Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 321761, 6 pages
doi:10.1155/2008/321761

Research Article

Continuity in Partially Ordered Sets

Venu G. Menon

Department of Mathematics, University of Connecticut, Stamford, CT 06901, USA

Correspondence should be addressed to Venu G. Menon, venu.menon@uconn.edu

Received 12 September 2007; Revised 8 October 2007; Accepted 4 December 2007

Recommended by Pentti Haukkanen

The notion of a continuous domain is generalized to include posets which are not dcpos and in which the set of elements way below an element is not necessarily directed. We show that several of the pleasing algebraic and topological properties of domains carry over to this setting.

1. Introduction

Continuous lattices and their generalizations, continuous domains, have been studied for more than three decades. Continuous lattices are complete lattices where each element is the supremum of elements way below it. For a poset to be a continuous domain, it needs to have sups of only directed sets in addition to the following two conditions: (i) each element is the sup of elements way below it, and (ii) for each element, the set of elements way below it is a directed set. A continuous poset [1] is any poset in which the conditions (i) and (ii) are satisfied. In a complete lattice, in fact in any sup-semilattice, the condition (ii) above is automatically satisfied. In [2], the authors have studied lattices which are not complete but satisfy the conditions (i) and (ii) above. The purpose of this paper is to study posets which need not be dcpos or lattices but which satisfy the condition that each element is the sup of elements way below it. The exact definition will be given in Section 2.

Here we recall some basic definitions and terminology from domain theory; more details can be found in [1]. For $x, y \in P$, a partially ordered set, we say that x is way below y (written $x \ll y$), if whenever $y \leq \sup D$, for a directed set D, there exists $d \in D$ such that $x \leq d$. A *continuous poset* is a partially ordered set P in which the following two conditions are satisfied. (i) For each $x \in P$, $x = \sup\{y : y \ll x\}$; and (ii) for each $x \in P$, $\{y : y \ll x\}$ is a directed set. A continuous poset in which every directed set has a least upper bound (such posets are called dcpos) is called a continuous domain. A continuous poset which is also a complete lattice is called a continuous lattice. In a complete lattice, since the second condition above is automatically satisfied, one needs to verify only the first condition. In any continuous poset P, the way below relation has the interpolation property, that is, for $x, y \in P$ with $x \ll y$, there exists an element $z \in P$, such that $x \ll z \ll y$. For more information about continuous lattices, and continuous domains, the standard reference is [1]. A subset S of a poset P is called *up-complete* if for any directed subset D of S, for which $\sup D$ exists in P, $\sup D$ is contained in S. We use the following notations: $\downarrow A = \{x : x \leq a \text{ for some } a \in A\}$; $\Downarrow A = \{x : x \ll a \text{ for some } a \in A\}$; and $\Downarrow x = \Downarrow \{x\}$. A subset J of a poset P is called a lower set if and only if $J = \downarrow J$. Recall that a subset U of a poset P is Scott open if and only if U is an upper set satisfying the property that if the supremum of any directed set D is in U, then D itself intersects U. The lower topology on a poset P has subbasic closed sets of the form $\uparrow x, x \in P$. The join of the lower topology and the Scott topology on P is the Lawson topology and is denoted by $\lambda(P)$.

2. C-posets

The following simple but crucial result is the motivation for our definition of a C-poset.

Theorem 2.1. *Let L be a continuous poset. Then for each $x, y \in L$, with $x \nleq y$, there exists $u \in L$ and an up-complete lower set $J \subseteq P$ such that $u \nleq y$, $x \notin J$, and $\uparrow u \cup J = L$.*

Proof. Suppose that L is a continuous poset. Let $x, y \in L$ with $x \nleq y$. Then $\exists u \ll x$ with $u \nleq y$. By the interpolation property, $\exists v \in L$ such that $u \ll v \ll x$. Clearly $v \nleq y$. Define $J = L \setminus \Uparrow v$. If D is a directed subset of J, then v is not way below any element of D. If $v \ll \sup D$, then there exists w such that $v \ll w \ll \sup D$. This implies that $\exists d \in D$ such that $v \ll w \leq d$, a contradiction. This shows that the supremum of D belongs to J, and thus J is up-complete. It is immediate that $u \nleq y$, $x \notin J$, and $\uparrow u \cup J = L$. □

Definition 2.2. A partially ordered set P is called a *C-poset* if for any $x, y \in P$, with $x \nleq y$, $\exists\, u \nleq y$ and an up-complete lower set J such that $x \notin J$, and $\uparrow u \cup J = P$.

Example 2.3. (i) If S is an infinite set and if P is the poset of all finite and cofinite subsets of S, then P is a C-poset. Suppose that A, B are in P such that $A \nsubseteq B$. Let $x \in A$ such that $x \notin B$. If we define $J = \{A : A \in P,\ x \notin A\}$, and $u = \{x\}$, then this J and u satisfy the conditions in the definition of a C-poset.

(ii) Let X be a T_0 space for which the lattice of open sets is a continuous lattice. Here X is endowed with the specialization order. Let L be a continuous domain with a least element endowed with the Scott topology. Let $P = [X \to L]$ be the set of all Scott continuous functions. It can be shown that P is a C-poset but not a continuous poset, see [1, page 200].

(iii) Let P_i, $i \in I$ be a collection of C-posets with 0 and 1. Then the product of $P_i s$ is a C-poset. If $(x_i) \nleq (y_i)$, then there exists $j \in I$ such that $x_j \nleq y_j$. Since P_j is a C-poset, there exists $a \in P_j$ such that $a \nleq y_j$ and an up-complete lower set K of P_j such that $x_j \notin K$ and $P_j = \uparrow a \cup K$. Define $u = (u_i)$, where $u_i = 0$ if $i \neq j$ and $u_j = a$. Also define $J = \downarrow S$, where $S = \{(x_i) : x_i = 1 \text{ if } i \neq j \text{ and } x_j \in K\}$. It is easy to see that the above u and J satisfy the conditions in the definition of a C-poset.

The next theorem establishes that C-posets do satisfy the approximation property, that is, each element is the supremum of elements way below it.

Theorem 2.4. *If P is a C-poset, then for each $x \in P$, $x = \sup\{z : z \ll x\} = \sup \Downarrow x$.*

Proof. We need to prove this only for $x \neq 0$. If $x \neq 0$, then there exists $y \in P$ such that $x \nleq y$. Therefore, by the definition of a C-poset, there exists $u \nleq y$, and an up-complete lower set J such that $x \notin J$, and $\uparrow u \cup J = P$. We will show that $u \ll x$. Suppose that D is a directed set in P such that $\sup D \geq x$. If $D \cap \uparrow u = \varnothing$, then $D \subseteq J$ and since J is up-complete, $\sup D \in J$. Since J is a lower set, this forces $x \in J$, a contradiction. This shows that $u \ll x$. Thus the set $\{z : z \ll x\}$ is nonempty, and x is an upper bound of it. Let w be any upper bound of that set. We will show that $x \leq w$. Indeed, if $x \nleq w$, then $\exists u' \nleq w$ and an up-complete lower set J' such that $x \notin J'$, and $\uparrow u' \cup J' = P$. But then $u' \ll x$, which contradicts the assumption that w is an upper bound of the set $\{z : z \ll x\}$. Therefore $x \leq w$. This completes the proof of the theorem. □

Corollary 2.5. *If a C-poset is a sup-semilattice, then it is a continuous poset; if a C-poset is a complete lattice, then it is a continuous lattice.*

Theorem 2.6. *Let P be a C-poset, and let $\{x_{i,k} : i \in I,\ k \in K(i)\}$ be a nonempty family of elements of P such that $\{x_{i,k} : k \in K(i)\}$ is a directed set for all $i \in I$. Then the following identity holds whenever the specified sups and infs exist:*

$$\bigwedge_{i \in I} \bigvee_{K(i)} x_{i,k} = \bigvee_{f \in M} \bigwedge_{i \in I} x_{i,f(i)}, \tag{2.1}$$

where M is the set of all choice functions $f : I \to \bigcup_{i \in I} K(i)$ with $f(i) \in K(i)$ for all i

Proof. Since the right hand side is always less than or equal to the left-hand side, we need to prove only the reverse inequality. By Theorem 2.4, it is enough to show that whenever $x \ll \bigwedge_{i \in I} \bigvee_{K(i)} x_{i,k}$, we have $x \ll \bigvee_{f \in M} \bigwedge_{i \in I} x_{i,f(i)}$.

Suppose that $x \ll \bigwedge_{i \in I} \bigvee_{K(i)} x_{i,k}$. Then, $x \ll \bigvee_{K(i)} x_{i,k}$, for all $i \in I$. Therefore, by the definition of the way below relation, we can choose a $g(i) \in K(i)$ with $x \leq x_{i,g(i)}$ for all $i \in I$. This implies that $x \leq \bigwedge_{i \in I} x_{i,g(i)}$, and hence x is less than or equal to the right hand side of the identity. This completes the proof of the theorem. □

Let $g : P \to Q$, and let $d : Q \to P$ be monotone functions between posets. The pair (g, d) is called a Galois connection if for $x \in P$, and $y \in Q$, $g(x) \geq y$ if and only if $x \geq d(y)$. Here g is called an upper adjoint, and d is called a lower adjoint. A monotone function $g : P \to Q$ between posets is called an upper adjoint if there exists a (necessarily unique) monotone function $d : Q \to P$ such that (g, d) is a Galois connection. Basic properties of Galois connections can be found in [1].

Definition 2.7. A function from a C-poset to another C-poset is called a *homomorphism* if it is an upper adjoint which preserves sups of directed sets. A subposet S of a C-poset P is called a *subalgebra* if the inclusion map $i : S \to P$ is a homomorphism.

Theorem 2.8. (a) *Let P be a C-poset and let Q be any poset. If $f : P \to Q$ is a surjective homomorphism, then Q is a C-poset.*

(b) *If P is a C-poset and if S is a subalgebra of P, then S is a C-poset.*

Proof. (a) Suppose that g is the lower adjoint of f. Let $y_2 \nleq y_1$ in Q. Pick $x_1 \in P$ with $f(x_1) = y_1$. Since g is a lower adjoint of f, $y_2 \nleq f(x_1)$ implies $g(y_2) \nleq x_1$. Then there exist $u \in P$ and an up-complete lower set J such that $u \nleq x_1$, $g(y_2) \notin J$, and $P = \uparrow u \cup J$. Define $I := g^{-1}(J)$, and $v := f(u)$. Since the lower adjoint g preserves sups, I is up-complete, and $y_2 \notin I$. If $x \in J$, then

$f(x) \leq f(x)$ implies $gf(x) \leq x$, which implies $gf(x) \in J$. Thus $f(x) \in g^{-1}(J) = I$. If $x \notin J$, then $u \leq x$ which implies $f(u) \leq f(x)$. Thus $\uparrow f(u) \cup I = Q$. This completes the proof that Q is a C-poset.

(b) Let $i : S \to P$ be the inclusion map and let d be its lower adjoint. Let $x, y \in S$ such that $x \nleq y$. Then there exists $u \in P$, and an up-complete lower set $J \subseteq P$, such that $u \nleq y$, $x \notin J$, and $P =\uparrow u \cup J$. Since $u \nleq y = i(y)$, $d(u) \nleq y$. Define $I := i^{-1}(J)$. Since i preserves sups of directed sets, I is an up-complete lower set, and $x \notin I$. For $t \in S$, $t \notin J$, implies $i(t) = t \notin J$, which implies that $u \leq t$. Thus $d(u) \leq d(t) = t$. Therefore, $\uparrow_S d(u) \cup I = S$. This completes the proof that S is a subalgebra. □

Notice that the proof of (a) above did not use the assumption that f preserves the sups of directed sets.

Definition 2.9. Let P be a C-poset. A subposet B of P is called a basis of P if given any $x, y \in P$ with $x \nleq y$, there exist $b \in B$ and an up-complete lower set $J \subseteq P$ such that $b \nleq y$, $x \notin J$, and $\uparrow b \cup J = P$.

Theorem 2.10. *If B is a basis of a C-poset P, then for each $x \in P$, $x = \sup\{b : b \in B,\ b \ll x\}$.*

Proof. We need to prove this only for $x \neq 0$. If $x \neq 0$, then there exists $y \in P$ such that $x \nleq y$. Then, by the definition of the basis, there exists $b \in B$ such that $b \nleq y$, and an up-complete lower set J such that $x \notin J$, and $\uparrow b \cup J = P$. It follows immediately that $b \ll x$. Thus, $\{b : b \in B,\ b \ll x\}$ is nonempty and x is an upper bound of it. Suppose that v is any upper bound of the same set. If $x \nleq v$, then there exist $b' \in B$ and an up-complete lower set J' such that $x \notin J'$, and $\uparrow u' \cup J' = P$. As shown above $b' \ll x$ which contradicts the assumption that v is an upper bound of $\{b : b \in B,\ b \ll x\}$. Thus $x \leq u$. This completes the proof. □

Proposition 2.11. (1) *Let P, Q be C-posets with B as a basis of P. If $g : P \to Q$ is a surjective homomorphism, then $g(B)$ is a basis of Q.*

(2) *If S is a subalgebra of a C-poset P, and if B is a basis of P, then $d(B)$ is a basis of S, where d is the lower adjoint of the inclusion map.*

Proof. (1) Let d be the lower adjoint of g. Suppose that $y_1, y_2 \in Q$ such that $y_2 \nleq y_1$. Let $x_1 \in P$, such that $g(x_1) = y_1$. Since $y_2 \nleq g(x_1)$, $d(y_2) \nleq x_1$. Since B is a basis of the C-poset P, there exist $b \in B$ and an up-complete lower set $J \subseteq P$ such that $b \nleq x_1$, $d(y_2) \notin J$, and $\uparrow b \cup J = P$. Let $I = d^{-1}(J)$ and $c = f(b)$. Since the lower adjoint preserves sups, I is an up-complete lower set, and $y_2 \notin I$. If $x \in J$, $f(x) \leq f(x)$ implies $df(x) \leq x$. This implies that $df(x) \in J$, and hence $f(x) \in d^{-1}(J) = I$. If $x \notin J$, then $b \leq x$, which implies $f(b) = c \leq f(x)$. Thus $\uparrow c \cup I = Q$. This completes the proof. □

The proof of (2) is similar and hence omitted.

Though algebraic lattices were studied for several decades before the introduction of continuous lattices, algebraic lattices, and their generalizations, algebraic domains, have played an important role in domain theory. Algebraic domains are continuous domains in which every element is the sup of compact elements (an element x is compact if $x \ll x$) below it, in which the set of compact elements below every element is a directed set. We generalize this notion to posets which are not necessarily dcpos in which the set of compact elements below any element is not necessarily directed. The following result is the motivation for our definition.

Theorem 2.12. *If L is an algebraic poset, then for $x, y \in L$, with $x \not\leq y$, there exist $u \in L$ and an up-complete lower set $J \subseteq L$, such that $u \not\leq y$, $x \notin J$, $\uparrow u \cup J = L$, and $\uparrow u \cap J = \varnothing$.*

Definition 2.13. A C-poset is called an *A-poset* if the condition of Theorem 2.12 is satisfied.

Proposition 2.14. *A C-poset P is an A-poset if and only if the set of all compact elements of P is a basis of P.*

3. Topology on C-posets

The wealth of topological structures in continuous domains and the interplay between the topological and algebraic properties of continuous domains are well documented [1]. In this section, we look at C-posets endowed with the Lawson topology.

Theorem 3.1. *A C-poset endowed with the Lawson topology is a pospace and hence Hausdorff. An A-poset endowed with the Lawson topology is totally order-disconnected.*

Proof. Let P be a C-poset. Suppose that $x, y \in P$ such that $x \not\leq y$. Then, there exist $u \in P$, and an up-complete lower set J such that $u \not\leq y$, $x \notin J$, and $\uparrow u \cup J = P$. Let $U = P \setminus J$, and $V = P \setminus \uparrow u$. Then U is a Lawson open-upper set, and V is a Lawson open lower set. Clearly $x \in U$ and $y \in V$. Moreover $U \cap V = (P \setminus J) \cap (P \setminus \uparrow u) = P \setminus (J \cup \uparrow u) = \varnothing$. This shows that P is a pospace.

If P is an A-poset, then for $x, y \in P$, $x \not\leq y$ implies the existence of $u \in P$ and an up-complete lower set J as in the previous paragraph with the additional condition that $\uparrow u \cap J = \varnothing$. This means that $\uparrow u$ is both closed and open. This completes the proof of the theorem. □

An inf-semilattice P is called *meet continuous* if for all directed subsets D of P for which $\sup D$ exists, and for all $x \in P$, $x \sup D = \sup xD$.

Proposition 3.2. *If a C-poset P is also an inf-semilattice, then P is meet-continuous.*

Proof. Let D be a directed subset of P such that $\sup D$ exists, and let $x \in P$. We need to show only that $x \sup D \leq \sup xD$ since the reverse inequality is always true. Suppose that $x \sup D \not\leq \sup xD$. Then there exist $u \not\leq \sup xD$ and an up-complete lower set J such that $x \sup D \notin J$ and $\uparrow u \cup J = P$. Since J is a lower set, this means that $x \notin J$, and $\sup D \notin J$. Then $x \in \uparrow u$, and since J is up-complete, $\exists d \in D$ such that $d \in \uparrow u$. Thus for this $d, xd \in \uparrow u$ which contradicts $u \not\leq \sup xD$. This completes the proof. □

Proposition 3.3. *Let P be a C-poset which is also an inf-semilattice. Then the inf-operation is a continuous function $(L \times L, \lambda(L \times L)) \rightarrow (L, \lambda(L))$.*

Proof. First, consider $V = \wedge^{-1}(P \setminus \uparrow x) = \{(y, z) : x \not\leq y \wedge z\}$. We will show that V is open. If $(y, z) \in V$, then $x \not\leq y \wedge z$. Therefore, there exist a $u \not\leq y \wedge z$ and an up-complete lower set J such that $x \notin J$, and $\uparrow u \cup J = P$. Let $S = (P \times P) \setminus \uparrow (u, u)$. Then clearly S is open in $P \times P$ and $(y, z) \in S$. We will show that $S \subseteq V$. If $(t, r) \in S$, then either $u \not\leq t$ or $u \not\leq r$. Therefore, $u \not\leq t \wedge r$, which implies that $t \wedge r \in J$. If $x \leq t \wedge r$, then, since J is a lower set, $x \in J$, a contradiction. Thus $x \not\leq t \wedge r$ which means that $S \subseteq V$.

Let O be any Scott open subset of P, and consider $U = \wedge^{-1}(O)$. We will show that $U = \{(y, z) : y \wedge z \in O\}$ is a Scott open subset of $P \times P$. Suppose that D is a directed subset of $P \times P$ such that $\sup D = (a, b) \in U$. Then, $a \wedge b \in O$. Let $D_1 = \{d_1 : \exists d_2 \text{ such that } (d_1, d_2) \in D\}$, and let

$D_2 = \{d_2 : \exists d_1 \text{ such that } (d_1, d_2) \in D\}$. Then D_1 and D_2 are directed sets such that $\sup D_1 = a$, and $\sup D_2 = b$. Since P is meet-continuous by Proposition 3.2, $a \wedge b = \sup D_1 \wedge \sup D_2 = \sup (D_1 D_2) \in O$. Since $D_1 D_2$ is a directed set and since O is Scott-open, there exist $d_1 \in D_1$ and $d_2 \in D_2$ such that $d_1 \wedge d_2 \in O$, which implies that $(d_1, d_2) \in U$. This completes the proof of the proposition. □

Acknowledgment

The author wishes to thank the referee for several suggestions to improve this paper.

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