f as $N \rightarrow \infty$.

Remark 3.3. Because of the definitions of f^N and g^N , these integrals are over finite intervals and therefore converge.

Proof. Applying the identities (3.1) and (3.2) in Lemma 3.1, we have

$$\begin{aligned}
 g_n(x) &= \int_0^\infty K_1(y)[f^N(|x+y-1|) + f^N(|x-y+1|) - f^N(x+y+1) - f^N(|x-y-1|)]dy \\
 &\quad + \int_0^\infty K_2(y)[f^N(x+y) + f^N(|x-y|)]dy \\
 &= \int_0^\infty f^N(y)[K_1(x+y+1) + \text{sign}(x-y+1)K_1(|x-y+1|) \\
 &\quad - \text{sign}(x-y-1)K_1(|x-y-1|) - \text{sign}(x+y-1)K_1(|x+y-1|)]dy \\
 &\quad + \int_0^\infty f^N(y)[k_1(x+y) + K_1(|x-y|)]dy \\
 &= \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty f^N(u)[k_1(x+u+1) + \text{sign}(x-u+1)k_1(|x-u+1|) \right. \\
 &\quad \left. - \text{sign}(x-u-1)k_1(|x-u-1|) \right. \\
 &\quad \left. - \text{sign}(x+u-1)k_1(|x+u-1|)]du \right. \\
 &\quad \left. + \int_0^\infty f^N(y)[k_1(x+y) + k_1(|x-y|)]dy \right\}. \tag{3.7}
 \end{aligned}$$

It is legitimate to interchange the order of integration and differentiation since the integrals are actually over finite intervals. By applying Lemma 3.1 one more time, we obtain

$$\begin{aligned}
 g_N(x) &= \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty k_1(y)[f^N(|x+y-1|) + f^N(|x-y+1|) \right. \\
 &\quad \left. - f^N(x+y+1) - f^N(|x-y-1|)]dy \right. \\
 &\quad \left. + \int_0^\infty k_2(y)[f^N(x+y) + f^N(|x-y|)]dy \right\}. \tag{3.8}
 \end{aligned}$$

From this and in view of Theorem 2.2, we conclude that $g_N \in L_2(\mathbb{R}_+)$. Let g be the transform of f under the transformation (2.6). Then Theorem 2.2 guarantees that $g \in L_2(\mathbb{R}_+)$, $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$, and the reciprocal formula (2.7) holds. For $g - g_N$, we have

$$\begin{aligned}
 (g - g_N)(x) &= \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty k_1(y)[(f - f^N)(|x+y-1|) + (f - f^N)(|x-y+1|) \right. \\
 &\quad \left. - (f - f^N)(x+y+1) - (f - f^N)(|x-y-1|)]dy \right. \\
 &\quad \left. + \int_0^\infty k_2(y)[(f - f^N)(x+y) + (f - f^N)(|x-y|)]dy \right\}. \tag{3.9}
 \end{aligned}$$

Again by Theorem 2.2, $(g - g_N)(x) \in L_2(\mathbb{R}_+)$ and

$$\|g - g_N\|_{L_2(\mathbb{R}_+)} = \|f - f^N\|_{L_2(\mathbb{R}_+)}. \quad (3.10)$$

And since $\|f - f^N\|_{L_2(\mathbb{R}_+)} \rightarrow 0$ as $N \rightarrow \infty$ then g_N converges in $L_2(\mathbb{R}_+)$ norm to $g \in L_2(\mathbb{R}_+)$.

Similarly, one can obtain the second part of the theorem. $\square$

THEOREM 3.4. *Let k_1 and k_2 be functions satisfying condition (2.5) and suppose that $K_1(x)$ and $K_2(x)$ defined as in the previous theorem are bounded on $\mathbb{R}_+$. Let $1 \leq p \leq 2$ and q be its conjugate exponent $1/p + 1/q = 1$. Then the transformation $f \mapsto g$, where g is defined by*

$$g(x) = \lim_{N \rightarrow \infty} \left\{ \int_0^\infty K_1(y) [f^N(|x+y-1|) + f^N(|x-y+1|) - f^N(x+y+1) - f^N(|x-y-1|)] dy + \int_0^\infty K_2(y) [f^N(x+y) + f^N(|x-y|)] dy \right\}, \quad (3.11)$$

is a bounded operator from $L_p(\mathbb{R}_+)$ into $L_q(\mathbb{R}_+)$. Here the limit is understood in $L_q(\mathbb{R}_+)$ norm.

Proof. From the boundedness of K_1 and K_2 , it is clear that transformation (3.11) is a bounded operator from $L_1(\mathbb{R}_+)$ into $L_\infty(\mathbb{R}_+)$.

On the other hand, Theorem 3.2 shows that transformation (3.11) defines a bounded operator from $L_2(\mathbb{R}_+)$ into $L_2(\mathbb{R}_+)$. Hence, Riesz's interpolation theorem implies that (3.11) is a bounded operator from $L_p(\mathbb{R}_+)$, $1 \leq p \leq 2$, into $L_q(\mathbb{R}_+)$, where q is the conjugate exponent of p . $\square$

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