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Research Article

On Integral Operator Defined by Convolution Involving Hybergeometric Functions

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For $\lambda > -1$ and $\mu \geq 0$, we consider a liner operator I_λ^μ on the class $\mathcal{A}$ of analytic functions in the unit disk defined by the convolution $(f_\mu)^{(-1)} * f(z)$, where $f_\mu = (1-\mu)z_2F_1(a,b,c;z) + \mu z(z_2F_1(a,b,c;z))'$, and introduce a certain new subclass of $\mathcal{A}$ using this operator. Several interesting properties of these classes are obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$.

If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg\left(\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U},\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1), \tag{1.2}$$

then $f(z)$ is said to be strongly starlike of order β and type α in $\mathbb{U}$, and denoted by $S^*(\alpha, \beta)$.

If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U},\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1), \tag{1.3}$$

then $f(z)$ is said to be strongly convex of order β and type α in $\mathbb{U}$, and denoted by $C(\alpha,\beta)$. It is obvious that $f(z) \in \mathcal{A}$ belongs to $C(\alpha,\beta)$ if and if $zf'(z) \in S^*(\alpha,\beta)$. Further, we note that $S^*(\alpha,1) \equiv S^*(\alpha)$ and $C(\alpha,1) \equiv C(\alpha)$ which are, respectively, starlike and convex univalent functions of order α.

Let P denote the class of functions of the form $p(z) = 1 + p_1 z + \cdots$ analytic in $\mathbb{U}$ which satisfy the condition $\operatorname{Re}\{P(z)\} > 0$.

For functions f given by (1.1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, let $(f*g)(z)$ denote the Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by

$$(f*g)(z) = f(z)*g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.4}$$

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathbb{U}$ such that $f(z) = g(w(z))$ [1].

Let $f \in \mathcal{A}$. Denote by $D^{\lambda} : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$D^{\lambda} = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1). \tag{1.5}$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, and

$$D^{\delta} f(z) = \frac{z\left(z^{\delta-1} f(z)\right)^{(\delta)}}{\delta!} \quad \left(\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right). \tag{1.6}$$

The operator $D^{\delta} f$ is called the δth-order Ruscheweyh derivative of f. Recently, K. I. Noor [2] and K. I. Noor and M. A. Noor [3] defined and studied an integral operator $I_n : \mathcal{A} \to \mathcal{A}$, analogous to $D^{\delta} f(z)$ as follows.

Let $f_n = z/(1-z)^{n+1}$, $(n \in \mathbb{N}_0)$, and $f_n^{(-1)}(z)$ be defined such that

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{(1-z)^2}. \tag{1.7}$$

Then

$$I_n f(z) = f_n^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f(z) \quad (f \in \mathcal{A}). \tag{1.8}$$

We note that $I_0 f(z) = zf'(z)$, $I_1 f(z) = f(z)$. The operator I_n is called the Noor integral of nth order of f (see [4, 5]), which is an important tool in defining several classes of analytic functions. In recent years, it has been shown that Noor integral operator has fundamental and significant applications in the geometric function theory.

For real or complex numbers a, b, c other than $0, -1, -2, \ldots$, the hypergeometric series is defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k, \tag{1.9}$$

where $(x)_k$ is Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\cdots(x+k-1) \quad \text{for } k = 1,2,3,\ldots,\ x \in \mathbb{C},\ (x)_0 = 1. \tag{1.10}$$

We note that the series (1.9) converges absolutely for all $z \in \mathbb{U}$ so that it represents an analytic function in $\mathbb{U}$. Also an incomplete beta function $\phi(a,c;z)$ is related to Gauss hypergeometric function $z_2F_1(a,b;c;z)$ as

$$\phi(a,c;z) = z_2F_1(1,a;c;z), \tag{1.11}$$

and we note that $\phi(a,1;z) = z/(1-z)^a$, where $\phi(2,1;z)$ is Koebe function. Using $\phi(a,c;z)$, a convolution operator [6], was defined by Carlson and Shaferr. Furthermore, Hohlov [7] introduced a convolution operator using ${}_2F_1(a,b;c;z)$.

N. Shukla and P. Shukla [8] studied the mapping properties of a function f_μ to be as given in

$$f_\mu(a,b,c)(z) = (1-\mu)z_2F_1(a,b,c;z) + \mu z\big(z_2F_1(a,b,c;z)\big)' \quad (\mu \geq 0), \tag{1.12}$$

and investigated the geometric properties of an integral operator of the form

$$I(z) = \int_0^z \frac{f_\mu(t)}{t}\,dt. \tag{1.13}$$

Kim and Shon [9] considered linear operator $L_\mu : \mathcal{A} \to \mathcal{A}$ defined by $L_\mu(a,b,c)f(z) = f_\mu(a,b,c)(z)*f(z)$.

We now introduce a function $(f_\mu)^{(-1)}$ given by

$$f_\mu(a,b,c)(z)*\big(f_\mu(a,b,c)(z)\big)^{(-1)} = \frac{z}{(1-z)^{\lambda+1}} \quad (\mu \geq 0,\ \lambda > -1), \tag{1.14}$$

and obtain the following linear operator:

$$I_\mu^\lambda(a,b,c)f(z) = \big(f_\mu(a,b,c)(z)\big)^{(-1)}*f(z). \tag{1.15}$$

The operator I_μ^λ is known as the generalized integral operator. For $\mu = 0$ in (1.14), $I_\lambda(a,b;c)f(z) := I_\mu^\lambda(a,b;c)f(z)$, which was introduced by K. I. Noor [10].

Now we find the explicit form of the function $(f_\mu)^{(-1)}$. It is well known that $\lambda > -1$

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1} \quad (z \in \mathbb{U}). \tag{1.16}$$

Putting (1.9) and (1.16) in (1.14), we get

$$\sum_{k=0}^{\infty} \frac{(\mu k+1)(a)_k(b)_k}{(c)_k(1)_k} z^{k+1}*\big(f_\mu\big)^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1}. \tag{1.17}$$

Therefore, the function $(f_\mu)^{(-1)}$ has the following form:

$$\left(f_\mu(a,b,c)(z)\right)^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu k+1)(a)_k (b)_k} z^{k+1} \quad (z \in \mathbb{U}). \tag{1.18}$$

Now we note that

$$I_\mu^\lambda(a,b,c)f(z) = z + \sum_{k=1}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu k+1)(a)_k (b)_k} a_{k+1} z^{k+1}. \tag{1.19}$$

From (1.19), we note that

$$I_0^\lambda(a,\lambda+1,a)f(z) = f(z), \qquad I_0^1(a,1,a)f(z) = zf'(z). \tag{1.20}$$

Also it can easily be verified that

$$z\left(I_\mu^\lambda(a,b,c)f(z)\right)' = (\lambda+1)I_\mu^{\lambda+1}(a,b,c)f(z) - \lambda I_\mu^\lambda(a,b,c)f(z), \tag{1.21}$$

$$z\left(I_\mu^\lambda(a+1,b,c)f(z)\right)' = aI_\mu^\lambda(a,b,c)f(z) - (a-1)I_\mu^\lambda(a+1,b,c)f(z). \tag{1.22}$$

Now we introduce the following classes in term of the new operator $I_\mu^\lambda(a,b,c)$. For $\lambda > -1$, $\mu \geq 0$, $0 \leq \alpha < 1$, and $0 < \beta \leq 1$, let $\mathcal{S}_\mu^\lambda(a,b,c;\alpha,\beta)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\left|\arg\left(\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}). \tag{1.23}$$

Observe that $I_\mu^\lambda(a,b,c)f(z) \in S^*(\alpha,\beta)$ and $z(I_\mu^\lambda(a,b,c)f(z))'/I_\mu^\lambda(a,b,c)f(z) \neq \alpha$. Also, for $\lambda > -1$, $\mu \geq 0$, $0 \leq \alpha < 1$, and $0 < \beta \leq 1$, let $\mathcal{C}_\mu^\lambda(a,b,c;\alpha,\beta)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\left|\arg\left(1 + \frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)''}{\left(I_\mu^\lambda(a,b,c)f(z)\right)'} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}). \tag{1.24}$$

Observe that $I_\mu^\lambda(a,b,c)f(z) \in C(\alpha,\beta)$ and $1 + z(I_\mu^\lambda(a,b,c)f(z))''/(I_\mu^\lambda(a,b,c)f(z))' \neq \alpha$.

Clearly, $f \in \mathcal{C}_\mu^\lambda(a,b,c;\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{S}_\mu^\lambda(a,b,c;\alpha,\beta)$.

Note that $\mathcal{S}_0^\lambda(a,\lambda+1,a;\alpha,\beta) \equiv S^*(\alpha,\beta)$, $\mathcal{S}_0^\lambda(a,\lambda+1,a;\alpha,1) \equiv S^*(\alpha)$, $\mathcal{C}_0^\lambda(a,\lambda+1,a;\alpha,\beta) \equiv C(\alpha,\beta)$, and $\mathcal{C}_0^\lambda(a,\lambda+1,a;\alpha,1) \equiv C(\alpha)$.

Finally, let $\mathcal{K}_\mu^\lambda(a,b,c;\alpha,\beta,\gamma;A,B)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\left|\arg\left(\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)g(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}) \tag{1.25}$$

for $\lambda > -1$, $\mu \geq 0$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and $g \in \mathcal{Q}_\mu^\lambda(a,b,c;\gamma;A,B)$, where

$$\begin{aligned} &\mathcal{Q}_\mu^\lambda(a,b,c;\gamma;A,B) \\ &= \left\{g \in \mathcal{A} : \frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^\lambda(a,b,c)g(z)\right)'}{I_\mu^\lambda(a,b,c)g(z)} - \gamma\right) \prec \frac{1+Az}{1+Bz}\right\} \quad (z \in \mathbb{U};\ 0 \leq \gamma < 1,\ -1 \leq B < A \leq 1). \end{aligned} \tag{1.26}$$

We also note that $\mathcal{K}_0^{\lambda}(a, \lambda+1, a; \alpha, 1, \gamma; 1, -1)$ and $\mathcal{K}_0^{\lambda}(a, 1, a; \alpha, 1, \gamma; 1, -1)$ are the classes of quasiconvex and close-to-convex functions of order α and type γ, respectively, introduced and studied by Noor and Alkhorasani [11] and Silverman [12].

2. Main results

In order to give our results, we need the following lemmas.

Lemma 2.1 (see [13]). *Let β, γ be complex numbers. Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\operatorname{Re}[\beta\phi(z) + \gamma] > 0$, $z \in \mathbb{U}$ and $q \in \mathcal{A}$ with $q(z) \prec \phi(z)$, $z \in \mathbb{U}$. If $p \in P$ is analytic in $\mathbb{U}$, then*

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.1}$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}). \tag{2.2}$$

Lemma 2.2 (see [14]). *Let δ, η be complex numbers. Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\operatorname{Re}[\delta\phi(z) + \eta] > 0$, $z \in \mathbb{U}$. If $p \in P$ is analytic in $\mathbb{U}$, then*

$$p(z) + \frac{zp'(z)}{\delta p(z) + \eta} \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.3}$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}). \tag{2.4}$$

Lemma 2.3 (see [15]). *Let $\phi(z)$ be convex univalent in $\mathbb{U}$ and let $E \geq 0$. Suppose $B(z)$ is analytic in $\mathbb{U}$ with $\operatorname{Re} B(z) \geq E$, If $g \in P$ is analytic in $\mathbb{U}$, then*

$$Ez^2 g''(z) + B(z)zg'(z) + g(z) \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.5}$$

implies

$$g(z) \prec \phi(z) \quad (z \in \mathbb{U}). \tag{2.6}$$

Theorem 2.4. *Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\operatorname{Re}\ \phi(z) \geq 0$. If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\gamma}\left(\frac{z\big(I_\mu^{\lambda+1}(a,b,c)f(z)\big)'}{I_\mu^{\lambda+1}(a,b,c)f(z)} - \gamma\right) \prec \phi(z) \quad (z \in \mathbb{U}), \tag{2.7}$$

then

$$\frac{1}{1-\gamma}\left(\frac{z\big(I_\mu^{\lambda}(a,b,c)f(z)\big)'}{I_\mu^{\lambda}(a,b,c)f(z)} - \gamma\right) \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.8}$$

for $\lambda > -1$, $\mu \geq 0$, and $0 \leq \gamma < 1$.

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)f(z)} - \gamma\right), \tag{2.9}$$

where $p \in P$. By using (1.21) in (2.9) and then differentiating, we get

$$\frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^{\lambda+1}(a,b,c)f(z)\right)'}{I_\mu^{\lambda+1}(a,b,c)f(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{(\lambda+1)q(z)}, \tag{2.10}$$

where $q(z) = I_\mu^{\lambda+1}(a,b,c)f(z)/I_\mu^\lambda(a,b,c)f(z)$ and $q(z) \prec \phi(z)$. Hence by applying Lemma 2.1, we obtain the required result. □

Theorem 2.5. *Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and* Re $\phi(z) \geq 0$. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)f(z)} - \gamma\right) \prec \phi(z) \quad (z \in \mathbb{U}), \tag{2.11}$$

then

$$\frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^\lambda(a+1,b,c)f(z)\right)'}{I_\mu^\lambda(a+1,b,c)f(z)} - \gamma\right) \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.12}$$

for $\lambda > -1$, $\mu \geq 0$, and $0 \leq \gamma < 1$.

Proof. By using the same technique in the proof of Theorem 2.4 and using (1.22) and applying Lemma 2.2, we obtain the required result. □

Taking $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 2.4 and in Theorem 2.5, we have the following.

Corollary 2.6. *It holds that*

$$\begin{gathered} Q_\mu^{\lambda+1}(a,b,c;\gamma;A,B) \subset Q_\mu^\lambda(a,b,c;\gamma;A,B), \\ Q_\mu^\lambda(a,b,c;\gamma;A,B) \subset Q_\mu^\lambda(a+1,b,c;\gamma;A,B) \end{gathered} \tag{2.13}$$

for $\lambda > -1$, $\mu \geq 0$, $0 \leq \gamma < 1$, and Re $a > 1 - \gamma$.

Also, by taking $\phi(z) = ((1 + z)/(1 - z))^\beta$ $(0 < \beta \leq 1)$ in Theorem 2.4 and in Theorem 2.5, we have the following.

Corollary 2.7. *It holds that*

$$\begin{gathered} \mathcal{S}_\mu^{\lambda+1}(a,b,c;\gamma,\beta) \subset \mathcal{S}_\mu^\lambda(a,b,c;\gamma,\beta), \\ \mathcal{S}_\mu^\lambda(a,b,c;\gamma,\beta) \subset \mathcal{S}_\mu^\lambda(a+1,b,c;\gamma,\beta) \end{gathered} \tag{2.14}$$

for $\lambda > -1$, $\mu \geq 0$, $0 \leq \gamma < 1$, and Re $a > 1 - \beta$.

Corollary 2.8. *For $\lambda > -1$, $\mu \geq 0$, $0 \leq \gamma < 1$, and* Re $a > 1 - \beta$, *one has*

$$\begin{gathered} \mathcal{C}_{\mu}^{\lambda+1}(a,b,c;\gamma,\beta) \subset \mathcal{C}_{\mu}^{\lambda}(a,b,c;\gamma,\beta), \\ \mathcal{C}_{\mu}^{\lambda}(a,b,c;\gamma,\beta) \subset \mathcal{C}_{\mu}^{\lambda}(a+1,b,c;\gamma,\beta). \end{gathered} \tag{2.15}$$

Proof. We will proof the first relation and by the same method we can proof the second relation

$$\begin{aligned} f(z) \in \mathcal{C}_{\mu}^{\lambda+1}(a,b,c;\gamma,\beta) &\Longleftrightarrow zf'(z) \in \mathcal{S}_{\mu}^{\lambda+1}(a,b,c;\gamma,\beta) \\ &\Longleftrightarrow zf'(z) \in \mathcal{S}_{\mu}^{\lambda}(a,b,c;\gamma,\beta) \\ &\Longleftrightarrow I_{\mu}^{\lambda}(a,b,c)\left(zf'(z)\right) \in S^{*}(\gamma,\beta) \\ &\Longleftrightarrow z\left(I_{\mu}^{\lambda}(a,b,c)f(z)\right)' \in S^{*}(\gamma,\beta) \\ &\Longleftrightarrow I_{\mu}^{\lambda}(a,b,c)f(z) \in C(\gamma,\beta) \\ &\Longleftrightarrow f(z) \in \mathcal{C}_{\mu}^{\lambda}(a,b,c;\gamma,\beta). \end{aligned} \tag{2.16}$$

□

Theorem 2.9. *Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and* Re $\phi(z) \geq 0$. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\gamma}\left(\frac{z\left(I_{\mu}^{\lambda}(a,b,c)f(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)f(z)} - \gamma\right) \prec \phi(z) \quad (0 \leq \gamma < 1;\ z \in \mathbb{U}), \tag{2.17}$$

then

$$\frac{1}{1-\gamma}\left(\frac{z\left(I_{\mu}^{\lambda}(a,b,c)F(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)F(z)} - \gamma\right) \prec \phi(z) \quad (0 \leq \gamma < 1;\ z \in \mathbb{U}), \tag{2.18}$$

where F be the integral operator defined by

$$F(z) = \frac{c+1}{z^{c}} \int_{0}^{z} t^{c-1} f(t)\,dt \quad (c > -1). \tag{2.19}$$

Proof. From (2.19), we have

$$z\left(I_{\mu}^{\lambda}(a,b,c)F(z)\right)' = (c+1)I_{\mu}^{\lambda}(a,b,c)f(z) - cI_{\mu}^{\lambda}(a,b,c)F(z). \tag{2.20}$$

Now, let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z\left(I_{\mu}^{\lambda}(a,b,c)F(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)F(z)} - \gamma\right), \tag{2.21}$$

where $p \in P$. Then by using (2.20), we get

$$(1-\gamma)p(z) + c + \gamma = \frac{(c+1)I_{\mu}^{\lambda}(a,b,c)f(z)}{I_{\mu}^{\lambda}(a,b,c)F(z)}. \tag{2.22}$$

Differentiating both sides of (2.22) logarithmically, we obtain

$$p(z) + \frac{zp'(z)}{c+\gamma+(1-\gamma)p(z)} = \frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)f(z)} - \gamma\right). \tag{2.23}$$

Then, by Lemma 2.2, we obtain that

$$\frac{1}{1-\gamma}\left(\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)f(z)} - \gamma\right) \prec \phi(z) \quad (0 \le \gamma < 1;\ z \in \mathbb{U}). \tag{2.24}$$

□

Now, by letting $\phi(z) = (1+Az)/(1+Bz)$ $(-1 \le B < A \le 1)$ in Theorem 2.9, we have the following.

Corollary 2.10. *For $\lambda > -1$, $\mu \ge 0$, $c > -\gamma$, and $0 \le \gamma < 1$. If $f \in Q_\mu^\lambda(a,b,c;\gamma;A,B)$, then $F \in Q_\mu^\lambda(a,b,c;\gamma;A,B)$, where F given by* (2.19).

Also, by taking $\phi(z) = ((1+z)/(1-z))^\beta$ $(0 < \beta \le 1)$ in Theorem 2.9, we have the following.

Corollary 2.11. *For $\lambda > -1$, $\mu \ge 0$, $c > -\beta$, $0 < \beta \le 1$, and $0 \le \gamma < 1$. If $f \in \mathcal{S}_\mu^\lambda(a,b,c;\gamma,\beta)$, then $F \in \mathcal{S}_\mu^\lambda(a,b,c;\gamma,\beta)$.*

Corollary 2.12. *For $\lambda > -1$, $\mu \ge 0$, $c > -\beta$, $0 < \beta \le 1$, and $0 \le \gamma < 1$. If $f \in \mathcal{C}_\mu^\lambda(a,b,c;\gamma,\beta)$, then $F \in \mathcal{C}_\mu^\lambda(a,b,c;\gamma,\beta)$.*

Proof. It holds that

$$\begin{aligned} f(z) \in \mathcal{C}_\mu^\lambda(a,b,c;\gamma,\beta) &\Longleftrightarrow F\left(zf'(z)\right) \in \mathcal{S}_\mu^\lambda(a,b,c;\gamma,\beta) \\ &\Longleftrightarrow z\left(F(z)\right)' \in \mathcal{S}_\mu^\lambda(a,b,c;\gamma,\beta) \\ &\Longleftrightarrow F(z) \in \mathcal{C}_\mu^\lambda(a,b,c;\gamma,\beta). \end{aligned} \tag{2.25}$$

□

Theorem 2.13. *Let $f \in \mathcal{A}$. Then*

$$\mathcal{K}_\mu^{\lambda+1}(a,b,c;\alpha,\beta,\gamma;A,B) \subset \mathcal{K}_\mu^\lambda(a,b,c;\alpha,\beta,\gamma;A,B) \tag{2.26}$$

for Re $a > 1-\beta$, $0 < \beta \le 1$, $0 \le \alpha < 1$, $0 \le \gamma < 1$, *and* $-1 \le B < A \le 1$.

Proof. Let $f \in \mathcal{K}_\mu^{\lambda+1}(a,b,c;\alpha,\beta,\gamma;A,B)$, then by the definition, we can write

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_\mu^{\lambda+1}(a,b,c)f(z)\right)'}{I_\mu^{\lambda+1}(a,b,c)g(z)} - \alpha\right) \prec \left(\frac{1+z}{1-z}\right)^\beta \quad (z \in \mathbb{U}) \tag{2.27}$$

for some $g \in Q_\mu^{\lambda+1}(a,b,c;\gamma;A,B)$.

Letting $h(z) = z(I_\mu^\lambda(a,b,c)f(z))'/I_\mu^\lambda(a,b,c)g(z)$ and $H(z) = z(I_\mu^\lambda(a,b,c)g(z))'I_\mu^\lambda(a,b,c)g(z)$, we observe that $h(z), H(z) \in P(z)$. Now by Corollary 2.6, $g \in Q_\mu^\lambda(a,b,c;\gamma;A,B)$ and so $\operatorname{Re} H(z) > \gamma$. Also, note that

$$z\left(I_\mu^\lambda(a,b,c)f(z)\right)' = \left(I_\mu^\lambda(a,b,c)g(z)\right)h(z). \tag{2.28}$$

Differentiating both sides in (2.28) yields

$$\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)g(z)} = \frac{z\left(I_\mu^\lambda(a,b,c)g(z)\right)'}{I_\mu^{\lambda+1}(a,b,c)g(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z). \tag{2.29}$$

Now by using the identity (1.21), we obtain

$$\begin{aligned}
&\frac{z\left(I_\mu^{\lambda+1}(a,b,c)f(z)\right)'}{I_\mu^{\lambda+1}(a,b,c)g(z)}\\
&\quad= \frac{I_\mu^{\lambda+1}(a,b,c)\left(zf'(z)\right)}{I_\mu^{\lambda+1}(a,b,c)g(z)}\\
&\quad= \frac{z\left(I_\mu^\lambda(a,b,c)\left(zf'(z)\right)\right)' + \lambda I_\mu^\lambda(a,b,c)\left(zf'(z)\right)}{z\left(I_\mu^\lambda(a,b,c)g(z)\right)' + \lambda I_\mu^\lambda(a,b,c)g(z)}\\
&\quad= \frac{z\left(I_\mu^\lambda(a,b,c)\left(zf'(z)\right)\right)'/I_\mu^\lambda(a,b,c)g(z) + \lambda\left(I_\mu^\lambda(a,b,c)\left(zf'(z)\right)/I_\mu^\lambda(a,b,c)g(z)\right)}{z\left(I_\mu^\lambda(a,b,c)g(z)\right)'/I_\mu^\lambda(a,b,c)g(z) + \lambda}\\
&\quad= \frac{H(z)h(z) + zh'(z) + \lambda h(z)}{H(z) + \lambda}\\
&\quad= h(z) + \frac{zh'(z)}{H(z) + \lambda}.
\end{aligned} \tag{2.30}$$

From (2.27), (2.28), and (2.30), we conclude that

$$\frac{1}{1-\alpha}\left(h(z) + \frac{zh'(z)}{H(z)+\lambda} - \alpha\right) \prec \left(\frac{1+z}{1-z}\right)^\beta. \tag{2.31}$$

Letting $E = 0$ and $B(z) = (1/(1-\alpha))(1/(H(z)+\lambda))$, we obtain

$$\operatorname{Re}\left[B(z)\right] = \frac{1}{1-\alpha}\frac{1}{\left|H(z)+\lambda\right|^2}\operatorname{Re}\left[H(z)+\lambda\right] > 0. \tag{2.32}$$

The above inequality satisfies the conditions required by Lemma 2.3. Hence $\phi(z) \prec ((1+z)/(1-z))^\beta$ and so the proof is complete. □

Theorem 2.14. *Let $f \in \mathcal{A}$. Then*

$$\mathcal{K}_\mu^\lambda(a,b,c;\alpha,\beta,\gamma;A,B) \subset \mathcal{K}_\mu^\lambda(a+1,b,c;\alpha,\beta,\gamma;A,B) \tag{2.33}$$

for $\operatorname{Re} a > 1-\beta$, $0 < \beta \le 1$, $0 \le \alpha < 1$, $0 \le \gamma < 1$, *and* $-1 \le B < A \le 1$.

Proof. By using the same technique as in the proof of Theorem 2.13, we get

$$\frac{1}{1-\alpha}\left(h(z)+\frac{zh'(z)}{H(z)+(\alpha-1)}-\alpha\right)\prec\left(\frac{1+z}{1-z}\right)^{\beta}. \tag{2.34}$$

By letting $E=0$ and $B(z)=(1/(1-\alpha))(1/(H(z)+(a-1)))$, we obtain

$$\operatorname{Re}\left[B(z)\right]=\frac{1}{1-\alpha}\frac{1}{\left|H(z)+(a-1)\right|^{2}}\operatorname{Re}\left[H(z)+(a-1)\right]>0. \tag{2.35}$$

Then, by applying Lemma 2.3, we obtain the required result. □

Theorem 2.15. *Let $c>-\beta$, $0<\beta\leq 1$, $0\leq\alpha<1$, $0\leq\gamma<1$, and $-1\leq B<A\leq 1$. If $f\in\mathcal{K}_{\mu}^{\lambda}(a,b,c;\alpha,\beta,\gamma;A,B)$, then $F\in\mathcal{K}_{\mu}^{\lambda}(a,b,c;\alpha,\beta,\gamma;A,B)$, where F is given by* (2.19)*.*

Proof. Also, by using the same technique as in the proof of Theorem 2.13, we get

$$\frac{1}{1-\alpha}\left(\frac{zh'(z)}{H(z)+c}+h(z)-\alpha\right)\prec\left(\frac{1+z}{1-z}\right)^{\beta}. \tag{2.36}$$

By letting $E=0$ and $B(z)=(1/(1-\alpha))(1/(H(z)+c))$, we obtain

$$\operatorname{Re}\left[B(z)\right]=\frac{1}{1-\alpha}\frac{1}{\left|H(z)+c\right|^{2}}\operatorname{Re}\left[H(z)+c\right]>0. \tag{2.37}$$

Then, applying Lemma 2.3, we obtain the required result. □

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