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Research Article

Three-Dimensional Pseudomanifolds on Eight Vertices

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A normal pseudomanifold is a pseudomanifold in which the links of simplices are also pseudomanifolds. So, a normal 2-pseudomanifold triangulates a connected closed 2-manifold. But, normal d-pseudomanifolds form a broader class than triangulations of connected closed d-manifolds for $d \geq 3$. Here, we classify all the 8-vertex neighbourly normal 3-pseudomanifolds. This gives a classification of all the 8-vertex normal 3-pseudomanifolds. There are 74 such 3-pseudomanifolds, 39 of which triangulate the 3-sphere and other 35 are not combinatorial 3-manifolds. These 35 triangulate six distinct topological spaces. As a preliminary result, we show that any 8-vertex 3-pseudomanifold is equivalent by proper bistellar moves to an 8-vertex neighbourly 3-pseudomanifold. This result is the best possible since there exists a 9-vertex nonneighbourly 3-pseudomanifold which does not allow any proper bistellar moves.

1. Introduction

Recall that a *simplicial complex* is a collection of nonempty finite sets (sets of *vertices*) such that every nonempty subset of an element is also an element. For $i \geq 0$, the elements of size $i+1$ are called the *i-simplices* (or *i-faces*) of the complex.

A simplicial complex is usually thought of as a prescription for construction of a topological space by pasting geometric simplices. The space thus obtained from a simplicial complex K is called the *geometric carrier* of K and is denoted by $|K|$. We also say that K *triangulates* $|K|$. A *combinatorial* 2-*manifold* (resp., *combinatorial* 2-*sphere*) is a simplicial complex which triangulates a closed surface (resp., the 2-sphere S^2).

For a simplicial complex K, the maximum of k such that K has a k-simplex, is called the *dimension* of K. A d-dimensional simplicial complex K is called *pure* if each simplex of K is contained in a d-simplex of K. A d-simplex in a pure d-dimensional simplicial complex is called a *facet*. A d-dimensional pure simplicial complex K is called a *weak pseudomanifold* if each $(d-1)$-simplex of K is contained in exactly two facets of K.

With a pure simplicial complex K of dimension $d \geq 1$, we associate a graph $\Lambda(K)$ as follows. The vertices of $\Lambda(K)$ are the facets of K and two vertices of $\Lambda(K)$ are adjacent if the corresponding facets intersect in a $(d-1)$-simplex of K. If $\Lambda(K)$ is connected, then K is called *strongly connected*. A strongly connected weak pseudomanifold is called a *pseudomanifold*. Thus, for a d-pseudomanifold K, $\Lambda(K)$ is a connected $(d+1)$-regular graph. This implies that K has no proper subcomplex which is also a d-pseudomanifold; (or else, the facets of such a subcomplex would provide a disconnection of $\Lambda(X)$).

For any set V with $\#(V) = d+2$ $(d \geq 0)$, let K be the simplicial complex whose simplexes are all the nonempty proper subsets of V. Then K is a d-pseudomanifold and triangulates the d-sphere S^d. This d-pseudomanifold K is called the *standard d-sphere* and is denoted by $S^d_{d+2}(V)$ (or S^d_{d+2}). By convention, S^0_2 is the only 0-pseudomanifold.

If σ is a face of a simplicial complex K, then the *link* of σ in K, denoted by $\mathrm{lk}_K(\sigma)$ (or $\mathrm{lk}(\sigma)$), is by definition the simplicial complex whose faces are the faces τ of K such that τ is disjoint from σ and $\sigma \cup \tau$ is a face of K. Clearly, the link of an i-face in a weak d-pseudomanifold is a weak $(d-i-1)$-pseudomanifold. For $d \geq 1$, a connected weak d-pseudomanifold is said to be a *normal d-pseudomanifold* if the links of all the simplices of dimension $\leq d-2$ are connected. Thus, any connected triangulated d-manifold (triangulation of a closed d-manifold) is a normal d-pseudomanifold. Clearly, the normal 2-pseudomanifolds are just the connected combinatorial 2-manifolds; but normal d-pseudomanifolds form a broader class than connected triangulated d-manifolds for $d \geq 3$.

Observe that if X is a normal pseudomanifold, then X is a pseudomanifold. (If $\Lambda(X)$ is not connected, then, since X is connected, $\Lambda(X)$ has two components G_1 and G_2 and two intersecting facets σ_1, σ_2 such that $\sigma_i \in G_i$, $i = 1, 2$. Choose σ_1, σ_2 among all such pairs such that $\dim(\sigma_1 \cap \sigma_2)$ is maximum. Then $\dim(\sigma_1 \cap \sigma_2) \leq d-2$ and $\mathrm{lk}_X(\sigma_1 \cap \sigma_2)$ is not connected, a contradiction.) Notice that all the links of positive dimensions (i.e., the links of simplices of dimension $\leq d-2$) in a normal d-pseudomanifold are normal pseudomanifolds. Thus, if K is a normal 3-pseudomanifold, then the link of a vertex in K is a combinatorial 2-manifold. A vertex v of a normal 3-pseudomanifold K is called *singular* if the link of v in K is not a 2-sphere. The set of singular vertices is denoted by $\mathrm{SV}(K)$. Clearly, the space $|K| \setminus \mathrm{SV}(K)$ is a pl 3-manifold. If $\mathrm{SV}(K) = \varnothing$ (i.e., the link of each vertex is a 2-sphere), then K is called a *combinatorial* 3-*manifold*. A *combinatorial* 3-*sphere* is a combinatorial 3-manifold which triangulates the topological 3-sphere S^3.

Let M be a weak d-pseudomanifold. If α is a $(d-i)$-face of M, $0 < i \leq d$, such that $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$ and β is not a face of M (such a face α is said to be a *removable* face of M), then consider the weak d-pseudomanifold (denoted by $\kappa_\alpha(M)$) whose facet-set is $\{\sigma : \sigma$ a facet of $M, \alpha \nsubseteq \sigma\} \cup \{\beta \cup \alpha \setminus \{v\} : v \in \alpha\}$. The operation $\kappa_\alpha : M \mapsto \kappa_\alpha(M)$ is called a *bistellar i-move*. For $0 < i < d$, a bistellar i-move is called a *proper bistellar move*. If κ_α is a proper bistellar i-move and $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$, then β is a removable i-face of $\kappa_\alpha(M)$ (with $\mathrm{lk}_{\kappa_\alpha(M)}(\beta) = S^{d-i-1}_{d-i+1}(\alpha)$) and $\kappa_\beta : \kappa_\alpha(M) \mapsto M$ is an bistellar $(d-i)$-move. For a vertex u, if $\mathrm{lk}_M(u) = S^{d-1}_{d+1}(\beta)$, then the bistellar d-move $\kappa_{\{u\}} : M \mapsto \kappa_{\{u\}}(M) = N$ deletes the vertex u (we also say that N is obtained from M by *collapsing* the vertex u). The operation $\kappa_\beta : N \mapsto M$ is called a bistellar 0-*move* (we also say that M is obtained from N by *starring* the vertex u in the facet β of N). The 10-vertex combinatorial 3-manifold A^3_{10} in Example 3.15 is not neighbourly and does not allow any bistellar 1-move. In [1], Bagchi and Datta have shown that if the number of vertices in a nonneighbourly combinatorial 3-manifold is at most 9, then the 3-manifold admits a bistellar 1-move. Existence of the 9-vertex 3-pseudomanifold B^3_9 in Example 3.16 shows that Bagchi and Datta's result is not true for 9-vertex 3-pseudomanifolds. Here we prove the following theorem.

Theorem 1.1. *If M is an 8-vertex 3-pseudomanifold, then there exists a sequence of bistellar 1-moves $\kappa_{A_1}, \dots, \kappa_{A_m}$, for some $m \geq 0$, such that $\kappa_{A_m}(\cdots(\kappa_{A_1}(M)))$ is a neighbourly 3-pseudomanifold.*

In [2], Altshuler has shown that every combinatorial 3-manifold with at most 8 vertices is a combinatorial 3-sphere. In [3], Grünbaum and Sreedharan have shown that there are exactly 37 polytopal 3-spheres on 8 vertices (namely, $S^3_{8,1}, \dots, S^3_{8,37}$ in Examples 3.1 and 3.3). They have also constructed the nonpolytopal sphere $S^3_{8,38}$. In [4], Barnette proved that there is only one more nonpolytopal 8-vertex 3-sphere (namely, $S^3_{8,39}$). In [5], Emch constructed an 8-vertex normal 3-pseudomanifold (namely, N_1 in Example 3.5) as a block design. This is not a combinatorial 3-manifold and its automorphism group is PGL(2,7) (cf. [6]). In [7], Altshuler has constructed another 8-vertex normal 3-pseudomanifold (namely, N_5 in Example 3.5). In [8], Lutz has shown that there exist exactly three 8-vertex normal 3-pseudomanifolds which are not combinatorial 3-manifolds (namely, N_1, N_5 and N_6 in Example 3.5) with vertex-transitive automorphism groups. Here we prove the following theorem.

Theorem 1.2. *Let $S^3_{8,35}, \dots, S^3_{8,38}$, $N_1, \dots, N_{15}$ be as in Examples 3.1 and 3.5.*

(i) *Then $S^3_{8,i} \not\cong S^3_{8,j}$, $N_k \not\cong N_l$, and $S^3_{8,m} \not\cong N_n$ for $35 \leq i < j \leq 38$, $1 \leq k < l \leq 15$, $35 \leq m \leq 38$, and $1 \leq n \leq 15$.*

(ii) *If M is an 8-vertex neighbourly normal 3-pseudomanifold, then M is isomorphic to one of $S^3_{8,35}, \dots, S^3_{8,38}$, $N_1, \dots, N_{15}$.*

Corollary 1.3. *There are exactly 39 combinatorial 3-manifolds on 8 vertices, all of which are combinatorial 3-spheres.*

Corollary 1.4. *There are exactly 35 normal 3-pseudomanifolds on 8 vertices which are not combinatorial 3-manifolds. These are $N_1, \dots, N_{35}$ defined in Examples 3.5 and 3.8.*

The topological properties of these normal 3-pseudomanifolds are given in Section 3.

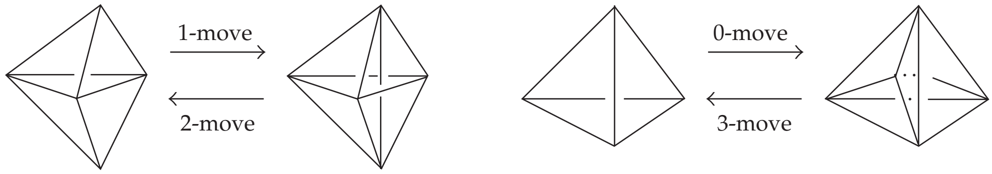
2. Preliminaries

All the simplicial complexes considered in this paper are finite (i.e., with finite vertex-set). The vertex-set of a simplicial complex K is denoted by $V(K)$. We identify the 0-faces of a complex with the vertices. The 1-faces of a complex K are also called the *edges* of K.

If K, L are two simplicial complexes, then an *isomorphism* from K to L is a bijection $\pi : V(K) \to V(L)$ such that for $\sigma \subseteq V(K)$, σ is a face of K if and only if $\pi(\sigma)$ is a face of L. Two complexes K, L are called *isomorphic* when such an isomorphism exists. We identify two complexes if they are isomorphic. An isomorphism from a complex K to itself is called an *automorphism* of K. All the automorphisms of K form a group under composition, which is denoted by $\mathrm{Aut}(K)$.

For a face σ in a simplicial complex K, the number of vertices in $\mathrm{lk}_K(\sigma)$ is called the *degree* of σ in K and is denoted by $\deg_K(\sigma)$ (or by $\deg(\sigma)$). If every pair of vertices of a simplicial complex K form an edge, then K is called *neighbourly*. For a simplicial complex K, if $U \subseteq V(K)$, then $K[U]$ denotes the induced complex of K on the vertex-set U.

If the number of i-faces of a d-dimensional simplicial complex K is $f_i(K)$ $(0 \leq i \leq d)$, then the number $\chi(K) := \sum_{i=0}^{d}(-1)^i f_i(K)$ is called the *Euler characteristic* of K.



Bistellar moves in dimension 3

Figure 1

A *graph* is a simplicial complex of dimension ≤ 1. A finite 1-pseudomanifold is called a *cycle*. An n-cycle is a cycle on n vertices and is denoted by C_n (or by $C_n(a_1, \ldots, a_n)$ if the edges are $a_1a_2, \ldots, a_{n-1}a_n, a_na_1$).

For a simplicial complex K, the graph consisting of the edges and vertices of K is called the *edge-graph* of K and is denoted by $\mathrm{EG}(K)$. The complement of $\mathrm{EG}(K)$ is called the *nonedge graph* of K and is denoted by $\mathrm{NEG}(K)$. For a weak 3-pseudomanifold M and an integer $n \geq 3$, we define the graph $G_n(M)$ as follows. The vertices of $G_n(M)$ are the vertices of M. Two vertices u and v form an edge in $G_n(M)$ if uv is an edge of degree n in M. Clearly, if M and N are isomorphic, then $G_n(M)$ and $G_n(N)$ are isomorphic for each n.

If M is a weak 3-pseudomanifold and $\kappa_\alpha : M \mapsto \kappa_\alpha(M) = N$ is a bistellar 1-move, then, from the definition, $(f_0(N), f_1(N), f_2(N), f_3(N)) = (f_0(M), f_1(M)+1, f_2(M)+2, f_3(M)+1)$ and $\deg_N(v) \geq \deg_M(v)$ for any vertex v. If $\kappa_\alpha : M \mapsto \kappa_\alpha(M) = L$ is a bistellar 3-move, then $(f_0(L), f_1(L), f_2(L), f_3(L)) = (f_0(M)-1, f_1(M)-4, f_2(M)-6, f_3(M)-3)$.

Consider the binary relation "$\leq$" on the set of weak 3-pseudomanifolds as $M \leq N$ if there exists a finite sequence of bistellar 1-moves $\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_m}$, for some $m \geq 0$, such that $N = \kappa_{\alpha_m}(\cdots \kappa_{\alpha_1}(M))$. Clearly, this $\leq$ is a partial order relation.

Two weak d-pseudomanifolds M and N are *bistellar equivalent* (denoted by $M \sim N$) if there exists a finite sequence of bistellar operations leading from M to N. If there exists a finite sequence of proper bistellar operations leading from M to N, then we say M and N are *properly bistellar equivalent* and we denote this by $M \approx N$. Clearly, "$\sim$" and "$\approx$" are equivalence relations on the set of pseudomanifolds. It is easy to see that $M \sim N$ implies that $|M|$ and $|N|$ are pl homeomorphic.

For two simplicial complexes X and Y with disjoint vertex sets, the simplicial complex $X * Y := X \cup Y \cup \{\sigma \cup \tau : \sigma \in X, \tau \in Y\}$ is called the *join* of X and Y.

Let K be an n-vertex (weak) d-pseudomanifold. If u is a vertex of K and v is not a vertex of K, then consider the simplicial complex $\Sigma_{uv}K$ on the vertex set $V(K) \cup \{v\}$ whose set of facets is $\{\sigma \cup \{u\} : \sigma$ is a facet of K and $u \notin \sigma\} \cup \{\tau \cup \{v\} : \tau$ is a facet of $K\}$. Then $\Sigma_{uv}K$ is a (weak) $(d+1)$-pseudomanifold and $|\Sigma_{uv}K|$ is the topological suspension $S|K|$ of $|K|$ (cf. [9]). It is easy to see that the links of u and v in $\Sigma_{uv}K$ are isomorphic to K. This $\Sigma_{uv}K$ is called the *one-point suspension* of K.

For two d-pseudomanifolds X and Y, a simplicial map $f : X \to Y$ is called a *k-fold branched covering* (with discrete branch locus) if $|f||_{|X| \setminus f^{-1}(U)} : |X| \setminus f^{-1}(U) \to |Y| \setminus U$ is a k-fold covering for some $U \subseteq V(Y)$. (We say that X is a *branched cover* of Y and Y is a *branched quotient* of X.) The smallest such U (so that $|f||_{|X| \setminus f^{-1}(U)} : |X| \setminus f^{-1}(U) \to |Y| \setminus U$ is a covering) is called the *branch locus*. If N is a k-fold branched quotient of M and $\widetilde{N}$ is obtained from N by collapsing a vertex (resp., starring a vertex in a facet), then $\widetilde{N}$ is the branched quotient of $\widetilde{M}$, where $\widetilde{M}$ can be obtained from M by collapsing k vertices (resp., starring k vertices in k facets). For proper bistellar moves we have the following lemma.

Lemma 2.1. *Let M and N be two d-pseudomanifolds and $f : M \to N$ be a k-fold branched covering. For $1 \leq l < d-1$, if α is a removable l-face, then $f^{-1}(\alpha)$ consists of k removable l-faces $\alpha_1, \ldots, \alpha_k (say)$ and $\kappa_{\alpha_k}(\cdots(\kappa_{\alpha_1}(M)))$ is a k-fold branched cover of $\kappa_\alpha(N)$.*

Proof. Let $\mathrm{lk}_N(\alpha) = S^{d-l-1}_{d-l+1}(\beta)$. Since the dimension of α is > 0, $f^{-1}(\alpha)$ consists of kl-faces, $\alpha_1, \ldots, \alpha_k$ (say) of M. Let $\mathrm{lk}_M(\alpha_i) = S^{d-l-1}_{d-l+1}(\beta_i)$ and $M_i := M[\alpha_i \cup \beta_i]$ for $1 \leq i \leq k$. Since f is simplicial, β_i is not a face of M and hence α_i is removable for each i. Since $0 < l < d-1$, it follows that M_i is neighbourly. For $i \neq j$, if $x \neq y \in V(M_i) \cap V(M_j)$, then xy is an edge in $M_i \cap M_j$ and hence the number of edges in $f^{-1}(f(x)f(y))$ is less than k, a contradiction. So, $\#(V(M_i) \cap V(M_j)) \leq 1$ for $i \neq j$. This implies that β_i is not a face in $\kappa_{\alpha_j}(M)$ and hence α_i is removable in $\kappa_{\alpha_j}(M)$ for $i \neq j$. The result now follows. □

Remark 3.14 shows that Lemma 2.1 is not true for $l = d-1$ (i.e., for bistellar 1-moves) in general.

Example 2.2. In Figure 2, we present some weak 2-pseudomanifolds on at most seven vertices. The degree sequences are presented parenthetically below the figures. Each of $S_1, \ldots, S_9$ triangulates the 2-sphere, each of $R_1, \ldots, R_4$ triangulates the real projective plane and T triangulates the torus. Observe that P_1, P_2 are not pseudomanifolds.

We know that if K is a weak 2-pseudomanifold with at most six vertices, then K is isomorphic to $S_1, \ldots, S_4$ or R_1 (cf. [9]). In [10], we have seen the following.

Proposition 2.3. *There are exactly 13 distinct 2-dimensional weak pseudomanifolds on 7 vertices, namely, $S_5, \ldots, S_9$, $R_2, \ldots, R_4$, $T, P_1, \ldots, P_3$, and P_4.*

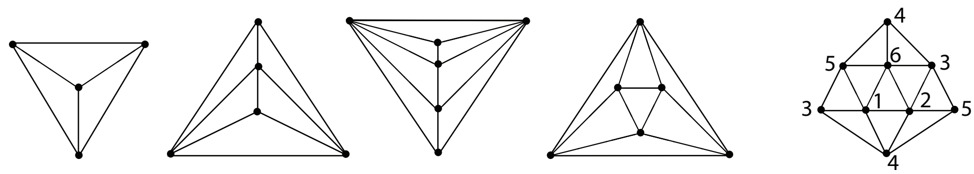
3. Examples

We identify a weak pseudomanifold with the set of facets in it.

Example 3.1. These four neighbourly 8-vertex combinatorial 3-manifolds were found by Grünbaum and Sreedharan (in [3], these are denoted by P^8_{35}, P^8_{36}, P^8_{37} and $\mathcal{M}$, resp.). It follows from Lemma 3.4 that these are combinatorial 3-spheres. It was shown in [3] that the first three of these are polytopal 3-spheres and the last one is a nonpolytopal sphere:

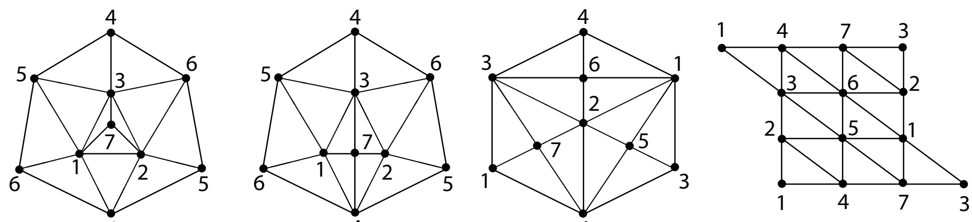
$$\begin{aligned}
S^3_{8,35} = \{&1234, 1267, 1256, 1245, 2345, 2356, 2367, 3467, 3456, 4567, 1238, 1278, 2378,\\
&1348, 3478, 1458, 4578, 1568, 1678, 5678\},\\
S^3_{8,36} = \{&1234, 1256, 1245, 1567, 2345, 2356, 2367, 3467, 3456, 4567, 1268, 1678, 2678,\\
&1238, 2378, 1348, 3478, 1458, 1578, 4578\},\\
S^3_{8,37} = \{&1234, 1256, 1245, 1457, 2345, 2356, 2367, 3467, 3456, 4567, 1568, 1578, 5678,\\
&1268, 2678, 1238, 2378, 1348, 1478, 3478\},\\
S^3_{8,38} = \{&1234, 1237, 1267, 1347, 1567, 2345, 2367, 3467, 3456, 4567, 2358, 2368, 3568,\\
&1268, 1568, 1248, 2458, 1478, 1578, 4578\}.
\end{aligned} \tag{3.1}$$

Lemma 3.2. *$S^3_{8,i} \not\cong S^3_{8,j}$ for $35 \leq i < j \leq 38$.*



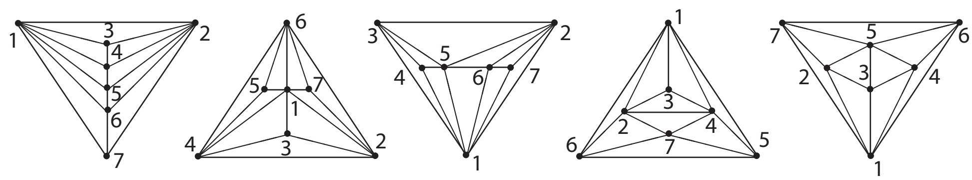
$S_1 = S^2_4\ (3^4)$ $S_2\ (4^3.3^2)$ $S_3\ (5^2.4^2.3^2)$ $S_4\ (4^6)$ $R_1 = \mathbb{R}P^2_6\ (5^6)$

(a)



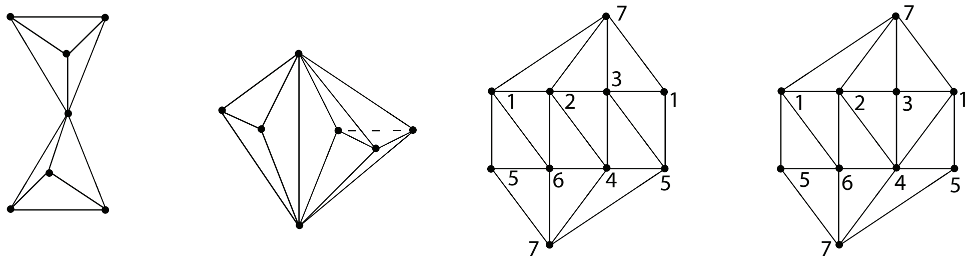
R_2 $(6^3.5^3.3)$ R_3 $(6^2.5^4.4)$ R_4 $(6^4.4^3)$ $T\ (6^7)$

(b)



$S_5\ (6^2.4^3.3^2)$ $S_6\ (6.5^3.3^3)$ $S_7\ (6.5^2.4^2.3^2)$ $S_8\ (5^3.4^3.3)$ $S_9\ (5^2.4^5)$

(c)



$P_1\ (6.3^6)$ $P_2\ (6^2.4^3.3^2)$ $P_3\ (6.5^6)$ $P_4\ (6^3.5^2.4^2)$

(d)

Figure 2

Proof. Observe that $G_6(S^3_{8,35}) = C_8(1,2,\ldots,8)$, $G_6(S^3_{8,36}) = (V, \{23,34,45,67,78,81\})$, $G_6(S^3_{8,37}) = (V, \{23,34,56,78,81\})$, and $G_6(S^3_{8,38}) = (V, \{17,23,58\})$, where $V = \{1,\ldots,8\}$. Since $K \cong L$ implies $G_6(K) \cong G_6(L)$, $S^3_{8,i} \ncong S^3_{8,j}$, for $35 \le i < j \le 38$. $\square$

Example 3.3. Some nonneighbourly 8-vertex combinatorial 3-manifolds. It follows from Lemma 3.4 that these are combinatorial 3-spheres. For $1 \leq i \leq 34$, the sphere $S^3_{8,i}$ is isomorphic to the polytopal sphere P^8_i in [3] and the sphere $S^3_{8,39}$ is isomorphic to the nonpolytopal sphere found by Barnette in [4]. We consecutively define

$$
\begin{gathered}
S^3_{8,39} = \kappa_{46}(S^3_{8,38}), \quad S^3_{8,33} = \kappa_{27}(S^3_{8,37}), \quad S^3_{8,32} = \kappa_{48}(S^3_{8,37}), \quad S^3_{8,31} = \kappa_{58}(S^3_{8,37}), \\
S^3_{8,30} = \kappa_{24}(S^3_{8,37}), \quad S^3_{8,29} = \kappa_{27}(S^3_{8,31}), \quad S^3_{8,28} = \kappa_{24}(S^3_{8,31}), \quad S^3_{8,27} = \kappa_{13}(S^3_{8,31}), \\
S^3_{8,25} = \kappa_{57}(S^3_{8,31}), \quad S^3_{8,24} = \kappa_{48}(S^3_{8,31}), \quad S^3_{8,23} = \kappa_{35}(S^3_{8,31}), \quad S^3_{8,26} = \kappa_{46}(S^3_{8,27}), \\
S^3_{8,22} = \kappa_{24}(S^3_{8,25}), \quad S^3_{8,21} = \kappa_{68}(S^3_{8,25}), \quad S^3_{8,20} = \kappa_{48}(S^3_{8,25}), \quad S^3_{8,19} = \kappa_{17}(S^3_{8,25}), \\
S^3_{8,18} = \kappa_{27}(S^3_{8,25}), \quad S^3_{8,12} = \kappa_{15}(S^3_{8,25}), \quad S^3_{8,11} = \kappa_{35}(S^3_{8,25}), \quad S^3_{8,17} = \kappa_{24}(S^3_{8,19}), \\
S^3_{8,34} = \kappa_{27}(S^3_{8,26}) = S^0_3(1,3) * S^0_3(2,7) * S^0_3(4,6) * S^0_3(5,8), \quad S^3_{8,16} = \kappa_{13}(S^3_{8,19}), \\
S^3_{8,15} = \kappa_{28}(S^3_{8,18}), \quad S^3_{8,14} = \kappa_{47}(S^3_{8,20}), \quad S^3_{8,10} = \kappa_{15}(S^3_{8,19}), \quad S^3_{8,9} = \kappa_{35}(S^3_{8,19}), \\
S^3_{8,8} = \kappa_{47}(S^3_{8,19}), \quad S^3_{8,13} = \kappa_{38}(S^3_{8,16}), \quad S^3_{8,7} = \kappa_{24}(S^3_{8,8}), \quad S^3_{8,6} = \kappa_{35}(S^3_{8,8}), \\
S^3_{8,5} = \kappa_{48}(S^3_{8,8}), \quad S^3_{8,4} = \kappa_{15}(S^3_{8,8}), \quad S^3_{8,3} = \kappa_{48}(S^3_{8,4}), \\
S^3_{8,2} = \kappa_{48}(S^3_{8,6}), \quad S^3_{8,1} = \kappa_{16}(S^3_{8,4}).
\end{gathered} \tag{3.2}
$$

Lemma 3.4. (a) $S^3_{8,i} \approx S^3_{8,j}$, *for* $1 \leq i, j \leq 39$, (b) $S^3_{8,m}$ *is a combinatorial* 3*-sphere for* $1 \leq m \leq 39$, *and* (c) $S^3_{8,k} \not\cong S^3_{8,l}$ *for* $1 \leq k < l \leq 39$.

Proof. For $0 \leq i \leq 6$, let $\mathcal{S}_i$ denote the set of $S^3_{8,j}$'s with i nonedges. Then $\mathcal{S}_0 = \{S^3_{8,35}, S^3_{8,36}, S^3_{8,37}, S^3_{8,38}\}$, $\mathcal{S}_1 = \{S^3_{8,30}, S^3_{8,31}, S^3_{8,32}, S^3_{8,33}, S^3_{8,39}\}$, $\mathcal{S}_2 = \{S^3_{8,23}, S^3_{8,24}, S^3_{8,25}, S^3_{8,27}, S^3_{8,28}, S^3_{8,29}\}$, $\mathcal{S}_3 = \{S^3_{8,11}, S^3_{8,12}, S^3_{8,18}, S^3_{8,19}, S^3_{8,20}, S^3_{8,21}, S^3_{8,22}, S^3_{8,26}\}$, $\mathcal{S}_4 = \{S^3_{8,8}, S^3_{8,9}, S^3_{8,10}, S^3_{8,14}, S^3_{8,15}, S^3_{8,16}, S^3_{8,17}, S^3_{8,34}\}$, $\mathcal{S}_5 = \{S^3_{8,4}, S^3_{8,5}, S^3_{8,6}, S^3_{8,7}, S^3_{8,13}\}$, and $\mathcal{S}_6 = \{S^3_{8,1}, S^3_{8,2}, S^3_{8,3}\}$.

From the proof of Lemma 4.7, $S^3_{8,35} \approx S^3_{8,30} \approx S^3_{8,36} \approx S^3_{8,30} \approx S^3_{8,37} \approx S^3_{8,32} \approx S^3_{8,38}$. Thus, $S^3_{8,i} \approx S^3_{8,j}$ for $35 \leq i, j \leq 38$. Now, if $S^3_{8,i} \in \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6$, then, from the definition of $S^3_{8,i}$, $S^3_{8,i} \approx S^3_{8,31} \approx S^3_{8,37}$. This proves part (a).

Since $S^3_{8,34}$ is a join of spheres, $S^3_{8,34}$ is a combinatorial 3-sphere. Clearly, if $M \approx N$ and M is a combinatorial 3-sphere, then N is so. Part (b) now follows from part (a).

Since the nonedge graphs of the members of $\mathcal{S}_6$ (resp., $\mathcal{S}_5$) are pairwise nonisomorphic, the members of $\mathcal{S}_6$ (resp., $\mathcal{S}_5$) are pairwise nonisomorphic.

For $S^3_{8,i}, S^3_{8,j} \in \mathcal{S}_4$ $(i < j)$ and $\mathrm{NEG}(S^3_{8,i}) \cong \mathrm{NEG}(S^3_{8,j})$ imply $(i, j) = (8, 9)$ or $(14, 15)$. Since $M \cong N$ implies $G_6(M) \cong G_6(N)$ and $G_6(S^3_{8,8}) \not\cong G_6(S^3_{8,9})$, $G_6(S^3_{8,14}) \not\cong G_6(S^3_{8,15})$, the members of $\mathcal{S}_4$ are pairwise nonisomorphic.

For $S^3_{8,i} \neq S^3_{8,j} \in \mathcal{S}_3$ and $\mathrm{NEG}(S^3_{8,i}) \cong \mathrm{NEG}(S^3_{8,j})$ imply $\{i, j\} = \{11, 12\}$ or $18 \leq i \neq j \leq 21$. Let $\sum_1 = \{S^3_{8,11}, S^3_{8,12}\}$, $\sum_2 = \{S^3_{8,18}, S^3_{8,19}, S^3_{8,20}, S^3_{8,21}\}$, $\sum_3 = \{S^3_{8,22}\}$ and $\sum_4 = \{S^3_{8,26}\}$. Since the nonedge graph of a member in Σ_i is nonisomorphic to the nonedge graph of a member of Σ_j for $i \neq j$, a member of Σ_i is nonisomorphic to a member of Σ_j. Observe that $G_6(S^3_{8,11}) \not\cong G_6(S^3_{8,12})$ and for $18 \leq i < j \leq 21$, $G_6(S^3_{8,i}) \cong G_6(S^3_{8,j})$ implies $(i, j) = (18, 19)$. Since $G_3(S^3_{8,18}) \not\cong G_3(S^3_{8,19})$, the members of $\mathcal{S}_3$ are pairwise nonisomorphic.

Since $G_3(S^3_{8,i}) \not\cong G_3(S^3_{8,j})$ for $S^3_{8,i} \neq S^3_{8,j} \in \mathcal{S}_2$, the members of $\mathcal{S}_2$ are pairwise nonisomorphic. By the same reasoning, the members of $\mathcal{S}_1$ are pairwise nonisomorphic.

By Lemma 3.2, the members of $\mathcal{S}_0$ are pairwise nonisomorphic. Since a member of $\mathcal{S}_i$ is nonisomorphic to a member of $\mathcal{S}_j$ for $i \neq j$, the above imply part (c). □

Example 3.5. Some 8-vertex neighbourly normal 3-pseudomanifolds:

$$\begin{aligned}
N_1 = \{&1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468, 3568, 4578, 4678,\\
&1247, 1257, 1367, 1467, 2347, 2567, 3457, 3567, 1236, 2346, 1345, 1235, 1456, 2456\},\\
N_2 = \{&1248, 2458, 2358, 3568, 3468, 4678, 4578, 1578, 1568, 1268, 2678,\\
&2378, 1378, 1348, 1247, 2457, 2357, 3567, 3467, 1567, 1267, 1347\} = \Sigma_{78}T,\\
N_3 = \{&1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468, 3568,\\
&4578, 4678, 1234, 2347, 2456, 2467, 3456, 3457, 1235, 1256, 1357\},\\
N_4 = \{&1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468,\\
&3568, 4578, 4678, 1245, 1256, 2356, 2367, 3467, 1347, 1457\},\\
N_5 = \{&1258, 1268, 1358, 1378, 1468, 1478, 2368, 2378, 2458, 2478, 3458, 3468,\\
&1257, 1267, 1367, 1457, 2357, 2467, 3457, 3467, 2356, 2456, 1356, 1456\},\\
N_6 = \{&1358, 1378, 1468, 1478, 1568, 2368, 2378, 2458, 2478, 2568, 3458, 3468,\\
&1235, 1245, 1457, 1567, 2357, 2567, 3457, 1236, 1246, 1367, 2467, 3467\},\\
N_7 = \{&1268, 1258, 1358, 1378, 1478, 1468, 2378, 2368, 2458, 2478, 3468,\\
&3458, 1356, 1367, 2357, 2356, 3467, 3457, 1256, 1467, 2457\},
\end{aligned}$$

$$\begin{gathered}
N_8 = \kappa_{348}(\kappa_{238}(\kappa_{56}(\kappa_{67}(N_7)))), \qquad N_9 = \kappa_{235}(\kappa_{67}(N_7)),\\
N_{10} = \kappa_{148}(\kappa_{67}(N_7)), \qquad N_{11} = \kappa_{348}(\kappa_{56}(N_{10})), \qquad N_{12} = \kappa_{457}(\kappa_{23}(N_9)),\\
N_{13} = \kappa_{567}(\kappa_{23}(N_9)), \qquad N_{14} = \kappa_{138}(\kappa_{57}(N_8)) \cong \Sigma_{78}R_2, \qquad N_{15} = \kappa_{158}(\kappa_{23}(N_9)).
\end{gathered} \tag{3.3}$$

All the vertices of N_1 are singular and their links are isomorphic to the 7-vertex torus T. There are two singular vertices in N_2 and their links are isomorphic to T. The singular vertices in N_3 are 8, 3, 4, 2, 5 and their links are isomorphic to T, R_2, R_2, R_3, and R_3, respectively. There is only one singular vertex in N_4 whose link is isomorphic to T. All the vertices of N_5 (resp., N_6) are singular and their links are isomorphic to R_4 (resp., R_3). Each of $N_7, \ldots, N_{15}$ has exactly two singular vertices and their links are 7-vertex $\mathbb{R}P^2$'s. Thus, each N_i is a normal 3-pseudomanifold.

It follows from the definition that $N_i \approx N_j$ for $7 \leq i, j \leq 15$. Here we prove the following lemmas.

Lemma 3.6. (a) *The geometric carriers of N_1, N_2, N_3, N_4, N_5, and N_7 are distinct (nonhomeomorphic),* (b) $N_i \not\cong N_j$ *for* $1 \leq i < j \leq 7$, (c) $N_5 {\sim} N_6$.

Proof. For a normal 3-pseudomanifold X, let $n_s(X)$ denote the number of singular vertices. Clearly, if M and N are two normal 3-pseudomanifolds with homeomorphic geometric carriers, then $(n_s(M), \chi(M)) = (n_s(N), \chi(N))$. Now, $(n_s(N_1), \chi(N_1)) = (8,8)$, $(n_s(N_2), \chi(N_2)) = (2,2)$, $(n_s(N_3), \chi(N_3)) = (5,3)$, $(n_s(N_4), \chi(N_4)) = (1,1)$, $(n_s(N_5), \chi(N_5)) = (8,4)$, $(n_s(N_7), \chi(N_7)) = (2,1)$. This proves part (a).

Part (b) follows from the fact that N_i is neighbourly and has no removable edge and, hence, there is no proper bistellar move from N_i for $1 \le i \le 6$.

Let N_5' be obtained from N_5 by starring a new vertex 0 in the facet 1358. Let $N_5'' = \kappa_{\{0\}}(\kappa_{08}(\kappa_{156}(\kappa_{07}(\kappa_{03}(\kappa_{035}(\kappa_{68}(\kappa_{02}(\kappa_{268}(\kappa_{13}(\kappa_{135}(\kappa_{138}(\kappa_{158}(N_5')))))))))))))$, then N_5'' is isomorphic to N_6 via the map $(2,3)(5,8)$. This proves part (c). □

Lemma 3.7. *$N_k \not\cong N_l$ for $1 \le k < l \le 15$.*

Proof. Let n_s be as above. Clearly, if M and N are two isomorphic 3-pseudomanifolds, then $(n_s(M), f_3(M)) = (n_s(N), f_3(N))$. Now, $(n_s(N_1), f_3(N_1)) = (8, 28)$, $(n_s(N_2), f_3(N_2)) = (2, 22)$, $(n_s(N_3), f_3(N_3)) = (5, 23)$, $(n_s(N_4), f_3(N_4)) = (1, 21)$, $(n_s(N_5), f_3(N_5)) = (n_s(N_6), f_3(N_6)) = (8, 24)$, and $(n_s(N_i), f_3(N_i)) = (2, 21)$ for $7 \le i \le 15$. Since the links of each vertex in N_5 is isomorphic to R_4 and the links of each vertex in N_6 is isomorphic to R_3, it follows that $N_5 \not\cong N_6$. Thus, $N_i \not\cong N_j$ for $1 \le i \le 6$, $1 \le j \le 15$, $i \ne j$.

Observe that the singular vertices in N_i are 3 and 8 for $7 \le i \le 15$. Moreover, (i) $\mathrm{lk}_{N_7}(3) \cong \mathrm{lk}_{N_7}(8) \cong R_4$, (ii) $\mathrm{lk}_{N_8}(3) \cong R_4$ and $\mathrm{lk}_{N_8}(8) \cong R_3$, (iii) $\mathrm{lk}_{N_9}(3) \cong R_2$ and $\mathrm{lk}_{N_9}(8) \cong R_4$, (iv) $\mathrm{lk}_{N_{10}}(3) \cong \mathrm{lk}_{N_{10}}(8) \cong R_3$ and $\deg_{N_{10}}(38) = 6$, (v) $\mathrm{lk}_{N_{11}}(3) \cong \mathrm{lk}_{N_{11}}(8) \cong R_3$ and $\deg_{N_{11}}(38) = 5$, (vi) $\mathrm{lk}_{N_{12}}(3) \cong R_2$, $\mathrm{lk}_{N_{12}}(8) \cong R_3$ and $G_3(N_{12}) = (V, \{32, 21, 17, 75, 54, 46\})$, (vii) $\mathrm{lk}_{N_{13}}(3) \cong R_2$, $\mathrm{lk}_{N_{13}}(8) \cong R_3$ and $G_3(N_{13}) = (V, \{32, 21, 17, 75, 56, 67, 64, 42\})$, (viii) $\mathrm{lk}_{N_{14}}(3) \cong \mathrm{lk}_{N_{14}}(8) \cong R_2$ and $\deg_{N_{14}}(38) = 3$. (xi) $\mathrm{lk}_{N_{15}}(3) \cong \mathrm{lk}_{N_{15}}(8) \cong R_2$ and $\deg_{N_{15}}(38) = 6$. These imply that there is no isomorphism between N_i and N_j for $7 \le i < j \le 15$. This completes the proof. □

Example 3.8. Some 8-vertex nonneighbourly normal 3-pseudomanifolds:

$$
\begin{gathered}
N_{16} = \kappa_{67}(N_7), \quad N_{17} = \kappa_{24}(N_8), \quad N_{18} = \kappa_{238}(\kappa_{56}(\kappa_{67}(N_7))), \quad N_{19} = \kappa_{57}(N_8), \\
N_{20} = \kappa_{56}(N_{10}), \quad N_{21} = \kappa_{12}(N_9), \quad N_{22} = \kappa_{14}(N_{11}), \quad N_{23} = \kappa_{23}(N_9), \\
N_{24} = \kappa_{38}(N_{14}), \quad N_{25} = \kappa_{56}(N_{16}), \quad N_{26} = \kappa_{12}(N_{16}), \quad N_{27} = \kappa_{56}(N_{17}), \\
N_{28} = \kappa_{57}(N_{18}), \quad N_{29} = \kappa_{15}(N_{18}), \quad N_{30} = \kappa_{12}(N_{23}), \quad N_{31} = \kappa_{24}(N_{22}), \\
N_{32} = \kappa_{24}(N_{26}), \quad N_{33} = \kappa_{57}(N_{25}), \quad N_{34} = \kappa_{45}(N_{28}), \quad N_{35} = \kappa_{58}(N_{29}).
\end{gathered}
\tag{3.4}
$$

Lemma 3.9. (a) *$N_i \not\cong N_j$ for $1 \le i < j \le 35$ and* (b) *$N_k \approx N_l$ for $7 \le k,\ l \le 35$.*

Proof. For $0 \le i \le 3$, let $\mathcal{N}_i$ denote the set of 3-pseudomanifolds defined in Examples 3.5 and 3.8 with i nonedges. Then $\mathcal{N}_0 = \{N_1, \dots, N_{15}\}$, $\mathcal{N}_1 = \{N_{16}, \dots, N_{24}\}$, $\mathcal{N}_2 = \{N_{25}, \dots, N_{31}\}$, and $\mathcal{N}_3 = \{N_{32}, \dots, N_{35}\}$. The singular vertices in N_i are 3 and 8 for $7 \le i \le 35$.

By Lemma 3.7, the members of $\mathcal{N}_0$ are pairwise nonisomorphic.

Observe that (i) $\mathrm{lk}_{N_{16}}(3) \cong R_4$ and $\mathrm{lk}_{N_{16}}(8) \cong R_3$, (ii) $\mathrm{lk}_{N_{17}}(3) \cong \mathrm{lk}_{N_{17}}(8) \cong R_4$, (iii) $\mathrm{lk}_{N_{18}}(3) \cong \mathrm{lk}_{N_{18}}(8) \cong R_3$ and $G_6(N_{18}) = (V, \{73, 31, 18, 84\})$, (iv) $\mathrm{lk}_{N_{19}}(3) \cong \mathrm{lk}_{N_{19}}(8) \cong R_3$ and $G_6(N_{19}) = (V, \{63, 31, 18, 86\})$, (v) $\mathrm{lk}_{N_{20}}(3) \cong \mathrm{lk}_{N_{20}}(8) \cong R_3$ and $G_6(N_{20}) = (V, \{74, 28, 83, 31\})$, (vi) $\mathrm{lk}_{N_{21}}(3) \cong R_2$, $\mathrm{lk}_{N_{21}}(8) \cong R_3$ and $G_6(N_{21}) = (V, \{48, 83, 37, 36\})$, (vii) $\mathrm{lk}_{N_{22}}(3) \cong R_2$, $\mathrm{lk}_{N_{22}}(8) \cong R_3$ and $G_6(N_{22}) = (V, \{28, 86, 63, 37, 38\})$, (viii) $\mathrm{lk}_{N_{23}}(3) \cong R_1$ and $\mathrm{lk}_{N_{23}}(8) \cong R_3$, (ix) $\mathrm{lk}_{N_{24}}(3) \cong \mathrm{lk}_{N_{24}}(8) \cong R_1$. These imply that there is no isomorphism between any two members of $\mathcal{N}_1$.

Observe that (i) $\mathrm{lk}_{N_{25}}(3) \cong R_3$ and $\mathrm{lk}_{N_{25}}(8) \cong R_4$, (ii) $\mathrm{lk}_{N_{26}}(3) \cong \mathrm{lk}_{N_{26}}(8) \cong R_3$ and $G_6(N_{26}) = (V, \{53, 38, 84\})$, (iii) $\mathrm{lk}_{N_{27}}(3) \cong \mathrm{lk}_{N_{27}}(8) \cong R_3$, $G_6(N_{27}) = (V, \{78, 81, 13, 37\})$ and $\mathrm{NEG}(N_{27}) = \{24, 56\}$, (iv) $\mathrm{lk}_{N_{28}}(3) \cong \mathrm{lk}_{N_{28}}(8) \cong R_3$, $G_6(N_{28}) = (V, \{18, 84, 43, 31\})$ and

$\mathrm{NEG}(N_{28}) = \{75, 56\}$, (v) $\mathrm{lk}_{N_{29}}(3) \cong R_3$ and $\mathrm{lk}_{N_{29}}(8) \cong R_2$, (vi) $\mathrm{lk}_{N_{30}}(3) \cong R_1$ and $\mathrm{lk}_{N_{30}}(8) \cong R_3$, (vii) $\mathrm{lk}_{N_{31}}(3) \cong \mathrm{lk}_{N_{31}}(8) \cong R_2$. These imply that there is no isomorphism between any two members of $\mathcal{N}_2$.

Observe that (i) $\mathrm{lk}_{N_{32}}(3) \cong \mathrm{lk}_{N_{32}}(8) \cong R_3$, (ii) $\mathrm{lk}_{N_{33}}(3) \cong \mathrm{lk}_{N_{33}}(8) \cong R_4$, (iii) $\mathrm{lk}_{N_{34}}(3) \cong \mathrm{lk}_{N_{34}}(8) \cong R_2$, (iv) $\mathrm{lk}_{N_{35}}(3) \cong R_2$ and $\mathrm{lk}_{N_{35}}(8) \cong R_1$. These imply that there is no isomorphism between any two members of $\mathcal{N}_3$.

Since a member of $\mathcal{N}_i$ is nonisomorphic to a member of $\mathcal{N}_j$ for $i \neq j$, the above imply part (a). Part (b) follows from the definition of N_k for $8 \leq k \leq 35$. $\square$

The 3-dimensional *Kummer variety* K^3 is the torus $S^1 \times S^1 \times S^1$ modulo the involution $\sigma : x \mapsto -x$. It has 8 singular points corresponding to 8 elements of order 2 in the abelian group $S^1 \times S^1 \times S^1$. In [11], Kühnel showed that N_5 triangulates K^3. For a topological space X, $C(X)$ denotes a cone with base X. Let $H = D^2 \times S^1$ denote the solid torus. As a consequence of the above lemmas we get.

Corollary 3.10. *All the* 8*-vertex normal* 3*-pseudomanifolds triangulate seven distinct topological spaces, namely,* $|S^3_{8,j}| = S^3$ *for* $1 \leq j \leq 38$, $|N_1|$, $|N_2| = S(S^1 \times S^1)$, $|N_3|$, $|N_4| = H \cup (C(\partial H))$, $|N_5| = |N_6| = K^3$, *and* $|N_i| = S(\mathbb{R}P^2)$ *for* $7 \leq i \leq 35$.

Proof. Let K be an 8-vertex normal 3-pseudomanifold. If K is a combinatorial 3-sphere, then it triangulates the 3-sphere S^3.

If K is not a combinatorial 3-sphere, then, by Lemma 3.9(b), $|K|$ is (pl) homeomorphic to $|N_1|, \ldots, |N_6|$, or $|N_7|$. Since $N_2 = \Sigma_{78}T$, $|N_2|$ is homeomorphic to the suspension $S(S^1 \times S^1)$. In N_4, the facets not containing the vertex 8 form a solid torus whose boundary is the link of 8. This implies that $|N_4| = H \cup (C(\partial H))$. It follows from Lemma 3.6(c) that $|N_6|$ is (pl) homeomorphic to $|N_5| = K^3$. Since N_{24} is isomorphic to the suspension $S^0_2 * R_1$, $|N_{24}| = S(\mathbb{R}P^2)$. Therefore, by Lemma 3.9(b), $|N_i|$ is (pl) homeomorphic to $|N_{24}| = S(\mathbb{R}P^2)$ for $7 \leq i \leq 35$. The result now follows from Lemma 3.6(a). $\square$

A 3-dimensional *pseudocomplex* K is an ordered pair (Δ, Φ), where Δ is a finite collection of disjoint tetrahedra and Φ is a family of affine isomorphisms between pairs of 2-faces of the tetrahedra in Δ. Let $|K|$ denote the quotient space obtained from the disjoint union $\sqcup_{\sigma \in \Delta} \sigma$ by setting $x = \varphi(x)$ for $\varphi \in \Phi$. The quotient of a tetrahedron $\sigma \in \Delta$ in $|K|$ is called a 3-*simplex* in $|K|$ and is denoted by $|\sigma|$. Similarly, the quotient of 2-faces, edges, and vertices of tetrahedra are called 2-*simplices*, *edges*, and *vertices* in $|K|$, respectively. If $|K|$ is homeomorphic to a topological space X, then K is called a *pseudotriangulation* of X. A 3-dimensional pseudocomplex $K = (\Delta, \Phi)$ is said to be *regular* if the following hold: (i) each 3-simplex in $|K|$ has four distinct vertices, and (ii) for $2 \leq i \leq 3$, no two distinct i-simplices in $|K|$ have the same set of vertices. So, for $2 \leq i \leq 3$, an i-simplex α in $|K|$ is uniquely determined by its vertices and denoted by $u_1 \cdots u_{i+1}$, where $u_1, \ldots, u_{i+1}$ are vertices of α. (But, the edges in $|K|$ may not form a simple graph.) So, we can identify a regular pseudocomplex $K = (\Delta, \Phi)$ with $\mathcal{K} := \{|\sigma| : \sigma \in \Delta\}$. Simplices and edges in $|K|$ are said to be simplices and edges of $\mathcal{K}$. Clearly, a pure 3-dimensional simplicial complex is a regular pseudocomplex.

Let $\mathcal{M}$ be a regular pseudotriangulation of X and $abcd$, $abce$ be two 3-simplices in $\mathcal{M}$. If ade, bde, cde are not 2-simplices in $\mathcal{M}$, then $\mathcal{N} := (\mathcal{M} \setminus \{abcd, abce\}) \cup \{abde, acde, bcde\}$ is also a regular pseudotriangulation of X. We say that $\mathcal{N}$ is obtained from $\mathcal{M}$ by the *generalized bistellar* 1*-move* κ_{abc}. If there is no edge between d and e in $\mathcal{M}$, then κ_F is called a *bistellar* 1*-move*. If there exist 3-simplices of the form $xyuv$, $xzuv$, $yzuv$ in a regular pseudotriangulation $\mathcal{P}$ of Y and xyz is not a 2-simplex, then $Q := (\mathcal{P} \setminus \{xyuv, xzuv, yzuv\}) \cup \{xyzu, xyzv\}$ is also a regular pseudotriangulation of Y. We say that Q is obtained from $\mathcal{P}$ by the *generalized bistellar* 2*-move* κ_E, where E is the common edge in $xyuv$, $xzuv$, and $yzuv$. If E is the only edge between u and v in $\mathcal{P}$, then κ_E is called a *bistellar* 2*-move*.

Let M be a pseudotriangulation of a closed 3-manifold and N a 3-pseudomanifold. A simplicial map $f : M \to N$ is said to be a *k-fold branched covering* (with discrete branch locus) if there exists $U \subseteq V(N)$ such that $|f||_{|M|\setminus f^{-1}(U)} : |M| \setminus f^{-1}(U) \to |N| \setminus U$ is a k-fold covering. The smallest such U (so that $|f||_{|M|\setminus f^{-1}(U)} : |M| \setminus f^{-1}(U) \to |N| \setminus U$ is a covering) is called the *branch locus*. It is known that N_1 can be regarded as a branched quotient of a regular hyperbolic tessellation (cf. [6]). In [11], Kühnel has shown that N_5 is a 2-fold branched quotient of a pseudotriangulation of the 3-dimensional torus. Here we prove the following theorem.

Theorem 3.11. (a) *N_{24} is a* 2*-fold branched quotient of a* 14*-vertex combinatorial* 3*-sphere.*

(b) *For $7 \leq i \leq 35$, N_i is a* 2*-fold branched quotient of a* 14*-vertex regular pseudotriangulation of the* 3*-sphere.*

Lemma 3.12. *Let M be a regular pseudotriangulation of a* 3*-manifold and N be a normal* 3*-pseudomanifold. Let $f : M \to N$ be a k-fold branched covering with at most two vertices in the branch locus. If $\kappa_e : N \mapsto \widetilde{N}$ is a bistellar* 2*-move, then there exist k generalized bistellar* 2*-moves $\kappa_{e_1}, \dots, \kappa_{e_k}$ such that $\kappa_{e_k}(\cdots(\kappa_{e_1}(M)))$ is a k-fold branched cover of $\widetilde{N}$.*

Proof. Let $\mathrm{lk}_N(e) = S^1_3(\{x, y, z\})$. Let $f^{-1}(e)$ consist of the edges $e_1, \dots, e_k$. Let the end points of e_i be u_i, v_i, the 3-simplices containing e_i be $u_iv_ix_iy_i$, $u_iv_ix_iz_i$, $u_iv_iy_iz_i$, and $f(x_i) = x$, $f(y_i) = y$, $f(z_i) = z$ for $1 \leq i \leq k$. Since xyz is not a simplex in N, it follows that $x_iy_iz_i$ is not a 2-simplex in M. Let M_i be the pseudocomplex consists of $u_iv_ix_iy_i$, $u_iv_ix_iz_i$, and $u_iv_iy_iz_i$. Since the number of vertices in the branched locus is at most 2, it follows that the number of vertices common in M_i and M_j is at most 2 for $i \neq j$. In particular, $\#(\{x_i, y_i, z_i\} \cap \{x_j, y_j, z_j\}) \leq 2$. Therefore, $x_jy_jz_j$ is not a 2-simplex in $\kappa_{e_i}(M)$. So, we can perform generalized bistellar 2-move κ_{e_j} on $\kappa_{e_i}(M) = (M \setminus M_i) \cup \{x_iy_iz_iu_i, x_iy_iz_iv_i\}$ for $i \neq j$. Clearly, $\widetilde{M} := \kappa_{e_k}(\cdots\kappa_{e_1}(M))$ is a k-fold branched cover of $\widetilde{N}$ (via the map $\widetilde{f}$, where $\widetilde{f}(w) = f(w)$ for $w \in V(\widetilde{M}) = V(M)$ and $\widetilde{f}(x_iy_iz_iu_i) = xyzu$ and $\widetilde{f}(x_iy_iz_iv_i) = xyzv$). □

Lemma 3.13. *Let M be a regular pseudotriangulation of a* 3*-manifold and N be a normal* 3*-pseudomanifold. Let $f : M \to N$ be a k-fold branched covering with at most two vertices in the branch locus. If $\kappa_F : N \mapsto \widetilde{N}$ is a bistellar* 1*-move, then there exist k generalized bistellar* 1*-moves $\kappa_{F_1}, \dots, \kappa_{F_k}$ such that $\kappa_{F_k}(\cdots(\kappa_{F_1}(M)))$ is a k-fold branched cover of $\widetilde{N}$.*

Proof. Let $F = xyz$ and $\mathrm{lk}_N(F) = \{u, v\}$. Let $f^{-1}(F)$ consist of the 2-simplices $F_1, \dots, F_k$. Let $F_i = x_iy_iz_i$ and the 3-simplices containing F_i be $x_iy_iz_iu_i$ and $x_iy_iz_iv_i$ and $f(x_i, y_i, z_i, u_i, v_i) = (x, y, z, u, v)$ for $1 \leq i \leq k$. Since f is simplicial, it follows that $x_iu_iv_i$, $y_iu_iv_i$, and $z_iu_iv_i$ are not 2-simplices in M. Let M_i be pseudocomplex $\{x_iy_iz_iu_i, x_iy_iz_iv_i\}$. Since the number of vertices in the branched locus is at most 2, it follows that $x_ju_jv_j$, $y_ju_jv_j$, and $z_ju_jv_j$ are not 2-simplices in $\kappa_{F_i}(M)$ for $i \neq j$. Then (by the similar arguments as in the proof of Lemma 3.12) $\kappa_{F_k}(\cdots\kappa_{F_1}(M))$ is a k-fold branched cover of $\widetilde{N}$. □

Proof of Theorem 3.11. If $\mathcal{I}$ denotes the boundary of the icosahedron, then there exists a simplicial 2-fold covering $f : \mathcal{I} \to R_1$. Consider the simplicial map $\widetilde{f} : S^0_2(\{a, b\}) * \mathcal{I} \to S^0_2(\{c, d\}) * R_1$

Table 1: 8-vertex normal 3-pseudomanifolds which are not combinatorial 3-manifolds.

X	f-vector (f_1, f_2, f_3)	$\chi(X)$	$n_s(X)$	links of singular vertices	Geometric carriers, Homology (H_1, H_2, H_3)
N_1	$(28, 56, 28)$	8	8	all are T	$\lvert N_1\rvert$ is simply connected, $(H_1, H_2, H_3) = (0, \mathbb{Z}^8, \mathbb{Z})$
N_2	$(28, 44, 22)$	2	2	both are T	$\lvert N_2\rvert = S(S^1 \times S^1)$
N_3	$(28, 46, 23)$	3	5	$T,\ R_2,\ R_2,\ R_3,\ R_3$	$(H_1, H_2, H_3) = (0, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$
N_4	$(28, 42, 21)$	1	1	T	$\lvert N_4\rvert = H \cup (C(\partial H))$
N_5	$(28, 48, 24)$	4	8	all are R_4	$\lvert N_5\rvert = K^3$
N_6	"	"	"	all are R_3	$\lvert N_6\rvert = K^3$
N_7	$(28, 42, 21)$	1	2	both are R_4	$\lvert N_7\rvert = S(\mathbb{R}P^2)$
N_i, $8 \le i \le 15$	"	"	"	both are in $\{R_1, \ldots, R_4\}$	$\lvert N_i\rvert = S(\mathbb{R}P^2)$
$N_i,\ 16 \le i \le 24$	$(27, 40, 20)$	"	"	"	"
$N_i,\ 25 \le i \le 31$	$(26, 38, 19)$	"	"	"	"
$N_i,\ 32 \le i \le 35$	$(25, 36, 18)$	"	"	"	"

[Here K^3 is the 3-dimensional Kummer variety, $H = D^2 \times S^1$ is the solid torus, $S(Y)$ is the topological suspension of Y, and $n_s(X)$ is the number of singular vertices in X.]

given by $\widetilde{f}(a) = c$, $\widetilde{f}(b) = d$ and $\widetilde{f}(u) = f(u)$ for $u \in V(\mathcal{O})$. Then $\widetilde{f}$ is a 2-fold branched covering with branch locus $\{c, d\}$. Since N_{24} is isomorphic to the suspension $S^0_2 * R_1$, it follows that N_{24} is a 2-fold branched quotient of the 14-vertex combinatorial 3-sphere $S^0_2(\{a, b\}) * \mathcal{O}$ (with branch locus $\{3, 8\}$). This proves part (a).

The result now follows from Lemmas 3.9(a), 3.12, and 3.13. (In fact, to obtain a 2-fold branched cover $\widetilde{N}_{14}$ of N_{14} from $R_1 * S^0_2$, one needs one bistellar 1-move and then one generalized bistellar 1-move; and all other moves required in the proof are bistellar moves on regular pseudotriangulations of S^3.) □

Remark 3.14. The combinatorial 3-sphere $R_1 * S^0_2$ is a 2-fold branched cover of N_{24} and N_{14} can be obtained from N_{24} by a bistellar 1-move. Now, if $f : M \to N_{14}$ is a 2-fold branched covering and M is a combinatorial 3-manifold, then (since $\mathrm{lk}_{N_{14}}(8)$ is a 7-vertex triangulated $\mathbb{R}P^2$) the link of any vertex in $f^{-1}(8)$ is a 14-vertex triangulated S^2 and hence $f_0(M) > 14$. (Similarly, for $i \neq 24$, if N_i is a branched quotient of a combinatorial 3-manifold M, then $f_0(M) > 14$.) So, there does not exist a combinatorial 3-sphere M which is a branched cover of N_{14} and which can be obtained from $R_1 * S^0_2$ by proper bistellar moves.

In [7], Altshuler observed that N_1 is orientable and $|N_1|$ is simply connected. In [8], Lutz showed that $(H_1(N_1), H_2(N_1), H_3(N_1)) = (0, \mathbb{Z}^8, \mathbb{Z})$. The normal 3-pseudomanifold N_3 is the only among all the 35 which has singular vertices of different types, namely, one singular vertex whose link is a triangulated torus and four singular vertices whose links are triangulated real projective planes. Using polymake [12], we find that $(H_1(N_3), H_2(N_3), H_3(N_3)) = (0, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$. We summarized all the findings about $N_1, \ldots, N_{35}$ in Table 1.

Example 3.15. For $d \ge 2$, let

$$K^d_{2d+3} = \{v_i \cdots v_{j-1} v_{j+1} \cdots v_{i+d+1} : i + 1 \le j \le i + d, 1 \le i \le 2d + 3\} \tag{3.5}$$

(additions in the suffixes are modulo $2d+3$). It was shown in [13] the following : (i) K^d_{2d+3} is a triangulated d-manifold for all $d \geq 2$, (ii) K^d_{2d+3} triangulates $S^{d-1} \times S^1$ for d even, and triangulates the twisted product $S^{d-1} \times_- S^1$ (the twisted S^{d-1}-bundle over S^1) for d odd. For $d \geq 3$, K^d_{2d+3} is the unique nonsimply connected $(2d+3)$-vertex triangulated d-manifold (cf. [14]). The combinatorial 3-manifolds K^3_9 was first constructed by Walkup in [15].

From K^3_9, we construct the following 10-vertex combinatorial 3-manifold:

$$\begin{aligned} A^3_{10} := {} & \left(K^3_9 \setminus \{v_1v_2v_3v_5, v_2v_3v_5v_6, v_3v_5v_6v_7, v_3v_4v_6v_7, v_4v_6v_7v_8\}\right) \\ & \cup \{v_0v_1v_2v_3, v_0v_1v_2v_5, v_0v_1v_3v_5, v_0v_2v_3v_6, v_0v_2v_5v_6, v_0v_3v_5v_7, v_0v_5v_6v_7, \\ & \quad v_0v_3v_4v_6, v_0v_3v_4v_7, v_0v_4v_6v_8, v_0v_4v_7v_8, v_0v_6v_7v_8\}. \end{aligned} \tag{3.6}$$

[Geometrically, first we remove a pl 3-ball consisting of five 3-simplices from $|K^3_9|$. This gives a pl 3-manifold with boundary and the boundary is a 2-sphere. Then we add a cone with base this boundary and vertex v_0. So, the new polyhedron $|A^3_{10}|$ is pl homeomorphic to $|K^3_9|$. This implies that the simplicial complex A^3_{10} is a combinatorial 3-manifold.]

The only nonedge in A^3_{10} is v_0v_9 and there is no common 2-face in the links of v_0 and v_9 in A^3_{10}. So, A^3_{10} does not allow any bistellar 1-move. So, A^3_{10} is a 10-vertex nonneighbourly combinatorial 3-manifold which does not admit any bistellar 1-move.

Similarly, from K^4_{11}, we construct the following 12-vertex triangulated 4-manifold:

$$\begin{aligned} A^4_{12} := {} & \left(K^4_{11} \setminus \{v_1v_2v_3v_4v_6, v_2v_3v_4v_6v_7, v_3v_4v_6v_7v_8, v_4v_6v_7v_8v_9, v_4v_5v_7v_8v_9, v_5v_7v_8v_9v_{10}\}\right) \\ & \cup \{v_0v_1v_2v_3v_4, v_0v_1v_2v_3v_6, v_0v_1v_2v_4v_6, v_0v_1v_3v_4v_6, v_0v_2v_3v_4v_7, v_0v_2v_3v_6v_7, v_0v_2v_4v_6v_7, \\ & \quad v_0v_3v_4v_6v_8, v_0v_3v_4v_7v_8, v_0v_3v_6v_7v_8, v_0v_4v_6v_7v_9, v_0v_4v_6v_8v_9, v_0v_4v_7v_8v_9, \\ & \quad v_0v_4v_5v_7v_9, v_0v_4v_5v_8v_9, v_0v_4v_7v_8v_9, v_0v_5v_7v_8v_{10}, v_0v_5v_7v_9v_{10}, v_0v_5v_8v_9v_{10}\}. \end{aligned} \tag{3.7}$$

The only nonedge in A^4_{12} is v_0v_{11} and there is no common 2-face in the links of v_0 and v_{11} in A^4_{12}. So, A^4_{12} does not allow any bistellar 1-move. So, A^4_{12} is a 12-vertex nonneighbourly triangulated 4-manifold which does not admit any bistellar 1-move.

By the same way, one can construct a $(2d+4)$-vertex nonneighbourly triangulated d-manifold A^d_{2d+4} (from K^d_{2d+3}) which does not admit any bistellar 1-move for all $d \geq 3$.

Example 3.16. Let N_3 be as in Example 3.5. Let M be obtained from N_3 by starring two vertices u and v in the facets 1248 and 3568, respectively, that is, $M = \kappa_{1248}(\kappa_{3568}(N_3))$. Then M is a 10-vertex normal 3-pseudomanifold. Let B^3_9 be obtained from M by identifying the vertices u and v. Let the new vertex be 9. Then

$$B^3_9 := \left(N_3 \setminus \{1248, 3568\}\right) \cup \{1249, 1289, 1489, 2489, 3569, 3589, 3689, 5689\}. \tag{3.8}$$

The degree 3 edges in B^3_9 are 16, 17, and 67; but none of these edges is removable. So, no bistellar 2-moves are possible from B^3_9. The only nonedge in B^3_9 is 79. Since there is no common 2-face in the links of 7 and 9, no bistellar 1-move is possible. So, B^3_9 is a 9-vertex nonneighbourly 3-pseudomanifold which does not admit any proper bistellar move.

4. Proofs

For $n \geq 4$, by an S^2_n we mean a combinatorial 2-sphere on n vertices. If $\kappa_\beta : M \mapsto N$ is a bistellar 1-move, then $\deg_N(v) \geq \deg_M(v)$ for $v \in V(M)$. Here we prove the following.

Lemma 4.1. *Let M be an n-vertex 3-pseudomanifold and u be a vertex of degree 4. If $n \geq 6$, then there exists a bistellar 1-move $\kappa_\beta : M \mapsto N$ such that $\deg_N(u) = 5$.*

Proof. Let $\mathrm{lk}_M(u) = S_4^2(\{a,b,c,d\})$ and $\beta = abc$. Let $\mathrm{lk}_M(\beta) = \{u,x\}$. If $x = d$, then the induced complex $K = M[\{u,a,b,c,d\}]$ is a 3-pseudomanifold. Since $n \geq 6$, K is a proper subcomplex of M. This is not possible. So, $x \neq d$ and hence ux is a nonedge in M. Then κ_β is a bistellar 1-move. Since ux is an edge in $\kappa_\beta(M)$, κ_β is a required bistellar 1-move. □

Lemma 4.2. *Let M be an n-vertex 3-pseudomanifold and u be a vertex of degree 5. If $n \geq 7$, then there exists a bistellar 1-move $\kappa_\beta : M \mapsto N$ such that $\deg_N(u) = 6$.*

Proof. Since $\deg_M(u) = 5$, the link of u in M is of the form $S_2^0(\{a,b\}) * S_3^1(\{x,y,z\})$ for some vertices a,b,x,y,z of M. If both $xyza$ and $xuzb$ are facets, then the induced subcomplex $M[\{x,y,z,u,a,b\}]$ is a 3-pseudomanifold. This is not possible since $n \geq 7$. So, without loss of generality, assume that $xyza$ is not a facet. Again, if $xyab$, $xzab$, and $yzab$ all are facets, then the induced subcomplex $M[\{u,x,y,z,a,b\}]$ is a 3-pseudomanifold, which is not possible. So, assume that $xyab$ is not a facet.

Consider the face $\beta = xya$. Suppose $\mathrm{lk}_M(\beta) = \{u,w\}$. From the above, $w \notin \{z,b\}$. So, uw is a nonedge and hence κ_β is a required bistellar 1-move. □

Lemma 4.3. *Let M be a nonneighbourly 8-vertex 3-pseudomanifold and u be a vertex of degree 6. If the degree of each vertex is at least 6, then there exists a bistellar 1-move $\kappa_\tau : M \mapsto N$ such that $\deg_N(u) = 7$.*

Proof. Let u be a vertex with $\deg_M(u) = 6$ and uv be a nonedge. Let $L = \mathrm{lk}_M(u)$.

Claim 1. There exists a 2-face τ such that $\tau \cup \{u\}$ and $\tau \cup \{v\}$ are facets.

First consider the case when there exists a vertex w such that $\deg_L(w) = 5$. Let $\mathrm{lk}_L(w) (= \mathrm{lk}_M(uw)) = C_5(1,2,3,4,5)$.

Let $K = \mathrm{lk}_M(w)$. Since $\deg(v) = 6$, vw is an edge. Thus K contains 7 vertices. If one of $12v, \ldots, 45v, 51v$ is a 2-face, say $12v$, then $12wv$ and $12wu$ are facets. In this case, $\tau = 12w$ serves the purpose. So, assume that $12v, \ldots, 45v, 51v$ are nonfaces in K. Then there are at least three 2-faces (not containing u) containing the edges $12, \ldots, 45, 51$ in K. Also, there are at least three 2-faces containing v in K. So, the number of 2-faces in K is at least 11. This implies that $\deg_K(v) = 3$ or 4 and K is a 7-vertex $\mathbb{R}P^2$ or P_4. Since $\deg_K(u) = 5$, it follows that K is isomorphic to R_2, R_3, or P_4 (defined in Section 2). In each case, (since $\deg_K(u) = 5$, $\deg_K(v) = 3$ or 4, and uv is a nonedge) there exists an edge α in K such that $\alpha \cup \{u\}$ and $\alpha \cup \{v\}$ are 2-faces in K and hence $\tau = \alpha \cup \{w\}$ serves the purpose.

Now, assume that L has no vertex of degree 5. Then L must be of the form $S_2^0(\{a_1,a_2\}) * S_2^0(\{b_1,b_2\}) * S_2^0(\{c_1,c_2\})$. If possible, let $a_ib_jc_kv$ is not a facet for $1 \leq i, j,k \leq 2$. Consider the 2-face $a_1b_1c_1$. There exists a vertex $x \neq u$ such that $a_1b_1c_1x$ is a facet. Assume, without loss of generality, that $a_1b_1c_1a_2$ is a facet. Since $\deg(c_1) > 5$ (resp., $\deg(b_1) > 5$), $a_1a_2b_2c_1$ (resp., $a_1a_2b_1c_2$) is not a facet. So, the facet (other than $a_1b_2c_1u$) containing $a_1b_2c_1$ must be $a_1b_2c_1c_2$. Similarly, the facet (other than $a_1b_1c_2u$) containing $a_1b_1c_2$ must be $a_1b_1b_2c_2$. Then $a_1b_2c_1c_2$, $a_1b_1b_2c_2$, and $a_1b_2c_2u$ are three facets containing $a_1b_2c_2$, a contradiction. This proves the claim.

By the claim, there exists a 2-simplex τ such that $\mathrm{lk}_M(\tau) = \{u,v\}$. Since uv is a nonedge of M, $\kappa_\tau : M \mapsto \kappa_\tau(M) = N$ is a bistellar 1-move. Since uv is an edge in N, it follows that $\deg_N(u) = 7$. □

Proof of Theorem 1.1. Let M be an 8-vertex 3-pseudomanifold. Then, by Lemma 4.1, there exist bistellar 1-moves $\kappa_{A_1}, \ldots, \kappa_{A_k}$, for some $k \geq 0$, such that the degree of each vertex in $\kappa_{A_k}(\cdots(\kappa_{A_1}(M))$ is at least 5. Therefore, by Lemma 4.2, there exist bistellar 1-moves $\kappa_{A_{k+1}}, \ldots, \kappa_{A_l}$, for some $l \geq k$, such that the degree of each vertex in $\kappa_{A_l}(\cdots\kappa_{A_k}(\cdots(\kappa_{A_1}(M)))$ is at least 6. Then, by Lemma 4.3, there exist bistellar 1-moves $\kappa_{A_{l+1}}, \ldots, \kappa_{A_m}$, for some $m \geq l$, such that the degree of each vertex in $\kappa_{A_m}(\cdots\kappa_{A_l}(\cdots\kappa_{A_k}(\cdots(\kappa_{A_1}(M))))$ is 7. This proves the theorem. □

Lemma 4.4. *Let K be an 8-vertex combinatorial 3-manifold. If K is neighbourly, then K is isomorphic to $S^3_{8,35}$, $S^3_{8,36}$, $S^3_{8,37}$, or $S^3_{8,38}$.*

Proof. Since K is a neighbourly combinatorial 3-manifold, by Proposition 2.3, the link of any vertex is isomorphic to $S_5, \ldots, S_8$, or S_9.

Claim 1. The links of all the vertices cannot be isomorphic to S_9 ($= S^0_2 * C_5$).

Otherwise, let $\mathrm{lk}(8) = S^0_2(6,7) * C_5(1,2,\ldots,5)$. Consider the vertex 2. Since the degree of 2 is 7, 1267 or 2367 is not a facet. Assume, without loss of generality, that 1267 is not a facet. Again, if 1236 is a facet, then $\deg_{\mathrm{lk}(2)}(6) = 3$ and hence $\mathrm{lk}(2) \not\cong S_9$. So, 1236 is not a facet. Similarly, 1256 is not a facet. Then the facet other than 1268 containing 126 must be 1246. Similarly, 1247 is a facet. This implies that $\mathrm{lk}(2) = S^0_2(6,7) * C_5(1,4,5,3,8)$. Thus $\deg(26) = 5$. Similarly, $\deg(16) = \deg(36) = \deg(46) = \deg(56) = 5$. Then, the 7-vertex 2-sphere $\mathrm{lk}(6)$ contains five vertices of degree 5. This is not possible. This proves the claim.

Case 1. Consider the case when K has a vertex, (say 8) whose link is isomorphic to S_8. Assume, without loss of generality, that the facets containing the vertex 8 are 1238, 1268, 1348, 1458, 1568, 2348, 2478, 2678, 4578, and 5678. Since $\deg(3) = 7$, $1234 \notin K$. Hence the facet other than 1238 containing the face 123 is one of 1235, 1236, or 1237.

If $1236 \in K$, then, clearly, $\deg(17) = 3$ or 4. If $\deg(17) = 4$, then on completing $\mathrm{lk}(1)$, we see that $1457, 1567 \in K$, thereby showing that $\deg(5) = 5$, an impossibility. Hence, $\deg(17) = 3$ and, therefore, $1457 \in K$. There are two possibilities for the completion of $\mathrm{lk}(1)$. If 1347, 1356, $1357 \in K$, from the links of 4 and 3, we see that $2346, 2467, 3467, 3567 \in K$. Here, $\deg(5) = 6$. If $1346, 1467, 1567 \in K$, then $\deg(5) = 5$. Thus, $1236 \notin K$.

Case 1.1. $1235 \in K$. Since $\deg(1) = 7$, either 1345 or 1256 is a facet. In the first case, $1257, 1267, 1567 \in K$. Here, $\deg(6) = 5$, a contradiction. So, $1256 \in M$ and hence $1347, 1357, 1457 \in K$. From the links of the vertices $1, 4, 7$ and 5, we see that $1256, 2346, 2467, 3467, 3567, 2356 \in K$. Here, $K \cong S^3_{8,38}$ by the map $(1,5,8,6)(2,7)(3,4)$.

Case 1.2. $1237 \in K$. By the same argument as in Case 1.1 (replace the vertex 1 by vertex 2), we get $1267, 2345, 2357, 2457 \in K$. From $\mathrm{lk}(1)$ and $\mathrm{lk}(7)$, $1346, 1456, 3456, 1367, 3567 \in K$. Here, $K \cong S^3_{8,38}$ by the map $(1,7,8,6)(2,5)(3,4)$.

Case 2. K has no vertex whose link is isomorphic to S_8 but has a vertex whose link is isomorphic to S_6. Using the same method as in Case 1.1, we find that $K \cong S^3_{8,37}$.

Case 3. K has no vertex whose link is isomorphic to S_8 or S_6 but has a vertex whose link is isomorphic to S_7. Using the same method as in Case 1.1, we find that $K \cong S^3_{8,36}$.

Case 4. K has no vertex whose link is isomorphic to S_6, S_7, or S_8 but has a vertex (say 8) whose link is isomorphic to S_5. The facets through 8 can be assumed to be 1238, 1278, 1348, 1458, 1568, 1678, 2348, 2458, 2568, and 2678. Clearly, $1234, 1267 \notin K$. If $\deg(15) = 6$, then from $\mathrm{lk}(1)$ and $\mathrm{lk}(5)$, we see that 1235, 1345, $2345 \in K$, thereby showing that $\deg(3) = 5$. Hence $1237 \in K$. Now, we can assume, without loss of generality, that the facets required to complete $\mathrm{lk}(1)$ are 1347, 1457, and 1567. Now, consider $\mathrm{lk}(2)$. If $\deg(27) = 6$, then after completing the links of 2 and 7, we observe that $\deg(4) = 6$. Hence $\deg(23) = 6$. The links of 2, 7, and 6 show that 2345, 2356, 2367, 3467, 4567, and $3456 \in K$. Here, $K \cong S^3_{8,35}$ by the map $(2,3,4,5,6,7,8)$. This completes the proof. □

Lemma 4.5. *Let K be an 8-vertex neighbourly normal 3-pseudomanifold. If K has one vertex whose link is the 7-vertex torus T, then K is isomorphic to N_1, N_2, N_3, or N_4.*

Proof. Let us assume that $V(K) = \{1, \dots, 8\}$ and the link of the vertex 8 is the 7-vertex torus T. So, the facets containing 8 are 1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468, 3568, 4578, and 4678. We have the following cases.

Case 1. There is a vertex (other than the vertex 8), say 7, whose link is isomorphic to T. Then $\mathrm{lk}(7)$ has no vertex of degree 3 and hence $2367, 1457, 1237, 1357 \notin K$. This implies that the facet (other than 1378) containing 137 is 1367 or 1347. In the first case, $\mathrm{lk}(17) = C_6(5,8,3,6,4,2)$. Thus, $1367, 1467, 1247, 1257 \in K$. Then, from the links of 67 and 37, we get $2567, 3567, 2347, 3457 \in K$. Now, from $\mathrm{lk}(34)$, $1346 \notin K$. Then, from the links of $36, 34, 23, 14$, and 26, we get $1236, 2346, 1345, 1235, 1456, 2456 \in K$. Here, $K = N_1$.

In the second case, $\mathrm{lk}(37) = C_6(2,8,1,4,6,5)$. Thus, $1347, 3467, 3567, 2357 \in K$. Now, from the links of 47 and 67, we get $1247, 2457, 1567, 1267 \in K$. Here, $K = N_2$.

Case 2. There is a vertex whose link is a 7-vertex $\mathbb{R}P^2$.

Claim 1. There exists a vertex in K whose link is isomorphic to R_2.

If there is vertex whose link is isomorphic to R_2, then we are done. Otherwise, since $\mathrm{Aut}(\mathrm{lk}(8))$ acts transitively on $\{1, \dots, 7\}$, assume that $\mathrm{lk}(4) \cong R_3$ (resp., R_4). Since $(1,2,5,7,6,3) \in \mathrm{Aut}(\mathrm{lk}(8))$, we may assume that the degree 4 vertex (resp., vertices) in $\mathrm{lk}(4)$ is 1 (resp., are 1, 5, 6). Then, from $\mathrm{lk}(4), 1247, 1347, 2467 \in K$. This implies that $\mathrm{lk}(7)$ is a nonsphere and $\deg(67) = 3$. Hence $\mathrm{lk}(7) \cong R_2$. This proves the claim.

By the claim, we can assume that $\mathrm{lk}(4) \cong R_2$. Again, we may assume that the vertex 1 is of degree 3 in $\mathrm{lk}(4)$. Then, from $1k(4), 1234, 2347, 2456, 2467, 3456, 3457 \in K$. Considering the links of the edges 36, 26, 27, 25, and 13, we get $1256, 1235, 1357 \in K$. Here, $K = N_3$.

Case 3. Only singular vertex in K is 8. So, the link of each vertex (other than vertex 8) is an S^2_7 (a 7-vertex 2-sphere). Since 8 is a degree 6 vertex in $\mathrm{lk}(u)$, it follows that $\mathrm{lk}(u)$ is isomorphic to one of S_5, S_6, or S_7 (defined in Example 2.2) for any vertex $u \neq 8$. If $\mathrm{lk}(1) \cong S_5$, then (since $(3,4,2,6,5,7) \in \mathrm{Aut}(\mathrm{lk}(8))$), we may assume that the other degree 6 vertex in $\mathrm{lk}(1)$ is 3. Then, from the links of 1 and 3, 1348, 1234, 1346 are facets containing 134, a contradiction. If $\mathrm{lk}(1) \cong S_6$, then (since $\mathrm{lk}(18) = C_6(3,4,2,6,5,7)$) we may assume that the degree 5 vertices in $\mathrm{lk}(1)$ are 2, 3, and 5. Then $\mathrm{lk}(3)$ cannot be an S^2_7, a contradiction. So, $\mathrm{lk}(1) \cong S_7$. Since $\mathrm{Aut}(\mathrm{lk}(8))$ acts transitively on $\{1, \dots, 7\}$, it follows that the link of each vertex is isomorphic to S_7.

Since $\mathrm{lk}(18) = C_6(3,4,2,6,5,7)$ and $(3,4,2,6,5,7) \in \mathrm{Aut}(\mathrm{lk}(8))$, we may assume that the degree 5 vertices in $\mathrm{lk}(1)$ are 4 and 5. Since $\mathrm{lk}(4) \cong S_7$, it follows that $1456 \notin K$. Then, from $\mathrm{lk}(1), 1245, 1256, 1347, 1457 \in K$. Now, from the links of 4 and 5, we get $3467, 2356 \in K$. Then, from $\mathrm{lk}(2), 2367 \in K$. Here $K = N_4$. This completes the proof. □

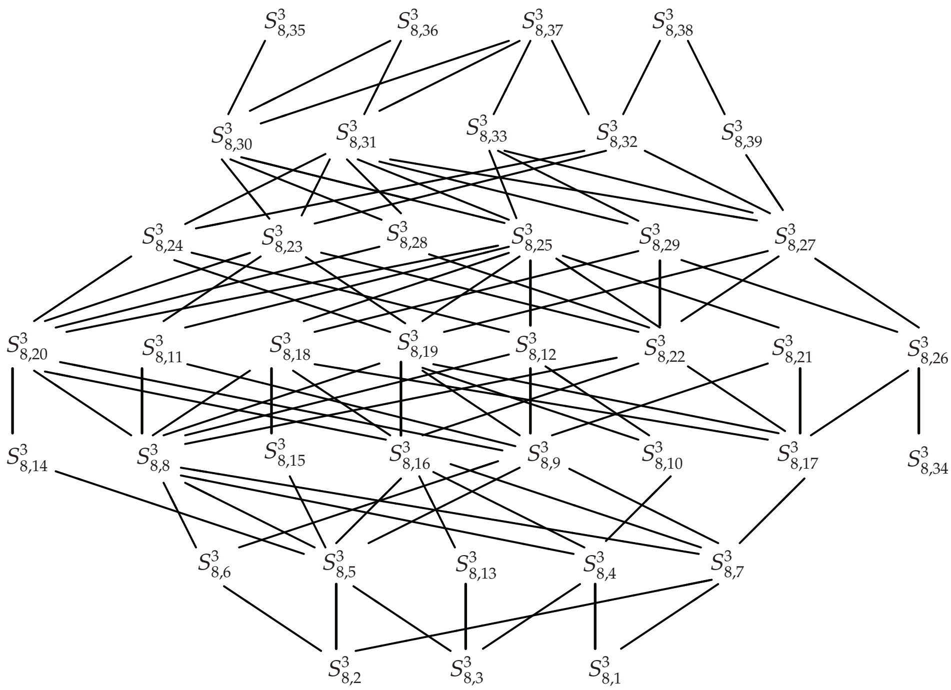


Figure 3: Hasse diagram of the poset of the 8-vertex combinatorial 3-manifolds (the partial order relation is as defined in Section 2).

Lemma 4.6. *Let K be an* 8*-vertex neighbourly normal* 3*-pseudomanifold. If K is not a combinatorial* 3*-manifold and has no vertex whose link is isomorphic to the* 7*-vertex torus T then K is isomorphic to* $N_5, \ldots, N_{14}$ *or* N_{15}*.*

Proof. Let n_s be the number of singular vertices in K. Since K is neighbourly, by Proposition 2.3, the link of any vertex is either a 7-vertex $\mathbb{R}P^2$ or a 7-vertex S^2. So, the number of facets through a singular (resp., nonsingular) vertex is 12 (resp., 10). Let f_3 be the number of facets of K. Consider the set $S = \{(v, \sigma) : \sigma \text{ is a facet of } K \text{ and } v \in \sigma \text{ is a vertex }\}$. Then $f_3 \times 4 = \#(S) = n_s \times 12 + (8 - n_s) \times 10 = 80 + 2n_s$. This implies n_s is even. Since K is not a combinatorial 3-manifold, it follows that $n_s \neq 0$ and hence $n_s \geq 2$. So, K has at least two vertices whose links are isomorphic to R_2, R_3, or R_4.

Case 1. There exist (at least) two vertices whose links are isomorphic to R_4. Assume that $\mathrm{lk}_M(8) = R_4$. Then $1258, 1268, 1358, 1378, 1468, 1478, 2368, 2378, 2458, 2478, 3458, 3468 \in K$. Since $(1,3,4)(5,6,7), (1,2)(3,4) \in \mathrm{Aut}(\mathrm{lk}(8))$, we may assume that $\mathrm{lk}(3)$ or $\mathrm{lk}(7) \cong R_4$.

Case 1.1. $\mathrm{lk}(7) \cong R_4$. Since $\mathrm{lk}_{\mathrm{lk}(7)}(8) = C_4(1,3,2,4)$, it follows that $1, 2, 3, 4$ are degree 5 vertices in $\mathrm{lk}(7)$. Since $(3,4)(5,6) \in \mathrm{Aut}(\mathrm{lk}(8))$, assume without loss that $136, 145 \in \mathrm{lk}(7)$. Then, from $\mathrm{lk}(7)$, we get $1257, 1267, 1367, 1457, 2357, 2467, 3457, 3467 \in K$. This shows that $\mathrm{lk}(2)$ is an $\mathbb{R}P^2_7$. Since $3457, 3458 \in K$, it follows that $2345 \notin K$. Then, from $\mathrm{lk}(2)$, $2356, 2456 \in K$. Then, from the links of 3 and 4, $1356, 1456 \in K$. Here $K = N_5$.

Case 1.2. $\mathrm{lk}(7) \not\cong R_4$. So, $\mathrm{lk}(3) \cong R_4$. Since $\mathrm{lk}_{\mathrm{lk}(3)}(8) = C_6(1,7,2,6,4,5)$, the degree 4 vertices in $\mathrm{lk}(3)$ are either $5, 6, 7$, or $1, 2, 4$. In the first case, on completion of $\mathrm{lk}(3)$, we observe that 56, 67, 57 remain nonedges in K. So, the degree 4 vertices in lk(3) are $1, 2$, and 3. Then 1356, 1367, 2356, 2357, 3457, and 3467 are facets. Since $\text{lk}(7) \ncong R_4$ and $\deg(78) = 4$, either $\text{lk}(7) \cong R_3$ or lk(7) is an S^2_7. In the former case, 2567 is a facet. This is not possible from lk(25). So, lk(7) is an S^2_7. Then, from $\text{lk}(7), 1467, 2457 \in K$. Now, from $\text{lk}(1), 1256 \in K$. Here, $K = N_7$.

Case 2. Exactly one vertex whose link is isomorphic to R_4 and there exists a vertex whose link is isomorphic to R_3. Using the same method as in Case 1, we find that $K \cong N_8$.

Case 3. Exactly one vertex whose link is isomorphic to R_4, there is no vertex whose link is isomorphic to R_3 and there exists (at least) a vertex whose link is isomorphic to R_2. Using the same method as in Case 1, we find that $K \cong N_9$.

Case 4. There is no vertex whose link is isomorphic to R_4 and there exist (at least) two vertices whose links are isomorphic to R_3. Assume that $\text{lk}_K(8) = R_4$, so that $\deg(78) = 4$. Using the same method as in Case 1, we get the following: (i) if $\text{lk}_K(7) \cong R_3$, then $K = N_6$ and (ii) if $\text{lk}_K(7) \ncong R_3$, then K is isomorphic to N_{10} or N_{11}.

Case 5. There is no vertex whose link is isomorphic to R_4, there exists exactly one vertex whose link is isomorphic to R_3 and there exists (at least) a vertex whose link is isomorphic to R_2. Using the same method as in Case 1, we find that K is isomorphic to N_{12} or N_{13}.

Case 6. There is no vertex whose link is isomorphic to R_4 or R_3 and there exist (at least) two vertices whose links are isomorphic to R_2. Using the same method as in Case 1, we find that K is isomorphic to N_{14} or N_{15}. This completes the proof. □

Proof of Theorem 1.2. Since $S^3_{8,m}$'s are combinatorial 3-manifolds and N_n's are not combinatorial 3-manifolds, $S^3_{8,m} \ncong N_n$ for $35 \leq m \leq 38$, $1 \leq n \leq 15$. Part (a) now follows from Lemmas 3.2, 3.7. Part (b) follows from Lemmas 4.4, 4.5, and 4.6. □

Lemma 4.7. *Let $\mathcal{S}_0, \ldots, \mathcal{S}_6$ be as in the proof of Lemma 3.4. If a combinatorial 3-manifold K is obtained from a member of $\mathcal{S}_j$ by a bistellar 2-move, then K is isomorphic to a member of $\mathcal{S}_{j+1}$ for $0 \leq j \leq 5$. Moreover, no bistellar 2-move is possible from a member of $\mathcal{S}_6$.*

Proof. Recall that $\mathcal{S}_0 = \{S^3_{8,35}, S^3_{8,36}, S^3_{8,37}, S^3_{8,38}\}$. The removable edges in $S^3_{8,37}$ are 13, 16, 17, 24, 27, 35, 46, 48, and 58. Since $(1,4)(2,7)(3,8) \in \text{Aut}(S^3_{8,37})$, up to isomorphisms, it is sufficient to consider the bistellar 2 -moves κ_{27}, κ_{24}, κ_{48}, κ_{58}, and κ_{46} only. Here $S^3_{8,33} := \kappa_{27}(S^3_{8,37})$, $S^3_{8,30} := \kappa_{24}(S^3_{8,37})$, $S^3_{8,32} := \kappa_{48}(S^3_{8,37})$, $S^3_{8,31} := \kappa_{58}(S^3_{8,37})$, and $\kappa_{46}(S^3_{8,37}) \cong S^3_{8,31}$ by the map $(1,4,5)(2,7)(3,6,8)$.

The removable edges in $S^3_{8,38}$ are 13, 38, 78, 27, 25, 15, and 46. Since $(1,2,8)(7,3,5), (1,2)(3,7)(4,6) \in \text{Aut}(S^3_{8,38})$, it is sufficient to consider the bistellar 2-moves κ_{46} and κ_{78} only. Here $S^3_{8,39} := \kappa_{46}(S^3_{8,36})$ and $\kappa_{78}(S^3_{8,38}) \cong S^3_{8,32}$ by the map $(1,7,8,4,6)(2,3)$.

The removable edges in $S^3_{8,36}$ are $13, 35, 58, 68, 46, 24, 27, 17$. Since $(1,5,6,2)(3,8,4,7)$ is an automorphism of $S^3_{8,36}$, it is sufficient to consider the bistellar 2-moves κ_{58} and κ_{68} only. Here $\kappa_{58}(S^3_{8,36}) = S^3_{8,31}$ and $\kappa_{68}(S^3_{8,36}) \cong S^3_{8,30}$ by the map $(1,6,4,8,2,5,7,3)$.

The removable edges in $S^3_{8,35}$ are $13, 35, 57, 71, 24, 46, 68$, and 82. Since $(1,2,\ldots,8), (1,8)(2,7)(3,6)(4,5) \in \text{Aut}(S^3_{8,35})$, it is sufficient to consider the bistellar 2-moves κ_{68} only. Here $\kappa_{68}(S^3_{8,35}) \cong S^3_{8,30}$ by the map $(1,7,3)(2,8,4,5,6)$. This proves the result for $j = 0$.

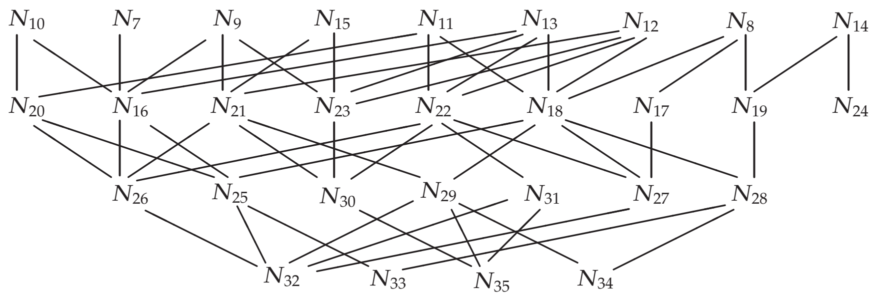


Figure 4: Hasse diagram of the poset of all the 3-pseudomanifolds $N_7, \ldots, N_{35}$.

By the same arguments as in the case for $j = 0$, one proves for the cases for $1 \leq j \leq 5$. We summarize these cases in Figure 3 below. Last part follows from the fact that none of $S^3_{8,1}$, $S^3_{8,3}$, or $S^3_{8,3}$ has any removable edges. □

Lemma 4.8. *Let $\mathcal{N}_0, \ldots, \mathcal{N}_3$ be as in the proof of Lemma 3.9. If a 3-pseudomanifold K is obtained from a member of $\mathcal{N}_j$ by a bistellar 2-move, then K is isomorphic to a member of $\mathcal{N}_{j+1}$ for $0 \leq j \leq 2$. Moreover, no bistellar 2-move is possible from a member of $\mathcal{N}_3$.*

Proof. Recall that $\mathcal{N}_0 = \{N_1, \ldots, N_{15}\}$. Since there are no degree 3 edges in N_1, N_2, N_5, and N_6, no bistellar 2-moves are possible from N_1, N_5, N_6, or N_2. The degree 3 edges in N_3 (resp., in N_4) are $14, 16, 17, 36, 67$ (resp., $13, 35, 57, 72, 24, 46, 61$). But, none of these edges is removable. So, bistellar 2-moves are not possible from N_3 or N_4.

The removable edges in N_7 are $12, 14, 24, 56, 57$, and 67. Since $(1,2)(6,7)$, $(1,2)(5,6)$, and $(1,5)(2,6)(3,8)(4,7)$ are automorphisms of N_7, it follows that up to isomorphisms, we only have to consider the bistellar 2-move κ_{67}. Here, $N_{16} = \kappa_{67}(N_7)$.

The removable edges in N_8 are $15, 17, 24, 56, 57$, and 67. Since $(1,6)(2,4), (1,6)(5,7), (2,4)(5,7) \in \mathrm{Aut}(N_8)$, we only consider the bistellar 2-moves κ_{24}, κ_{56}, and κ_{57}. Here, $N_{17} = \kappa_{24}(N_8)$, $N_{18} = \kappa_{56}(N_8)$, and $N_{19} = \kappa_{57}(N_8)$.

The removable edges in N_9 are $12, 23, 24$, and 67. Since $(1,4)(6,7) \in \mathrm{Aut}(N_9)$, we consider only κ_{12}, κ_{23}, and κ_{67}. Here, $N_{21} = \kappa_{12}(N_9)$, $N_{23} = \kappa_{23}(N_9)$, and $\kappa_{67}(N_9) = N_{16}$.

The removable edges in N_{10} are $12, 14, 24, 56, 57$, and 67. Since $(1,7)(2,5)(3,8)(4,6)$, $(1,4)(6,7) \in \mathrm{Aut}(N_{10})$, we consider the bistellar 2-moves κ_{56} and κ_{57} only. Here, $N_{20} = \kappa_{56}(N_{10})$ and $\kappa_{67}(N_{10}) = N_{16}$.

The removable edges of N_{11} are $14, 24, 56, 57$, and 67. Since $(1,2)(5,6)(3,8) \in \mathrm{Aut}(N_{11})$, we only consider the bistellar 2-moves κ_{14}, κ_{56}, and κ_{67}. Here, $N_{22} = \kappa_{14}(N_{11})$, $\kappa_{56}(N_{11}) = N_{20}$, and $\kappa_{67}(N_{11}) \cong N_{18}$ (by the map $(2,4)(5,7)$).

The removable edges in N_{12} are 12, 23, 45, and 57. Here, $\kappa_{12}(N_{12}) \cong N_{22}$ (by the map $(2,4,6)$), $\kappa_{23}(N_{12}) = N_{23}$, $\kappa_{45}(N_{12}) \cong N_{21}$ (by the map $(1,6,5,2,7,4)(3,8)$), and $\kappa_{57}(N_{12}) \cong N_{18}$ (by the map $(1,6,7,4)$).

The removable edges in N_{13} are $12, 23, 24, 56, 57$, and 67. Since $(1,4)(6,7) \in \mathrm{Aut}(N_{13})$, we only consider κ_{12}, κ_{23}, κ_{57}, and κ_{67}. Here, $\kappa_{12}(N_{13}) \cong N_{22}$ (by the map $(2,7,5,4)$), $\kappa_{23}(N_{13}) = N_{23}$, $\kappa_{57}(N_{13}) \cong N_{18}$ (by the map $(1,4)(6,7)$), and $\kappa_{67}(N_{13}) = N_{16}$.

The removable edges in N_{14} are $38, 56, 57, 67$. Since $(1,2,4)(5,6,7)(3,8) \in \mathrm{Aut}(N_{14})$, we only consider κ_{38} and κ_{57}. Here, $N_{24} = \kappa_{38}(N_{14})$ and $\kappa_{57}(N_{14}) = N_{19}$.

The removable edges in N_{15} are $15, 23, 24, 58$. Since $(1,7)(2,5)(3,8)(4,6) \in \mathrm{Aut}(N_{15})$, we only consider the bistellar 2-moves κ_{23} and κ_{24}. Here, $\kappa_{23}(N_{15}) = N_{23}$ and $\kappa_{24}(N_{15}) \cong N_{21}$ (by the map $(1,6,5,7,4)$). This proves the result for $j = 0$.

By the same arguments as in the case for $j = 0$, one proves the same for other cases (namely, for $j = 1, 2$) as well. We summarize these cases in Figure 4 . Last part follows from the fact that, for $N_i \in \mathcal{N}_3$, N_i has no removable edge. □

Proof of Corollary 1.3. Let $\mathcal{S}_0, \ldots, \mathcal{S}_6$ be as in the proof of Lemma 3.4. Let M be an 8-vertex combinatorial 3-manifold. Then, by Theorem 1.1, there exist bistellar 1-moves $\kappa_{A_1}, \ldots, \kappa_{A_m}$, for some $m \geq 0$, such that $M_1 := \kappa_{A_m}(\cdots(\kappa_{A_1}(M)))$ is a neighbourly 8-vertex 3-pseudomanifold. Since bistellar moves send a combinatorial 3-manifold to a combinatorial 3-manifold, M_1 is a combinatorial 3-manifold. Then, by Theorem 1.2, $M_1 \in \mathcal{S}_0$. In other words, $M = \kappa_{e_1}(\cdots(\kappa_{e_m}(M_1)))$, where $M_1 \in \mathcal{S}_0$ and $\kappa_{e_m} : M_1 \mapsto \kappa_{e_m}(M_1)$, $\kappa_{e_i} : \kappa_{e_{i+1}}(\cdots(\kappa_{e_m}(M_1))) \mapsto \kappa_{e_i}(\cdots(\kappa_{e_m}(M_1)))$, for $1 \leq i \leq m-1$, are bistellar 2-moves. Therefore, by Lemma 4.7, $M \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$. The result now follows from Lemma 3.4. □

Proof of Corollary 1.4. Let $\mathcal{N}_0, \ldots, \mathcal{N}_3$ be as in the proof of Lemma 3.9. Let M be an 8-vertex normal 3-pseudomanifold. Then, by Theorem 1.1, there exist bistellar 1-moves $\kappa_{A_1}, \ldots, \kappa_{A_m}$, for some $m \geq 0$, such that $M_1 := \kappa_{A_m}(\cdots(\kappa_{A_1}(M)))$ is a neighbourly 3-pseudomanifold. Since bistellar moves send a normal 3-pseudomanifold to a normal 3-pseudomanifold, M_1 is normal. Hence, by Theorem 1.2, $M_1 \in \mathcal{N}_0$. In other words, $M = \kappa_{e_1}(\cdots(\kappa_{e_m}(M_1)))$, where $M_1 \in \mathcal{N}_0$ and $\kappa_{e_m} : M_1 \mapsto \kappa_{e_m}(M_1)$, $\kappa_{e_i} : \kappa_{e_{i+1}}(\cdots(\kappa_{e_m}(M_1))) \mapsto \kappa_{e_i}(\cdots(\kappa_{e_m}(M_1)))$, for $1 \leq i \leq m-1$, are bistellar 2-moves. Therefore, by Lemma 4.8, $M \in \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. The result now follows from Lemma 3.9. □

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References

[1] B. Bagchi and B. Datta, "Uniqueness of walkup's 9-vertex 3-dimensional Klein bottle," *Discrete Mathematics*. In press.
[2] A. Altshuler, "Combinatorial 3-manifolds with few vertices," *Journal of Combinatorial Theory. Series A*, vol. 16, no. 2, pp. 165–173, 1974.
[3] B. Grünbaum and V. P. Sreedharan, "An enumeration of simplicial 4-polytopes with 8 vertices," *Journal of Combinatorial Theory*, vol. 2, pp. 437–465, 1967.
[4] D. Barnette, "The triangulations of the 3-sphere with up to 8 vertices," *Journal of Combinatorial Theory. Series A*, vol. 14, no. 1, pp. 37–52, 1973.
[5] A. Emch, "Triple and multiple systems, their geometric configurations and groups," *Transactions of the American Mathematical Society*, vol. 31, no. 1, pp. 25–42, 1929.
[6] W. Kühnel, "Topological aspects of twofold triple systems," *Expositiones Mathematicae*, vol. 16, no. 4, pp. 289–332, 1998.
[7] A. Altshuler, "3-pseudomanifolds with preassigned links," *Transactions of the American Mathematical Society*, vol. 241, pp. 213–237, 1978.
[8] F. H. Lutz, *Triangulated Manifolds with Few Vertices and Vertex-Transitive Group Actions*, Berichte aus der Mathematik, Shaker, Aachen, Germany, 1999, Dissertation, Technischen Universität Berlin.
[9] B. Bagchi and B. Datta, "A structure theorem for pseudomanifolds," *Discrete Mathematics*, vol. 188, no. 1–3, pp. 41–60, 1998.
[10] B. Datta, "Two-dimensional weak pseudomanifolds on seven vertices," *Boletín de la Sociedad Matemática Mexicanae. Tercera Serie*, vol. 5, no. 2, pp. 419–426, 1999.
[11] W. Kühnel, "Minimal triangulations of Kummer varieties," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 57, pp. 7–20, 1987.

[12] E. Gawrilow and M. Joswig, polymake, 1997–2007, version 2.3, http://www.math.tu-berlin.de/polymake.
[13] W. Kühnel, "Triangulations of manifolds with few vertices," in *Advances in Differential Geometry and Topology*, F. Tricerri, Ed., pp. 59–114, World Scientific, Teaneck, NJ, USA, 1990.
[14] B. Bagchi and B. Datta, "Minimal trialgulations of sphere bundles over the circle," *Journal of Combinatorial Theory. Series A*, vol. 115, no. 5, pp. 737–752, 2008.
[15] D. W. Walkup, "The lower bound conjecture for 3- and 4-manifolds," *Acta Mathematica*, vol. 125, no. 1, pp. 75–107, 1970.