Research Article

Elliptic Equations in Weighted Sobolev Spaces on Unbounded Domains

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We study in this paper a class of second-order linear elliptic equations in weighted Sobolev spaces on unbounded domains of $\mathbb{R}^n$, $n \geq 3$. We obtain an a priori bound, and a regularity result from which we deduce a uniqueness theorem.

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1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$. Assign in Ω the uniformly elliptic second-order linear differential operator

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \quad (1.1)$$

with coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega)$, $i, j = 1, \dots, n$, and consider the associate Dirichlet problem:

$$\begin{aligned} u &\in W^{2,p}(\Omega) \cap \overset{\circ}{W}^{1,p}(\Omega), \\ Lu &= f, \quad f \in L^p(\Omega), \end{aligned} \quad (1.2)$$

where $p \in]1, +\infty[$.

It is well known that if Ω is a bounded and sufficiently regular set, the above problem has been widely investigated by several authors under various hypotheses of discontinuity on the leading coefficients, in the case $p = 2$ or p sufficiently close to 2. In particular, some $W^{2,p}$ -bounds for the solutions of the problem (1.2) and related existence and uniqueness results have been obtained. Among the other results on this subject, we quote here those

proved in [1], where the author assumed that a_{ij} 's belong to $W^{1,n}(\Omega)$ (and considered the case $p = 2$) and in [2–4] (where the coefficients belong to some classes wider than $W^{1,n}(\Omega)$). More recently, a relevant contribution has been given in [5–8], where the coefficients a_{ij} are assumed to be in the class VMO and $p \in]1, +\infty[$; observe here that VMO contains the space $W^{1,n}(\Omega)$.

If the set Ω is unbounded and regular enough, under assumptions similar to those required in [1], problem (1.2) has for instance been studied in [9–11] with $p = 2$, and in [12] with $p \in]1, +\infty[$. Instead, in [13, 14] the leading coefficients satisfy restrictions similar to those in [5, 6].

In this paper, we extend some results of [13, 14] to a weighted case. More precisely, we denote by ρ a weight function belonging to a suitable class such that

$$\inf_{\Omega} \rho > 0, \quad \lim_{|x| \rightarrow +\infty} \rho(x) = +\infty, \quad (1.3)$$

and consider the Dirichlet problem:

$$\begin{aligned} u &\in W_s^{2,p}(\Omega) \cap W_s^{\circ 1,p}(\Omega), \\ Lu &= f, \quad f \in L_s^p(\Omega), \end{aligned} \quad (1.4)$$

where $s \in \mathbb{R}$, $W_s^{2,p}(\Omega)$, $W_s^{\circ 1,p}(\Omega)$, and $L_s^p(\Omega)$ are some weighted Sobolev spaces and the weight functions are a suitable power of ρ . We obtain an a priori bound for the solutions of (1.4). Moreover, we state a regularity result that allows us to deduce a uniqueness theorem for the problem (1.4). A similar weighted case was studied in [15] with the leading coefficients satisfying hypotheses of Miranda's type and when $p = 2$.

2. Weight functions and weighted spaces

Let G be any Lebesgue measurable subset of $\mathbb{R}^n$ and let $\Sigma(G)$ be the collection of all Lebesgue measurable subsets of G . If $F \in \Sigma(G)$, denote by $|F|$ the Lebesgue measure of F , by χ_F the characteristic function of F , by $F(x, r)$ the intersection $F \cap B(x, r)$ ($x \in \mathbb{R}^n$, $r \in \mathbb{R}_+$)—where $B(x, r)$ is the open ball of radius r centered at x —and by $\mathfrak{D}(F)$ the class of restrictions to F of functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\overline{F} \cap \text{supp } \zeta \subseteq F$. Moreover, if $X(F)$ is a space of functions defined on F , we denote by $X_{\text{loc}}(F)$ the class of all functions $g : F \rightarrow \mathbb{R}$, such that $\zeta g \in X(F)$ for any $\zeta \in \mathfrak{D}(F)$.

We introduce a class of weight functions defined on an open subset Ω of $\mathbb{R}^n$. Denote by $\mathcal{A}(\Omega)$ the set of all measurable functions $\rho : \Omega \rightarrow \mathbb{R}_+$, such that

$$\gamma^{-1} \rho(y) \leq \rho(x) \leq \gamma \rho(y) \quad \forall y \in \Omega, \quad \forall x \in \Omega(y, \rho(y)), \quad (2.1)$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y . Examples of functions in $\mathcal{A}(\Omega)$ are the function

$$x \in \mathbb{R}^n \longrightarrow 1 + a|x|, \quad a \in]0, 1[, \quad (2.2)$$

and, if $\Omega \neq \mathbb{R}^n$ and S is a nonempty subset of $\partial\Omega$, the function

$$x \in \Omega \longrightarrow a \text{ dist}(x, S), \quad a \in]0, 1[. \quad (2.3)$$

For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \{z \in \partial\Omega : \lim_{x \rightarrow z} \rho(x) = 0\}. \quad (2.4)$$

It is known that

$$\rho \in L_{\text{loc}}^\infty(\overline{\Omega}), \quad \rho^{-1} \in L_{\text{loc}}^\infty(\overline{\Omega} \setminus S_\rho) \quad (2.5)$$

(see [16, 17]).

We assign an unbounded open subset Ω of $\mathbb{R}^n$.

Let ρ_1 be a function, such that $\rho_1 \in \mathcal{A}(\mathbb{R}^n)$ and

$$\inf_{\Omega} \rho_1 > 0, \quad \lim_{|x| \rightarrow +\infty} \rho_1(x) = +\infty. \quad (2.6)$$

We put

$$\rho = \rho_1|_{\Omega}. \quad (2.7)$$

For any $a \in]0, 1]$ and $x \in \mathbb{R}^n$, we set

$$I_a(x) = \Omega(x, a\rho_1(x)). \quad (2.8)$$

If $k \in \mathbb{N}_0$, $1 \leq p < +\infty$, $s \in \mathbb{R}$, and $\rho \in \mathcal{A}(\Omega)$, consider the space $W_s^{k,p}(\Omega)$ of distributions u on Ω , such that $\rho^s \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$, equipped with the norm

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\rho^s \partial^\alpha u\|_{L^p(\Omega)}. \quad (2.9)$$

Moreover, denote by $\overset{\circ}{W}_s^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W_s^{k,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$. A more detailed account of properties of the above defined spaces can be found, for instance, in [18].

From [15, Lemmas 1.1 and 2.1], we deduce the following two lemmas, respectively.

Lemma 2.1. *For any $p \in [1, +\infty[$, $s \in \mathbb{R}$, and $a \in]0, 1]$, $g \in L_s^p(\Omega)$ if and only if $g \in L_{\text{loc}}^p(\overline{\Omega})$ and the function $x \in \mathbb{R}^n \rightarrow \rho_1^{s-n/p}(x) \|g\|_{L^p(I_a(x))}$ belongs to $L^p(\mathbb{R}^n)$. Moreover, there exist $c_1, c_2 \in \mathbb{R}_+$, such that*

$$c_1 \|g\|_{L_s^p(\Omega)}^p \leq \int_{\mathbb{R}^n} \rho_1^{sp-n}(x) \|g\|_{L^p(I_a(x))}^p dx \leq c_2 \|g\|_{L_s^p(\Omega)}^p \quad \forall g \in L_s^p(\Omega), \quad (2.10)$$

where c_1 and c_2 depend on n, p, s, a , and ρ .

Lemma 2.2. *If Ω has the segment property, then for any $k \in \mathbb{N}_0$, $p \in [1, +\infty[$, and $s \in \mathbb{R}$ one has*

$$W_s^{k,p}(\Omega) \cap \overset{\circ}{W}_{\text{loc}}^{k,p}(\overline{\Omega}) = \overset{\circ}{W}_s^{k,p}(\Omega). \quad (2.11)$$

3. Some embedding lemmas

We now recall the definitions of the function spaces in which the coefficients of the operator will be chosen. If Ω has the property

$$|\Omega(x, r)| \geq Ar^n \quad \forall x \in \Omega, \quad \forall r \in]0, 1], \quad (3.1)$$

where A is a positive constant independent of x and r , it is possible to consider the space $\text{BMO}(\Omega, \tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\text{loc}}(\overline{\Omega})$ such that

$$[g]_{\text{BMO}(\Omega, \tau)} = \sup_{\substack{x \in \Omega \\ r \in]0, \tau]}} \oint_{\Omega(x, r)} \left| g - \oint_{\Omega(x, r)} g \right| < +\infty, \quad (3.2)$$

where

$$\oint_{\Omega(x, r)} g = |\Omega(x, r)|^{-1} \int_{\Omega(x, r)} g. \quad (3.3)$$

If $g \in \text{BMO}(\Omega) = \text{BMO}(\Omega, \tau_A)$, where

$$\tau_A = \sup \left\{ \tau \in \mathbb{R}_+ : \sup_{\substack{x \in \Omega \\ r \in]0, \tau]}} \frac{r^n}{|\Omega(x, r)|} \leq \frac{1}{A} \right\}, \quad (3.4)$$

we will say that $g \in \text{VMO}(\Omega)$ if $[g]_{\text{BMO}(\Omega, \tau)} \rightarrow 0$ for $\tau \rightarrow 0^+$. A function $\eta[g] :]0, 1] \rightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $\text{VMO}(\Omega)$ if

$$[g]_{\text{BMO}(\Omega, \tau)} \leq \eta[g](\tau) \quad \forall \tau \in]0, 1], \quad \lim_{\tau \rightarrow 0^+} \eta[g](\tau) = 0. \quad (3.5)$$

For $t \in [1, +\infty[$ and $\lambda \in [0, n[$, we denote by $M^{t, \lambda}(\Omega)$ the set of all functions g in $L^t_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{t, \lambda}(\Omega)} = \sup_{\substack{r \in]0, 1] \\ x \in \Omega}} r^{-\lambda/t} \|g\|_{L^t(\Omega(x, r))} < +\infty, \quad (3.6)$$

endowed with the norm defined by (3.6). Then, we define $M^{t, \lambda}_o(\Omega)$ as the closure of $C^\infty_o(\Omega)$ in $M^{t, \lambda}(\Omega)$. In particular, we put $M^t(\Omega) = M^{t, 0}(\Omega)$, and $M^t_o(\Omega) = M^{t, 0}_o(\Omega)$. In order to define the modulus of continuity of a function g in $M^{t, \lambda}_o(\Omega)$, recall first that for a function $g \in M^{t, \lambda}(\Omega)$ the following characterization holds:

$$g \in M^{t, \lambda}_o(\Omega) \iff \lim_{\tau \rightarrow 0^+} (p_g(\tau) + \|(1 - \zeta_{1/\tau})g\|_{M^{t, \lambda}(\Omega)}) = 0, \quad (3.7)$$

where

$$p_g(\tau) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x, 1)| \leq \tau}} \|\chi_E g\|_{M^{t, \lambda}(\Omega)}, \quad \tau \in \mathbb{R}_+, \quad (3.8)$$

and $\zeta_r, r \in \mathbb{R}_+$, is a function in $C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \zeta_r \leq 1, \quad \zeta_r|_{B_r} = 1, \quad \text{supp } \zeta_r \subset B_{2r}, \quad (3.9)$$

with the position $B_r = B(0, r)$. Thus, the *modulus of continuity* of $g \in M_o^{t,\lambda}(\Omega)$ is a function

$$\sigma_o[g] :]0, 1] \longrightarrow \mathbb{R}_+, \quad (3.10)$$

such that

$$p_g(\tau) + \|(1 - \zeta_{1/\tau})g\|_{M^{t,\lambda}(\Omega)} \leq \sigma_o[g](\tau) \quad \forall \tau \in]0, 1], \quad \lim_{\tau \rightarrow 0^+} \sigma_o[g](\tau) = 0. \quad (3.11)$$

A more detailed account of properties of the above defined function spaces can be found in [9, 19, 20].

We consider the following condition:

(h_0) Ω has the cone property, $p \in]1, +\infty[, s \in \mathbb{R}, k, h, t$ are numbers such that

$$k \in \mathbb{N}, h \in \{0, 1, \dots, k-1\}, t \geq p, t > p \quad \text{if } p = \frac{n}{k-h}, g \in M^t(\Omega). \quad (3.12)$$

From [21, Theorem 3.1] we have the following.

Lemma 3.1. *If the assumption (h_0) holds, then for any $u \in W_s^{k,p}(\Omega)$ one has $g\partial^h u \in L_s^p(\Omega)$ and*

$$\|g\partial^h u\|_{L_s^p(\Omega)} \leq c\|g\|_{M^t(\Omega)}\|u\|_{W_s^{k,p}(\Omega)}, \quad (3.13)$$

with c dependent only on Ω, n, k, h, p , and t .

From [21, Theorem 3.2] it follows Lemma 3.2.

Lemma 3.2. *If the assumption (h_0) is satisfied and in addition $g \in M_o^t(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open set $\Omega_\varepsilon \subset\subset \Omega$, with the cone property, such that*

$$\|g\partial^h u\|_{L_s^p(\Omega)} \leq \varepsilon\|u\|_{W_s^{k,p}(\Omega)} + c(\varepsilon)\|u\|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W_s^{k,p}(\Omega), \quad (3.14)$$

where $c(\varepsilon), \Omega_\varepsilon$ depend on $\varepsilon, \Omega, n, k, h, p, t, \rho, s$, and $\sigma_o[g]$.

4. An a priori bound

Assume that Ω is an unbounded open subset of $\mathbb{R}^n, n \geq 3$, with the uniform $C^{1,1}$ -regularity property, and let ρ be the function defined by (2.7). Moreover, let $p \in]1, +\infty[$ and $s \in \mathbb{R}$. Consider in Ω the differential operator:

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \quad (4.1)$$

with the following conditions on the coefficients:

(h_1)

$$\begin{aligned} a_{ij} = a_{ji} &\in L^\infty(\Omega) \cap \text{VMO}_{\text{loc}}(\overline{\Omega}), \quad i, j = 1, \dots, n, \\ \exists v > 0 : \sum_{i,j=1}^n a_{ij} \xi_i \xi_j &\geq v |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \end{aligned} \quad (4.2)$$

there exist functions e_{ij} , $i, j = 1, \dots, n$, g and $\mu \in \mathbb{R}_+$ such that

(h_2)

$$\begin{aligned} e_{ij} = e_{ji} &\in L^\infty(\Omega), \quad (e_{ij})_{x_h} \in M_0^{t, n-t}(\Omega), \quad \text{with } t \in]2, n], \quad i, j, h = 1, \dots, n, \\ \sum_{i,j=1}^n e_{ij} \xi_i \xi_j &\geq \mu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \\ g &\in L^\infty(\Omega), \quad g_0 = \text{ess inf}_\Omega g > 0, \\ \lim_{r \rightarrow +\infty} \sum_{i,j=1}^n \|e_{ij} - g a_{ij}\|_{L^\infty(\Omega \setminus B_r)} &= 0, \end{aligned} \quad (4.3)$$

(h_3)

$$\begin{aligned} a_i &\in M_o^{t_1}(\Omega), \quad i = 1, \dots, n, \\ a = a' + a'', \quad a' &\in M_o^{t_2}(\Omega), \quad a'' \in L^\infty(\Omega), \quad \text{ess inf}_\Omega a'' = a''_0 > 0, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} t_1 &\geq n \quad \text{if } p < n, \quad t_1 > n \quad \text{if } p = n, \quad t_1 = p \quad \text{if } p > n, \\ t_2 &\geq n/2 \quad \text{if } p < n/2, \quad t_2 > n/2 \quad \text{if } p = n/2, \quad t_2 = p \quad \text{if } p > n/2. \end{aligned} \quad (4.5)$$

Observe that under the assumptions (h_1)–(h_3), it follows that the operator $L : W_s^{2,p}(\Omega) \rightarrow L_s^p(\Omega)$ is bounded from Lemma 3.1.

Theorem 4.1. *If the hypotheses (h_1), (h_2), and (h_3) are verified, then there exist a constant $c \in \mathbb{R}_+$ and a bounded open subset $\Omega_0 \subset \subset \Omega$, with the cone property, such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_0)} \right), \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \quad (4.6)$$

with c and Ω_0 depending on $n, p, \rho, s, \Omega, v, \mu, g_0, a''_0, t, t_1, t_2, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \eta[\xi_{2r_0} a_{ij}], \sigma_o[(e_{ij})_x], \sigma_o[a_i], \sigma_o[a'],$ where $r_0 \in \mathbb{R}_+$ depends on $n, p, \Omega, \mu, g_0, a''_0, t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_o[(e_{ij})_x]$.

Proof. We consider a function $\phi \in C_o^\infty(\mathbb{R}^n)$, such that

$$\begin{aligned} \phi|_{B_{1/2}} &= 1, \quad \text{supp } \phi \subset B_1, \\ \sup_{\mathbb{R}^n} |\partial^\alpha \phi| &\leq c_\alpha \quad \forall \alpha \in \mathbb{N}_0^n, \end{aligned} \quad (4.7)$$

where $c_\alpha \in \mathbb{R}_+$ depends only on α , fix $y \in \mathbb{R}^n$ and put

$$\psi = \psi_y : x \in \mathbb{R}^n \longrightarrow \phi\left(\frac{x-y}{\rho_1(y)}\right). \quad (4.8)$$

Clearly we have

$$\begin{aligned} \psi|_{B(y, (1/2)\rho_1(y))} &= 1, \quad \text{supp } \psi \subset B(y, \rho_1(y)), \\ \sup_{\mathbb{R}^n} |\partial^\alpha \psi| &\leq c_\alpha \rho_1^{-|\alpha|}(y) \quad \forall \alpha \in \mathbb{N}_0^n. \end{aligned} \quad (4.9)$$

Now, we put

$$L_0 = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad (4.10)$$

and fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega)$. Since $\psi u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}^{1,p}(\Omega)$, from [14, Theorem 3.3], it follows that there exist $c_1 \in \mathbb{R}_+$ and a bounded open subset $\Omega_1 \subset\subset \Omega$, with the cone property, such that

$$\|\psi u\|_{W^{2,p}(\Omega)} \leq c_1 (\|(L_0 + a'')(\psi u)\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)}), \quad (4.11)$$

with c_1 and Ω_1 depending on $n, p, \Omega, \nu, \mu, g_0, a_0'', t, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \eta[\xi_{2r_0} a_{ij}], \sigma_\circ[(e_{ij})_x]$, where $r_0 \in \mathbb{R}_+$ depends on $n, p, \Omega, \mu, g_0, a_0'', t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_\circ[(e_{ij})_x]$. Since

$$L_0(\psi u) = \psi L_0 u - 2 \sum_{i,j=1}^n a_{ij} \psi_{x_i} u_{x_j} - \sum_{i,j=1}^n a_{ij} \psi_{x_i x_j} u, \quad (4.12)$$

from (4.11) and (4.12), we have

$$\begin{aligned} &\|\psi u\|_{W^{2,p}(\Omega)} \\ &\leq c_2 \left(\|\psi(L_0 + a'')u\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|\psi_{x_i} u_{x_j}\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|\psi_{x_i x_j} u\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)} \right), \end{aligned} \quad (4.13)$$

with c_2 dependent on the same parameters of c_1 .

On the other hand, since $\rho \in L_{\text{loc}}^\infty(\overline{\Omega})$, we have that

$$\|\psi u\|_{L^p(\Omega_1)} \leq c_3 \rho_1^{-2}(y) \|u\|_{L^p(I_1(y))}, \quad (4.14)$$

where $c_3 \in \mathbb{R}_+$ depends only on ρ .

Therefore, by (4.13) and (4.14), we deduce the bound:

$$\begin{aligned} \|u\|_{W^{2,p}(I_{1/2}(y))} &\leq \|\psi u\|_{W^{2,p}(\Omega)} \\ &\leq c_4 (\|L_0 u + a'' u\|_{L^p(I_1(y))} + \rho_1^{-1}(y) \|u_x\|_{L^p(I_1(y))} + \rho_1^{-2}(y) \|u\|_{L^p(I_1(y))}), \end{aligned} \quad (4.15)$$

where $c_4 \in \mathbb{R}_+$ depends on the same parameters of c_2 and on ρ .

From (4.15) it follows

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho_1^{ps-n}(y) \|u\|_{W^{2,p}(I_{1/2}(y))}^p dy \\ & \leq c_5 \left(\int_{\mathbb{R}^n} \rho_1^{ps-n}(y) \|L_0 u + a'' u\|_{L^p(I_1(y))}^p dy \right. \\ & \quad \left. + \int_{\mathbb{R}^n} \rho_1^{ps-n-p}(y) \|u_x\|_{L^p(I_1(y))}^p dy + \int_{\mathbb{R}^n} \rho_1^{ps-n-2p}(y) \|u\|_{L^p(I_1(y))}^p dy \right), \end{aligned} \quad (4.16)$$

where $c_5 \in \mathbb{R}_+$ depends on the same parameters of c_4 .

Since

$$L_s^p(\Omega) \hookrightarrow L_{s-1}^p(\Omega), \quad L_s^p(\Omega) \hookrightarrow L_{s-2}^p(\Omega), \quad (4.17)$$

from (4.16) and from Lemma 2.1 we have that

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c_6 (\|L_0 u + a'' u\|_{L_s^p(\Omega)} + \|u_x\|_{L_{s-1}^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)}), \quad (4.18)$$

with $c_6 \in \mathbb{R}_+$ dependent on the same parameters of c_5 and also on s .

Moreover, from Lemma 3.2 it follows that for any $\varepsilon \in \mathbb{R}_+$, there exist $c'(\varepsilon), c''(\varepsilon) \in \mathbb{R}_+$, and two bounded open sets $\Omega'_\varepsilon, \Omega''_\varepsilon \subset \subset \Omega$, both with the cone property, such that

$$\begin{aligned} & \|u_x\|_{L_{s-1}^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c'(\varepsilon) \|u\|_{L^p(\Omega'_\varepsilon)}, \\ & \left\| \sum_{i=1}^n a_i u_{x_i} + a' u \right\|_{L_s^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c''(\varepsilon) \|u\|_{L^p(\Omega''_\varepsilon)}, \end{aligned} \quad (4.19)$$

where $c'(\varepsilon), \Omega'_\varepsilon$ depend on $\varepsilon, \Omega, n, p, \rho, s$, and $c''(\varepsilon), \Omega''_\varepsilon$ depend on $\varepsilon, \Omega, n, p, t_1, t_2, \rho, s, \sigma_\circ[a_i]$, and $\sigma_\circ[a']$.

From (4.18) and (4.19) it follows (4.6) and then we have the result. $\square$

5. A uniqueness result

In this section, we will prove our uniqueness theorem. We begin to prove a regularity result.

Lemma 5.1. *Suppose that the assumptions (h_1) , (h_2) , and (h_3) (with $t_1 > n$ and $t_2 > n/2$) hold, and let u be a solution of the problem*

$$\begin{aligned} u & \in W_{loc}^{2,q}(\overline{\Omega}) \cap \overset{\circ}{W}_{loc}^{1,q}(\overline{\Omega}) \cap L_m^p(\Omega), \\ Lu & \in L_s^p(\Omega), \end{aligned} \quad (5.1)$$

where $q \in]1, p]$ and $m \in \mathbb{R}$. Then, u belongs to $W_s^{2,p}(\Omega)$.

Proof. By [13, Lemma 4.1] we have

$$u \in W_{\text{loc}}^{2,p}(\overline{\Omega}) \cap \overset{\circ}{W}_{\text{loc}}^{1,p}(\overline{\Omega}). \quad (5.2)$$

We choose $r, r' \in \mathbb{R}_+$, with $r < r' < 1$, and a function $\phi \in C_0^\infty(\mathbb{R}^n)$, such that

$$\begin{aligned} \phi|_{B_r} &= 1, \quad \text{supp } \phi \subset B_{r'}, \\ \sup_{\mathbb{R}^n} |\partial^\alpha \phi| &\leq c_\alpha (r' - r)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}_0^n, \end{aligned} \quad (5.3)$$

where $c_\alpha \in \mathbb{R}_+$ depends only on α .

We fix $y \in \mathbb{R}^n$ and put

$$\psi = \psi_y : x \in \mathbb{R}^n \longrightarrow \phi\left(\frac{x - y}{\rho_1(y)}\right). \quad (5.4)$$

Clearly we have

$$\begin{aligned} \psi|_{B(y, r\rho_1(y))} &= 1, \quad \text{supp } \psi \subset B(y, r'\rho_1(y)), \\ \sup_{\mathbb{R}^n} |\partial^\alpha \psi| &\leq c_\alpha \rho_1^{-|\alpha|}(y) (r' - r)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}_0^n. \end{aligned} \quad (5.5)$$

Since $\psi u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}^{1,p}(\Omega)$, from [14, Theorem 3.3] it follows that there exist $c_1 \in \mathbb{R}_+$ and a bounded open subset $\Omega_1 \subset\subset \Omega$, with the cone property, such that

$$\|\psi u\|_{W^{2,p}(\Omega)} \leq c_1 (\|L(\psi u)\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)}), \quad (5.6)$$

with c_1 and Ω_1 depending on $n, p, \Omega, v, \mu, g_0, a_0'', t, t_1, t_2, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \eta[\xi_{2r_0} a_{ij}], \sigma_\circ[(e_{ij})_x], \sigma_\circ[a_i], \sigma_\circ[a']$, where $r_0 \in \mathbb{R}_+$ depends on $n, p, \Omega, \mu, g_0, a_0'', t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_\circ[(e_{ij})_x]$.

Since

$$\begin{aligned} L(\psi u) &= - \sum_{i,j=1}^n a_{ij}(\psi u)_{x_i x_j} + \sum_{i=1}^n a_i(\psi u)_{x_i} + a\psi u \\ &= \psi Lu - 2 \sum_{i,j=1}^n a_{ij}(\psi_{x_i} u)_{x_j} + \sum_{i,j=1}^n a_{ij} \psi_{x_i x_j} u + \sum_{i=1}^n a_i \psi_{x_i} u, \end{aligned} \quad (5.7)$$

from (5.6) and (5.7), we have

$$\begin{aligned} &\|\psi u\|_{W^{2,p}(\Omega)} \\ &\leq c_2 \left(\|\psi Lu\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|(\psi_{x_i} u)_{x_j}\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|\psi_{x_i x_j} u\|_{L^p(\Omega)} + \sum_{i=1}^n \|a_i \psi_{x_i} u\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)} \right), \end{aligned} \quad (5.8)$$

with c_2 dependent on the same parameters of c_1 .

From Lemma 3.1 with $s = 0$, we have that

$$\|a_i \psi_{x_i} u\|_{L^p(\Omega)} \leq c_3 \|a_i\|_{M^{t_1}(\Omega)} (\|\psi_{x_i} u\|_{L^p(\Omega)} + \|(\psi_{x_i} u)_x\|_{L^p(\Omega)}), \quad (5.9)$$

with c_3 dependent on Ω , n , p , and t_1 .

Using [22, Corollary 4.5], we can obtain the following interpolation estimate:

$$\|\psi_{x_i} u\|_{L^p(\Omega)} + \|(\psi_{x_i} u)_{x_j}\|_{L^p(\Omega)} \leq c_4 (\|(\psi_{x_i} u)_{xx}\|_{L^p(\Omega)}^{1/2} \|\psi_{x_i} u\|_{L^p(\Omega)}^{1/2} + \|\psi_{x_i} u\|_{L^p(\Omega)}), \quad (5.10)$$

where the constant c_4 depends on Ω , n , p .

Thus, by (5.8)–(5.10), with easy computations, we deduce the bound:

$$\begin{aligned} \|u\|_{W^{2,p}(I_r(y))} &\leq \|\psi u\|_{W^{2,p}(\Omega)} \leq c_5 (r' - r)^{-2} \\ &\quad \times (\|Lu\|_{L^p(I_{r'}(y))} + \|u\|_{W^{2,p}(I_{r'}(y))}^{1/2} (\rho_1^{-1}(y) \|u\|_{L^p(I_{r'}(y))})^{1/2} + \rho_1^{-1}(y) \|u\|_{L^p(I_{r'}(y))}), \end{aligned} \quad (5.11)$$

where $c_5 \in \mathbb{R}_+$ depends on n , p , ρ , Ω , v , μ , g_0 , a_0'' , t , t_1 , t_2 , $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}]$, $\sigma_\circ[(e_{ij})_x]$, $\|a_i\|_{M^{t_1}(\Omega)}$, $\sigma_\circ[a_i]$, $\sigma_\circ[a']$.

By a well-known lemma of monotonicity of Miranda (see [23, Lemma 3.1]), it follows from (5.11) that

$$\|u\|_{W^{2,p}(I_{1/2}(y))} \leq c_6 (\|Lu\|_{L^p(I_1(y))} + \rho_1^{-1}(y) \|u\|_{L^p(I_1(y))} + (\rho_1^{-1}(y) \|u\|_{L^p(I_1(y))})^{1/2} \|u\|_{W^{2,p}(I_{1/2}(y))}^{1/2}), \quad (5.12)$$

and then, using Young's inequality, we deduce from (5.12) that

$$\|u\|_{W^{2,p}(I_{1/2}(y))} \leq c_7 (\|Lu\|_{L^p(I_1(y))} + \rho_1^{-1}(y) \|u\|_{L^p(I_1(y))}), \quad (5.13)$$

with $c_6, c_7 \in \mathbb{R}_+$ dependent on the same parameters of c_5 .

From (5.13) it follows

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho_1^{ps-n}(y) \|u\|_{W^{2,p}(I_{1/2}(y))}^p dy \\ &\leq c_8 \left(\int_{\mathbb{R}^n} \rho_1^{ps-n}(y) \|Lu\|_{L^p(I_1(y))}^p dy + \int_{\mathbb{R}^n} \rho_1^{ps-n-p}(y) \|u\|_{L^p(I_1(y))}^p dy \right), \end{aligned} \quad (5.14)$$

where $c_8 \in \mathbb{R}_+$ depends on the same parameters of c_7 .

If $m \geq s - 1$, since

$$L_m^p(\Omega) \hookrightarrow L_{s-1}^p(\Omega), \quad (5.15)$$

from (5.14) and from Lemma 2.1 we have that

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c_9 (\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_{s-1}^p(\Omega)}), \quad (5.16)$$

with $c_9 \in \mathbb{R}_+$ dependent on the same parameters of c_8 and on s . Therefore, u belongs to $W_s^{2,p}(\Omega)$.

If $m < s - 1$, we denote by k the positive integer, such that

$$s - m - 1 \leq k < s - m. \quad (5.17)$$

Then, for $i = 1, \dots, k$, we have that

$$L_s^p(\Omega) \hookrightarrow L_{m+i}^p(\Omega). \quad (5.18)$$

Therefore, using (5.14) and (5.16) with $m + i$, $i = 1, \dots, k$, instead of s , we deduce that $u \in W_{m+1}^{2,p}(\Omega), \dots, u \in W_{m+k}^{2,p}(\Omega)$. On the other hand, we have that

$$W_{m+k}^{2,p}(\Omega) \hookrightarrow L_{s-1}^p(\Omega) \quad (5.19)$$

and then, since $u \in L_{s-1}^p(\Omega)$, (5.14) holds. Thus, u satisfies (5.16) and then $u \in W_s^{2,p}(\Omega)$. $\square$

Theorem 5.2. *If conditions (h_1) , (h_2) , and (h_3) (with $t_1 > n$ and $t_2 > n/2$) hold, and $a \geq a_0 > 0$ a.e. in Ω , then the problem*

$$u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \quad Lu = 0, \quad (5.20)$$

admits only the zero solution.

Proof. Fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega)$, such that $Lu = 0$. From Lemma 5.1 it follows that $u \in W^{2,p}(\Omega)$. On the other hand, since $u \in W^{1,p}(\Omega) \cap \overset{\circ}{W}_{\text{loc}}^{1,p}(\overline{\Omega})$, from Lemma 2.2 we have that $u \in \overset{\circ}{W}^{1,p}(\Omega)$. Thus, from [13, Theorem 5.2] we deduce that $u = 0$. $\square$

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