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Research Article

Norm Attaining Multilinear Forms on $L_1(\mu)$

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Given an arbitrary measure μ, this study shows that the set of norm attaining multilinear forms is not dense in the space of all continuous multilinear forms on $L_1(\mu)$. However, we have the density if and only if μ is purely atomic. Furthermore, the study presents an example of a Banach space X in which the set of norm attaining operators from X into X^* is dense in the space of all bounded linear operators $L(X, X^*)$. In contrast, the set of norm attaining bilinear forms on X is not dense in the space of continuous bilinear forms on X.

1. Introduction

The Bishop-Phelps theorem [1] asserts that the set of norm attaining linear functionals on a Banach space X is dense in the dual space X^*. Some authors have considered the question of the density of norm attaining multilinear forms. To present the problem more precisely, given real Banach spaces $X_1, \dots, X_N$, we denote by $\mathcal{L}^N(X_1, \dots, X_N)$ the space of all continuous N-linear mappings from $X_1 \times \cdots \times X_N$ into the scaler field. We say that $\varphi \in \mathcal{L}^N(X_1, \dots, X_N)$ attains its norm if there is $x_i \in B_{X_i}$ (the unit ball of X_i) for $i = 1, 2, \dots, N$, such that

$$\left|\varphi(x_1, \dots, x_N)\right| = \|\varphi\| = \sup\left\{\left|\varphi(y_1, \dots, y_N)\right| : (y_1, \dots, y_N) \in B_{X_1} \times \cdots \times B_{X_N}\right\}, \tag{1.1}$$

and we denote by $\mathcal{AL}^N(X_1, \dots, X_N)$ the set of all norm attaining N-linear forms. In the case where $X_1 = \cdots = X_N = X$, we write simply $\mathcal{L}^N(X)$ and $\mathcal{AL}^N(X)$.

Aron et al. [2] posed the question of when $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$, and gave sufficient conditions for this density to hold. The first example of a Banach space X such that $\mathcal{AL}^2(X)$ is not dense in $\mathcal{L}^2(X)$ was given in [3]. Shortly after, Choi [4] showed that $\mathcal{AL}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$. For additional results on this problem, we refer the reader to [5–9].

In this paper, we give some improvements on the results in [10]. More concretely, it was shown in that study that given an arbitrary finite measure μ, $\mathcal{AL}^2(L_1(\mu))$ is dense in $\mathcal{L}^2(L_1(\mu))$ if and only if μ is purely atomic. *In this note, we extend the above result to an arbitrary measure.* Namely, we proved that, *given any arbitrary measure* μ, $\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ if and only if μ is purely atomic. Also, we present a new example of a Banach space X such that the set of norm attaining operators from X into X^* is dense in the space of all bounded linear operators from X into X^*, but the set $\mathcal{AL}^2(X)$ is not dense in $\mathcal{L}^2(X)$. This can be shown by relating the main result in our work to the following theorem.

Theorem 1.1 (see [11, Theorem 1]). *Given an arbitrary measure μ and a localizable measure ν, the set of norm attaining operators from $L_1(\mu)$ into $L_\infty(\nu)$ is dense in the space $L(L_1(\mu), L_\infty(\nu))$.*

2. The results

We begin by recalling the isometric classification of L_1-spaces and a technical lemma which deals with the density of norm attaining bilinear forms on arbitrary l_1-sums of Banach spaces in order to reduce the proof of our problem to the case where μ is a finite measure. Recall that if μ is an arbitrary measure, $L_1(\mu)$ can be decomposed in the form

$$L_1(\mu) \cong \left(\oplus_{i\in I} L_1(\mu_i)\right)_{\ell_1} \tag{2.1}$$

where μ_i is a finite measure for all $i \in I$ (see, e.g., [12, Appendix B]). On the other hand, if ν is a localizable measure we have that $L_\infty(\nu) = L_1(\nu)^*$, and we get a set of finite measures $\{\nu_j : j \in J\}$ such that

$$L_\infty(\nu) \cong \left(\oplus_{j\in J} L_\infty(\nu_j)\right)_{\ell_\infty}. \tag{2.2}$$

In what follows, we may assume without loss of generality that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. The well-known representation of the space $\mathcal{L}^2(L_1(\mu))$ is nothing but $L_\infty(\mu \otimes \mu)$ "the space of all essential bounded measurable functions," where $\mu\otimes\mu$ denotes the product measure on $\Omega \times \Omega$. More concretely,

$$\mathcal{L}^2(L_1(\mu) \cong L(L_1(\mu), L_1(\mu)^*) \cong L(L_1(\mu), L_\infty(\mu)) \cong L_\infty(\mu \otimes \mu); \tag{2.3}$$

see [12, Example 3.27] . In view of the above, we get the integral representation for the continuous bilinear form $\widehat{h}$ on $\mathcal{L}^2(L_1(\mu))$ as follows:

$$\widehat{h}(f, g) = \int_{\Omega\times\Omega} h(x, y) f(x) g(y) d\mu(x) d\mu(y), \tag{2.4}$$

for $f, g \in L_1(\mu)$, $x, y \in \Omega$, and $h \in L_\infty(\mu\otimes\mu)$. Moreover, the application $h \mapsto \widehat{h}$ is linear isometric bijection from $L_\infty(\mu \otimes \mu)$ onto $\mathcal{L}^2(L_1(\mu))$; see [4].

To make the vision more comprehensive, we state the following technical lemmas that will be needed later. To simplify the notation, we consider the case $N = 2$. The proof for the general case is exactly the same.

Lemma 2.1 (see [10, Lemma 2.1]). *Let ν be an arbitrary nonzero finite measure and $\mu = \nu\otimes m$, where m denotes Lebesgue measure on $I = [0, 1]$. Then $\mathcal{AL}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$.*

The other technical lemma deals with l_1-sums of Banach spaces. By $Y \oplus_1 Z$ we denote the ℓ_1-sum of two Banach spaces Y and Z, that is, $\|y+z\| = \|y\| + \|z\|$ for arbitrary $y \in Y$, $z \in Z$.

Lemma 2.2 (see [10, Lemma 2.2]). *Let Y, Z be Banach spaces and $X = Y \oplus_1 Z$. If $\mathcal{AL}^2(X)$ is dense in $\mathcal{L}^2(X)$, then $\mathcal{AL}^2(Y)$ is dense in $\mathcal{L}^2(Y)$.*

Our first result of this paper is a characterization of those functions $h \in \mathcal{L}_\infty(\mu \otimes \mu)$, where $\widehat{h}$ its corresponding bilinear form in $\mathcal{L}^2(L_1(\mu))$ that attains its norm (see [4]).

Proposition 2.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, fixed $h \in L_\infty(\mu \otimes \mu)$, and let $\widehat{h}$ be its corresponding bilinear form as defined in* (2.4)

(1) *There exist sets $A, B \in \mathcal{A}$ with $\mu(A) > 0$, $\mu(B) > 0$ and a scalar t with $|t| = 1$ such that*

$$th(x,y) = \|h\|_\infty \tag{2.5}$$

for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$.

(2) *There are sets A, B like in (1) and measurable functions φ, ψ on Ω such that*

$$|\varphi(w)| = |\psi(w)| = 1, \tag{2.6}$$

where $w \in \Omega$ and $\varphi(x)\psi(y)h(x,y) = \|h\|_\infty$, for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$.

(3) *The bilinear form $\widehat{h} \in \mathcal{L}^2(L_1(\mu))$ corresponding to $h \in L_\infty(\mu \otimes \mu)$ attains its norm.*

$$\textit{Then } (1) \Longrightarrow (2) \Longleftrightarrow (3). \tag{2.7}$$

Moreover, in the real case all three statements are equivalent.

Proof. (1) ⇒ (2) is clear, just take $\varphi = t$ and $\psi = 1$.

For (2) ⇒ (3), just consider the functions $f = \varphi\chi_A/\mu(A)$, $g = \psi\chi_A/\mu(B)$ where f, g are in the unit sphere of $L_1(\mu)$, χ_A, χ_B denote the characteristic functions on A and B, respectively, and

$$\widehat{h}(f,g) = \frac{1}{\mu(A)\mu(B)} \int_{A\times B} h(x,y)\varphi(x)\psi(y)\,d\mu(x)\,d\mu(y) = \frac{1}{\mu(A)\mu(B)} \int_{A\times B} \|h\|_\infty d(\mu\otimes\mu) = \|h\|_\infty. \tag{2.8}$$

(3) ⇒ (2) Let $f, g \in L_1(\mu)$ be such that $\|f\|_1 = \|g\|_1 = 1$ and $\widehat{h}(f,g) = \|h\|_\infty$. Take

$$A = \{x \in \Omega : f(x) \neq 0\}, \qquad B = \{y \in \Omega : g(y) \neq 0\} \tag{2.9}$$

to be two measurable sets in Ω with $\mu(A) > 0$, $\mu(B) > 0$, and write f, g in the forms $f = \varphi|f|$, $g = \psi|g|$ where φ, ψ are measurable functions on Ω with $|\varphi| = 1$, $|\psi| = 1$, then we have

$$\|h\|_\infty = \widehat{h}(f,g) = \int_{A\times B} h(x,y)\varphi(x)|f(x)|\psi(y)|g(y)|\,d\mu(x)\,d\mu(y) \leq \|h\|_\infty \|f\|_1 \|g\|_1 = \|h\|_\infty, \tag{2.10}$$

from which we conclude that

$$h(x,y)\varphi(x)\psi(y) = \|h\|_\infty \tag{2.11}$$

for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$.

In the real case, the functions φ, ψ have only the values ± 1, then we can choose measurable subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $\mu(A_0)\mu(B_0) > 0$, where φ, ψ are constants on A_0, B_0, respectively. If $t = \pm 1$ is the product of these constants, then we have clearly $th(x,y) = \|h\|_\infty$ for $[\mu \otimes \mu]$-almost every $(x,y) \in A_0 \times B_0$, so we get that (3) $\Rightarrow$ (1), as required. □

In the special case $h = \chi_E$, the characteristic function of a measurable set $E \in \mathcal{A} \times \mathcal{A}$, we have the following result.

Corollary 2.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, let $E \in \mathcal{A} \times \mathcal{A}$ be a measurable set with $(\mu \otimes \mu)(E) > 0$, and consider the following bilinear form $\widehat{\chi}_E$ corresponding to the characteristic function of E. The following statements are equivalent:*

(1) $\widehat{\chi}_E \in \mathcal{AL}^2(L_1(\mu))$;

(2) $\widehat{\chi}_E \in \overline{\mathcal{AL}^2(L_1(\mu))}$;

(3) *There exist subsets $A, B \in \mathcal{A}$ with $\mu(A)\mu(B) > 0$ such that $[\mu \otimes \mu]((A \times B) \cap E) = \mu(A)\mu(B)$.*

Note that we can say that the measurable rectangle $A \times B$ is contained in the set E.

Proof. (1) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (3). Let $h \in L_\infty(\mu \otimes \mu)$ be such that $\|\chi_E - h\|_\infty < 1/2$, and $\widehat{h} \in \mathcal{AL}^2(L_1(\mu))$, then it is clear that $\|h\|_\infty > 1/2$. From the implication (3) $\Rightarrow$ (2) of Proposition 2.3 , we have two measurable sets $A, B \in \mathcal{A}$ with $\mu(A)\mu(B) > 0$, and measurable functions φ, ψ on Ω with $|\varphi(x)| = |\psi(y)| = 1$, such that

$$\varphi(x)\psi(y)h(x,y) = \|h\|_\infty, \tag{2.12}$$

then

$$\left|h(x,y)\right| = \|h\|_\infty > \frac{1}{2}, \tag{2.13}$$

for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$. Hence

$$\left|\chi_E(x,y)\right| \geq \left|h(x,y)\right| - \left|h(x,y) - \chi_E(x,y)\right| > \frac{1}{2} - \left\|h - \chi_E\right\|_\infty > 0. \tag{2.14}$$

for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$, from which we get that $\chi_E = 1$, for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$, which means that (3) holds.

(3) $\Rightarrow$ (1). If A, B are the sets that satisfy the conditions of the statement (3), then we may clearly see that the function $\chi_E = 1 = \|\chi_E\|_\infty$, for $[\mu \otimes \mu]$-almost every $(x,y) \in A \times B$, then the function $f = \chi_E$ verifies the statement (1) of Proposition 2.3 including the case $t = 1$. □

Remark 2.5. Let us point out the following consequence of the representation theory for L_1-spaces. Indeed, if ν is a finite measure, we may write

$$L_1(\nu) = \left(\oplus_{iI} X_i\right)_{\ell_1}, \tag{2.15}$$

where each space X_i is either 1-dimensional or of the form $L_1([0,1]^{\Lambda})$ and Λ is a finite or infinite set. For each coordinate interval, we consider the Lebesgue measure on the Borel subsets of $[0,1]$ and $[0,1]^{\Lambda}$ provided with the product measure on the Borel σ-algebra (see [13]).

We are now ready to provide the main result.

Theorem 2.6. *Given an arbitrary measure μ, the following statements are equivalents.*

(1) *μ is purely atomic.*

(2) *$\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for any number N.*

(3) *$\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for any number $N \geq 2$.*

(4) *$\mathcal{AL}^2(L_1(\mu))$ is dense in $\mathcal{L}^2(L_1(\mu))$.*

Proof. (1) $\Rightarrow$ (2). If μ is purely atomic, then $L_1(\mu)$ has the Radon-Nikodym property, and (2) follows from [2, Theorem 1].

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (4). This follows from [8, Proposition 2.1].

(4) $\Rightarrow$ (1). Given an arbitrary nonempty set Λ, consider the product $[0,1]^{\Lambda}$ of so many copies of $[0,1]$ as indicated by Λ with product measure. We have clearly $\mu = \nu \otimes m$, where ν is an arbitrary nonzero finite measure and m denotes the Lebesgue measure on $[0,1]$. Then it follows form Lemma 2.1 that $\mathcal{AL}^2(L_1[0,1]^{\Lambda})$ is not dense in $\mathcal{L}^2(L_1[0,1]^{\Lambda})$. Indeed, if μ is a finite measure satisfying statement (4) of the above theorem, then by Remark 2.5, $L_1(\mu) \cong (\oplus_{i\in I} X_i)_{\ell_1}$ for each $i \in I$, where X_i is 1-dimensional or of the form $L_1[0,1]^{\Lambda_i}$ for appropriate nonempty set Λ_i (see [13, Theorem 14]). It follows then from Lemma 2.2 that $\mathcal{AL}^2(X_i)$ is dense in $\mathcal{L}^2(X_i)$ for all $i \in I$. But in view of Remark 2.5, none of the spaces X_i are of the form $L_1[0,1]^{\Lambda_i}$. Then all X_i are 1-dimensional, and then $L_1(\mu) \cong \ell_1(I)$, which means that μ is purely atomic. Finally, if μ is not necessarily a finite measure satisfying (4) of our theorem, we recall that $L_1(\mu) \cong (\oplus_{i\in I} L_1(\mu_i))_{\ell_1}$, where μ_i is a finite measure for all $i \in I$. So by Lemma 2.2, we get that $\mathcal{AL}^2(L_1(\mu_i))$ is dense in $\mathcal{L}^2(L_1(\mu_i))$, and this proves that μ_i is purely atomic for each $i \in I$, which clearly means that μ is purely atomic. $\square$

Remark 2.7. Let us mention the relation between the $\mathcal{L}^2(X)$, the space of all continuous bilinear forms on the Banach space X, and $L(X, X^*)$, the space of all bounded linear operators from X into X^*, to see that just consider the canonical identification of $\mathcal{L}^2(X)$ with $L(X, X^*)$. The operator $T \in L(X, X^*)$ corresponding to a bilinear form $\varphi \in \mathcal{L}^2(X)$ is given by

$$[T(x)](y) = \varphi(x, y) \quad (x, y \in X). \tag{2.16}$$

The bilinear form φ attains its norm if and only if the operator T attains its norm at a point $x \in B_X$, that is, $T(x)$ also attains its norm as a functional on X, therefore, $T \in NA(X, X^*)$ whenever $\varphi \in \mathcal{AL}^N(X)$, but the converse is not true (see [4, 14, 15]). Connecting our main result in this paper with Theorem 1.1, we get a new example of a Banach space X such that the set of norm attaining bounded linear operators from X into X^* is dense in the space of all bounded linear operators from X into X^*, but $\mathcal{AL}^2(X)$ is not dense in $\mathcal{L}^2(X)$.

Therefore, the following result is inevitable.

Corollary 2.8. *If μ is a localizable and not purely atomic measure, then the set of norm attaining bounded linear operators from $L_1(\mu)$ into $L_\infty(\mu)$ is dense in the space $L(L_1(\mu), L_\infty(\mu))$ but $\mathcal{A}\mathcal{L}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$.*

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