Research Article

Some Estimates of Schrödinger-Type Operators with Certain Nonnegative Potentials

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We consider the Schrödinger-type operator $H = (-\Delta)^2 + V^2$, where the nonnegative potential V belongs to the reverse Hölder class B_{q_1} for $q_1 \geq n/2$, $n \geq 5$. The L^p estimates of the operator $\nabla^4 H^{-1}$ related to H are obtained when $V \in B_{q_1}$ and $1 < p \leq q_1/2$. We also obtain the weak-type estimates of the operator $\nabla^4 H^{-1}$ under the same condition of V .

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1. Introduction

In recent years, there has been considerable activity in the study of Schrödinger operators (see [1–4]). In this paper, we consider the Schrödinger-type operator

$$H = (-\Delta)^2 + V^2 \quad \text{on } \mathbb{R}^n, \quad n \geq 5, \quad (1.1)$$

where the potential V belongs to B_{q_1} for $q_1 \geq n/2$. We are interested in the L^p boundedness of the operator $\nabla^4 H^{-1}$, where the potential V satisfies weaker condition than that in [5, Theorem 1, (2)]. The estimates of some other operators related to Schrödinger-type operators can be found in [2, 5].

Note that a nonnegative locally L^q integrable function V on $\mathbb{R}^n$ is said to belong to B_q ($1 < q < \infty$) if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V(x)^q dx \right)^{1/q} \leq C \left(\frac{1}{|B|} \int_B V(x) dx \right) \quad (1.2)$$

holds for every ball B in $\mathbb{R}^n$.

It follows from [3] that the B_q class has a property of “self-improvement”, that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$.

We now give the main results for the operator $\nabla^4 H^{-1}$ in this paper.

Theorem 1.1. *Suppose $V \in B_{q_1}$, $q_1 \geq n/2$. Then for $1 < p \leq q_1/2$ there exists a positive constant C_p such that*

$$\|\nabla^4 H^{-1} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.3)$$

By the proof of Theorem 1.1, we obtain the following weak-type estimate.

Theorem 1.2. *Suppose $V \in B_{q_1}$, $q_1 \geq n/2$. Then for $1 < p \leq q_1/2$ there exists a positive constant C_1 such that*

$$|\{x \in \mathbb{R}^n : |\nabla^4 H^{-1} f(x)| \geq \lambda\}| \leq \frac{C_1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}. \quad (1.4)$$

Under a stronger condition on the potential V , Sugano [5] has obtained the following proposition.

Proposition 1.3. *Suppose $V \in B_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x, V)^2$. Then for $1 < p < \infty$ there exists a positive constant C_p such that*

$$\|\nabla^4 H^{-1} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.5)$$

As a direct consequence of our L^p estimates, we have the following corollary.

Corollary 1.4. *Suppose $V \in B_{q_1}$ for $q_1 \geq n/2$. Assume that $(-\Delta)^2 u + V^2 u = f$ in $\mathbb{R}^n$. Then*

$$\|\nabla^4 u\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 < p \leq \frac{q_1}{2}. \quad (1.6)$$

Throughout this paper, unless otherwise indicated, we will use C to denote constants, which are not necessarily the same at each occurrence. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$ such that $c \leq A/B \leq C$.

2. The auxiliary function $m(x, V)$ and estimates of fundamental solution

In this section, we firstly recall the definition of the auxiliary function $m(x, V)$ and some lemmas about the auxiliary function $m(x, V)$ which have been proven in [3].

Lemma 2.1. *If $V \in B_q$, $q > 1$, then the measure $V(x)dx$ satisfies the doubling condition, that is, there exists $C > 0$ such that*

$$\int_{B(x, 2r)} V(y) dy \leq C \int_{B(x, r)} V(y) dy \quad (2.1)$$

holds for all balls $B(x, r)$ in $\mathbb{R}^n$.

Lemma 2.2. For $0 < r < R < \infty$ and $V \in B_{q_1}$ for $q_1 \geq n/2$, there exists $C > 0$ such that

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq C \left(\frac{r}{R} \right)^{2-n/q_1} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y) dy. \quad (2.2)$$

Assume that $V \in B_{q_1}$, $q_1 \geq n/2$. The auxiliary function $m(x, V)$ is defined by

$$\frac{1}{m(x, V)} \doteq \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n. \quad (2.3)$$

Lemma 2.3. If $r = 1/m(x, V)$, then

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy = 1. \quad (2.4)$$

Moreover,

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \sim 1, \quad \text{iff } r \sim \frac{1}{m(x, V)}. \quad (2.5)$$

Lemma 2.4. There exists $l_0 > 0$ such that for any x and y in $\mathbb{R}^n$,

$$\frac{1}{C} (1 + m(x, V)|x - y|)^{-l_0} \leq \frac{m(x, V)}{m(y, V)} \leq C (1 + m(x, V)|x - y|)^{l_0/(l_0+1)}. \quad (2.6)$$

In particular, $m(x, V) \sim m(y, V)$, if $|x - y| < C/m(x, V)$.

Lemma 2.5. There exists $l_1 > 0$ such that

$$\int_{B(x,R)} \frac{V(y)}{|x - y|^{n-2}} dy \leq \frac{C}{R^{n-2}} \int_{B(x,R)} V(y) dy \leq C (1 + Rm(x, V))^{l_1}. \quad (2.7)$$

Lemma 2.6. There exists $C > 0$, $c > 0$, and $l_0 > 0$ such that, for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} c \{1 + |x - y|m(y, V)\}^{1/(l_0+1)} &\leq 1 + |x - y|m(x, V) \\ &\leq C \{1 + |x - y|m(y, V)\}^{l_0+1}. \end{aligned} \quad (2.8)$$

Refer to [3] for the proof of the above lemmas.

The next lemma has been obtained by Tao and Wang in [6].

Lemma 2.7. *Let $q > s \geq 0$, $q \geq \max\{1, sn/\alpha\}$, $\alpha > 0$, and k be sufficiently large, then there are positive constants k_0 , C , and C_k such that*

$$\begin{aligned} \int_{|x-y|<r} \frac{V(y)^s}{|x-y|^{n-\alpha}} dy &\leq Cr^{\alpha-2s} \{1 + rm(x, V)\}^{sk_0}, \\ \int_{\mathbb{R}^n} \frac{V(y)^s}{\{1 + m(x, V)|x-y|\}^k |x-y|^{n-\alpha}} dy &\leq C_k m(x, V)^{2s-\alpha} \end{aligned} \quad (2.9)$$

for any $r > 0$, $x \in \mathbb{R}^n$, and $V \in B_q$.

In order to prove Theorem 1.1, we need to give the estimates of the fundamental solution of H . Zhong has established the estimates of the fundamental solution of H in [2] when $V(x)$ is a nonnegative polynomial. Recently, Sugano [5] has obtained the polynomial decay estimates of the fundamental solution of H under a weaker condition on V in the following theorem.

Theorem 2.8. *Assume $V \in B_{n/2}$ and let $\Gamma_H(x, y)$ be the fundamental solution of H . For any positive integer N , there exists a constant C_N such that*

$$0 \leq \Gamma_H(x, y) \leq \frac{C_N}{(1 + m(x, V)|x-y|)^N} \frac{1}{|x-y|^{n-4}}. \quad (2.10)$$

3. Proof of the main results

In this section, we will prove Theorems 1.1 and 1.2.

Theorem 3.1. *Suppose $V \in B_{q_1}$, $q_1 \geq n/2$. Then for $1 < p \leq q_1/2$ there exists a positive constant C_p such that*

$$\|V^2 H^{-1} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \quad (3.1)$$

Proof. Let $f \in L^p(\mathbb{R}^n)$ and

$$u(x) = \int_{\mathbb{R}^n} \Gamma_H(x, y) f(y) dy. \quad (3.2)$$

We need to show that

$$\|V^2 u\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (3.3)$$

Write

$$\begin{aligned} u(x) &= \int_{|x-y|<r} \Gamma(x, y) f(y) dy + \int_{|x-y|\geq r} \Gamma(x, y) f(y) dy \\ &= u_1(x) + u_2(x), \end{aligned} \quad (3.4)$$

where $r = 1/m(x, V)$.

Because of the self-improvement of the B_{q_1} class, $V \in B_{q_0}$ for some $q_0 > q_1$, we have

$$\begin{aligned} |u_1(x)| &\leq C \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-4}} dy \\ &\leq C \left(\int_{|x-y|<r} |f(y)|^{q_0/2} dy \right)^{2/q_0} \left(\int_{|x-y|<r} |x-y|^{-(n-4)q'} dy \right)^{1/q'} \\ &= Cr^{4-2n/q_0} \left(\int_{|x-y|<r} |f(y)|^{q_0/2} dy \right)^{2/q_0}, \end{aligned} \quad (3.5)$$

where $1/q' + 2/q_0 = 1$.

Thus,

$$\begin{aligned} &\int_{\mathbb{R}^n} |V^2(y)u_1(y)|^{q_0/2} dy \\ &\leq C \int_{\mathbb{R}^n} \left(\int_{|x-y|<r} |f(y)|^{q_0/2} dy \right) V(x)^{q_0} m(x, V)^{n-2q_0} dx \\ &= C \int_{\mathbb{R}^n} |f(y)|^{q_0/2} \left(\int_{|x-y|<1/m(x, V)} V(x)^{q_0} m(x, V)^{n-2q_0} dx \right) dy. \end{aligned} \quad (3.6)$$

Now, let $R = 1/m(y, V)$. Then

$$\begin{aligned} \int_{|x-y|<1/m(x, V)} V(x)^{q_0} m(x, V)^{n-2q_0} dx &\leq CR^{2q_0-n} \int_{|x-y|<CR} V(x)^{q_0} dx \\ &\leq CR^{2q_0} \left(R^{-n} \int_{|x-y|<CR} V(x) dx \right)^{q_0} \\ &= C \left(\frac{1}{R^{n-2}} \int_{|x-y|<CR} V(x) dx \right)^{q_0} \\ &\leq C, \end{aligned} \quad (3.7)$$

where we used (1.2), Lemmas 2.3 and 2.4.

Hence, we have proved that for some $q_0 > q_1 \geq n/2$,

$$\int_{\mathbb{R}^n} |V^2(x)u_1(x)|^{q_0/2} dx \leq C \int_{\mathbb{R}^n} |f(x)|^{q_0/2} dx. \quad (3.8)$$

By choosing $s = 2$, $\alpha = 4$, and $r = 1/m(x, V)$ in Lemma 2.7, we immediately have

$$\int_{|x-y| < 1/m(x, V)} \frac{V^2(x)}{|x-y|^{n-4}} dx \leq 4^{k_0}. \quad (3.9)$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} |V^2(x)u_1(x)| dx &\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_{|x-y| < 1/m(x, V)} \frac{V^2(x)}{|x-y|^{n-4}} dx \right) dy \\ &\leq C_{k_0} \int_{\mathbb{R}^n} |f(y)| dy. \end{aligned} \quad (3.10)$$

Therefore, by using interpolation we have

$$\|V^2 u_1\|_{L^{p_1}(\mathbb{R}^n)} \leq C \|f\|_{L^{p_1}(\mathbb{R}^n)} \quad \text{for } 1 \leq p_1 \leq \frac{q_0}{2}. \quad (3.11)$$

Then we deal with u_2 .

For $1 < p \leq q_0/2$, by the Hölder inequality,

$$\begin{aligned} |u_2(x)| &\leq C \int_{|x-y| \geq r} \frac{|f(y)| dy}{(1 + |x-y|m(x, V))^N |x-y|^{n-4}} \\ &\leq C \left(\int_{|x-y| \geq r} \frac{|f(y)|^p dy}{(1 + |x-y|m(x, V))^N |x-y|^{n-4}} \right)^{1/p} \\ &\quad \times \left(\int_{|x-y| \geq r} \frac{dy}{(1 + |x-y|m(x, V))^N |x-y|^{n-4}} \right)^{1/p'} \\ &= C r^{4/p'} \left(\int_{|x-y| \geq r} \frac{|f(y)|^p dy}{(1 + |x-y|m(x, V))^N |x-y|^{n-4}} \right)^{1/p}, \end{aligned} \quad (3.12)$$

where $r = 1/m(x, V)$ and we apply the second inequality for $s = 0$ and $\alpha = 4$ in Lemma 2.7 to the last step.

Thus, for $1 \leq p \leq q_0/2$,

$$\begin{aligned} & \int_{\mathbb{R}^n} |V^2(x)u_2(x)|^p dx \\ & \leq C \int_{\mathbb{R}^n} |f(y)|^p \left(\int_{|x-y| \geq 1/m(x,V)} \frac{|V(x)|^{2p} dx}{m(x,V)^{4p-4} (1 + |x-y|m(x,V))^N |x-y|^{n-4}} \right) dy. \end{aligned} \quad (3.13)$$

Fix $y \in \mathbb{R}^n$ and let $R = 1/m(y, V)$. By Lemmas 2.4, 2.6, and 2.7,

$$\begin{aligned} & \int_{|x-y| \geq 1/m(x,V)} \frac{|V(x)|^{2p} dx}{m(x,V)^{4p-4} (1 + |x-y|m(x,V))^N |x-y|^{n-4}} \\ & \leq C \int_{|x-y| \geq 1/m(x,V)} \frac{|V(x)|^{2p} dx}{R^{4-4p} (1 + |x-y|R^{-1})^{N_1} |x-y|^{n-4}} \quad \left(N_1 = \frac{N-4(p-1)l_0}{l_0+1} \right) \\ & \leq C_k \frac{1}{R^{4-4p}} m(y, V)^{4p-4} \\ & \leq C \end{aligned} \quad (3.14)$$

if we choose N large enough.

From this, we have

$$\int_{\mathbb{R}^n} |V^2(x)u_2(x)|^p dx \leq \int_{\mathbb{R}^n} |f(x)|^p dx \quad \text{for } 1 \leq p \leq \frac{q_0}{2}. \quad (3.15)$$

Thus the theorem is proved. $\square$

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose $V \in B_{q_1}$ for some $q_1 \geq n/2$. By Theorem 3.1, we have

$$\|V^2((-\Delta)^2 + V^2)^{-1}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 \leq p \leq \frac{q_1}{2}. \quad (3.16)$$

It follows that

$$\|(-\Delta)^2((-\Delta)^2 + V^2)^{-1}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 \leq p \leq \frac{q_1}{2}. \quad (3.17)$$

Because $\nabla^4(-\Delta)^{-2}$ is a Calderón-Zygmund operator, for $1 < p \leq q_1/2$, we have

$$\|\nabla^4((-\Delta)^2 + V^2)^{-1}f\|_{L^p(\mathbb{R}^n)} \leq C_p \|(-\Delta)^2((-\Delta)^2 + V^2)^{-1}f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \quad (3.18)$$

$\square$

Proof of Theorem 1.2. Note that $\nabla^4(-\Delta)^{-2}$ satisfies

$$|\{x \in \mathbb{R}^n : |\nabla^4(-\Delta)^{-2}f(x)| \geq \lambda\}| \leq \frac{C_1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}. \quad (3.19)$$

Thus, by the proof of Theorem 1.1,

$$\begin{aligned} |\{x \in \mathbb{R}^n : |\nabla^4((-\Delta)^2 + V^2)^{-1}f(x)| \geq \lambda\}| &\leq \frac{C_1}{\lambda} \|(-\Delta)^2((-\Delta)^2 + V^2)^{-1}f\|_{L^1(\mathbb{R}^n)} \\ &\leq \frac{C_1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \quad (3.20)$$

□

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