Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 687815, 10 pages
doi:10.1155/2008/687815

Research Article

Convergence to Common Fixed Point for Generalized Asymptotically Nonexpansive Semigroup in Banach Spaces

Yali Li, Jianjun Liu, and Lei Deng

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Correspondence should be addressed to Lei Deng, denglei_math@sina.com

Received 13 May 2008; Accepted 14 August 2008

Recommended by Nils Ackermann

Let K be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with a uniformly *Gâteaux* differentiable norm, $\mathcal{F} = \{T(h) : h \geq 0\}$ a generalized asymptotically nonexpansive self-mapping semigroup of K, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. We prove that the following implicit and modified implicit viscosity iterative schemes $\{x_n\}$ defined by $x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n$ and $x_n = \alpha_n y_n + (1-\alpha_n)T(t_n)x_n$, $y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}$ strongly converge to $p \in F$ as $n \to \infty$ and p is the unique solution to the following variational inequality: $\langle f(p) - p, j(y-p)\rangle \leq 0$ for all $y \in F$.

1. Introduction

Let C be a closed convex subset of a Hilbert space H and T a nonexpansive mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Let $F(T)$ be nonempty and u an element of C. For each t with $0 < t < 1$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tu + (1-t)Tx$. Browder [1] showed that $\{x_t\}$ defined by $x_t = tu + (1-t)Tx_t$ converges strongly to the element of $F(T)$ which is nearest to u in $F(T)$ as $t \to 0$.

In 2004, for a contraction $f : C \to C$ and a nonexpansive mapping $T : C \to C$, Xu [2] proposed the following viscosity approximation method in Banach space:

$$x_t = tf(x_t) + (1-t)Tx_t, \quad t \in (0,1),\ t \longrightarrow 0, \tag{1.1}$$

and Song and Xu [3] studied the convergence of the following implicit viscosity iterative scheme:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$.

On the other hand, for a fixed Lipschitz strongly pseudocontractive mapping f and a continuous pseudocontractive mapping T, Song and Chen [4] proposed the following motivated implicit viscosity iterative scheme:

$$\begin{aligned} x_n &= \alpha_n y_n + (1-\alpha_n)Tx_n, \\ y_n &= \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}. \end{aligned} \tag{1.3}$$

In this paper, we will still study the implicit viscosity iterative scheme (1.2) and propose the following iterative scheme:

$$\begin{aligned} x_n &= \alpha_n y_n + (1-\alpha_n)T(t_n)x_n, \\ y_n &= \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}, \end{aligned} \tag{1.4}$$

where $\{T(h) : h \geq 0\}$ is a generalized asymptotically nonexpansive self-mappings semigroup and f a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$.

2. Preliminaries

Throughout this paper, we assume that E is a Banach space and K a nonempty closed convex subset of E. Let E^* be a dual space of E, $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$J(x) = \{f \in E^*,\ \langle x, f\rangle = \|x\| \cdot \|f\|,\ \|x\| = \|f\|\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Definition 2.1 (see [5]). A mapping $T : E \to E$ is said to be total asymptotically nonexpansive if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n > 0$, with $k_n^{(1)}$ and $k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous functions $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)}\phi(\|x - y\|) + k_n^{(2)} \quad \forall\ x, y \in K. \tag{2.2}$$

Remark 2.2. If $\phi(\lambda) = \lambda$, the total asymptotically nonexpansive mapping coincides with generalized asymptotically nonexpansive mapping. In addition, for all $n \in \mathbb{N}$, if $k_n^{(2)} = 0$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping; if $k_n^{(1)} = 0$, $k_n^{(2)} = \max\{0, p_n\}$, where $p_n := \sup_{x,y\in K}(\|T_n x - T_n y\| - \|x - y\|)$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping in the intermediate sense; if $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$, then we obtain from (2.2) the class of nonexpansive mapping.

Remark 2.3. In [5], for the total asymptotically nonexpansive mapping, the authors assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$, so for $M_0 = \max\{\phi(M), M^*\}$, $\phi(\lambda) \leq M_0(1 + \lambda)$ for all $\lambda \geq 0$, then the total asymptotically nonexpansive mapping studied by [5] coincides with generalized asymptotically nonexpansive mapping.

A (one-parameter) generalized asymptotically nonexpansive semigroup is a family $\mathcal{F} = \{T(h) : h \geq 0\}$ of self-mapping of K such that

(i) $T(0)x = x$ for $x \in K$;

(ii) $T(s+t)x = T(s)T(t)x$ for $t, s \geq 0$ and $x \in K$;

(iii) $\lim_{t\to 0} T(t)x = x$ for $x \in K$;

(iv) for each $h \geq 0$, $T(h)$ is generalized asymptotically nonexpansive, that is,

$$\|T(h)x - T(h)y\| \leq \left(1 + k_h^{(1)}\right)\|x - y\| + k_h^{(2)} \quad \forall\, x, y \in K. \tag{2.3}$$

We will denote by F the common fixed point set of $\mathcal{F}$, that is,

$$F := \mathrm{Fix}(\mathcal{F}) = \{x \in K : T(h)x = x,\ h \geq 0\} = \bigcap_{h\geq 0} \mathrm{Fix}(T(h)). \tag{2.4}$$

Definition 2.4. A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$.

Definition 2.5. Let $U = \{x \in E : \|x\| = 1\}$, the norm of E is said to be uniformly *Gâteaux* differentiable, if for each $y \in U$, $\lim_{t\to 0}((\|x+ty\| - \|x\|)/t)$ exists uniformly for $x \in U$.

Definition 2.6. Let μ be a continuous liner functional on l^∞ and let $(a_0, a_1, \ldots) \in l^\infty$. One writes $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. One calls μ a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \ldots) \in l^\infty$.

For a Banach limit μ, one knows that $\underline{\lim}_{n\to\infty} a_n \leq \mu_n(a_n) \leq \overline{\lim}_{n\to\infty} a_n$ for every $a = (a_0, a_1, \ldots) \in l^\infty$. So if $a = (a_0, a_1, \ldots) \in l^\infty$, $b = (b_0, b_1, \ldots) \in l^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, one has $\mu_n(a_n) = \mu_n(b_n)$.

Definition 2.7. Let K be a nonempty closed convex subset of a Banach space E, $\mathcal{F} = \{T(h) : h \geq 0\}$ a continuous operator semigroup on K. Then $\mathcal{F}$ is said to be uniformly asymptotically regular (in short, u.a.r.) on K if for all $h \geq 0$ and any bounded subset C of K, $\lim_{t\to\infty}\sup_{x\in C}\|T(h)(T(t)x) - T(t)x\| = 0$.

Lemma 2.8 (see [6]). *Let E be a Banach space with a uniformly Gâteaux differentiable norm, then the normalized duality mapping $J : E \to 2^{E^*}$ defined by* (2.1) *is single-valued and uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E.*

The single-valued normalized duality mapping is denoted by j.

Lemma 2.9. *Let E be a reflexive and strictly convex Banach space with a uniformly* Gâteaux *differentiable norm, K a nonempty closed convex subset of E. Suppose that $\{x_n\}$ is a bounded sequence in K, $\{T(h) : h \geq 0\}$ a continuous generalized asymptotically nonexpansive semigroup from K into itself such that $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$ for all $h \geq 0$. Define the set*

$$K^* = \left\{x \in K : \mu_n\|x_n - x\|^2 = \min_{y\in K} \mu_n\|x_n - y\|^2\right\}. \tag{2.5}$$

If $F \neq \varnothing$, then $K^* \cap F \neq \varnothing$.

Proof. Set $g(y) = \mu_n\|x_n - y\|^2$, then $g(y)$ is a convex and continuous function, and $g(y) \to \infty$ as $\|y\| \to \infty$. Using [7, Theorem 1.3.11], there exists $x \in K$ such that $g(x) = \inf_{y\in K} g(y)$ by the reflexivity of E, that is, K^* is nonempty. Clearly, K^* is closed convex by the convexity and continuity of $g(y)$.

Since $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$, $\lim_{h\to\infty} k_h^{(i)} = 0$ $(i = 1, 2)$, and $g(y)$ is continuous for all $z \in K^*$, we have

$$\begin{aligned} g\left(\lim_{h\to\infty} T(h)z\right) &= \lim_{h\to\infty} g(T(h)z) \\ &= \lim_{h\to\infty} \mu_n\|x_n - T(h)z\|^2 \\ &\le \lim_{h\to\infty} \mu_n\|T(h)x_n - T(h)z\|^2 \\ &\le \lim_{h\to\infty} \mu_n\left(\left(1 + k_h^{(1)}\right)\|x_n - z\| + k_h^{(2)}\right)^2 \\ &= \mu_n\|x_n - z\|^2. \end{aligned} \tag{2.6}$$

Hence $\lim_{h\to\infty} T(h)z \in K^*$.

Let $p \in F$. Since K^* is closed convex set, there exists a unique $v \in K^*$ such that

$$\|p - v\| = \min_{x\in K^*}\|p - x\|. \tag{2.7}$$

Since $p = \lim_{h\to\infty} T(h)p$ and $\lim_{h\to\infty} T(h)v \in K^*$,

$$\begin{aligned} \left\|p - \lim_{h\to\infty} T(h)v\right\| &= \left\|\lim_{h\to\infty} T(h)p - \lim_{h\to\infty} T(h)v\right\| \\ &= \lim_{h\to\infty}\|T(h)p - T(h)v\| \\ &\le \lim_{h\to\infty}\left(1 + k_h^{(1)}\right)\|p - v\| + k_h^{(2)} \\ &= \|p - v\|. \end{aligned} \tag{2.8}$$

Therefore, $\lim_{h\to\infty} T(h)v = v$. Since $T(s + t)x = T(s)T(t)x$ for all $x \in K$, then we have

$$v = \lim_{t\to\infty} T(t)v = \lim_{t\to\infty} T(s + t)v = \lim_{t\to\infty} T(s)T(t)v = T(s)\lim_{t\to\infty} T(t)v = T(s)v \tag{2.9}$$

for all $s \ge 0$. Therefore $v \in F$ and the proof is complete. □

Lemma 2.10 (see [8]). *Let K be a nonempty convex subset of a Banach space E with a uniformly* Gâteaux *differentiable norm, and $\{x_n\}$ a bounded sequence of E. If $z_0 \in K$, then*

$$\mu_n\|x_n - z_0\|^2 = \min_{y\in K} \mu_n\|x_n - y\|^2 \tag{2.10}$$

if and only if

$$\mu_n\langle y - z_0, J(x_n - z_0)\rangle \le 0 \quad \forall y \in K. \tag{2.11}$$

Lemma 2.11 (see [9]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following conditions:*

$$a_{n+1} \le (1 - \lambda_n)a_n + b_n + c_n \quad \forall n \ge n_0, \tag{2.12}$$

where n_0 is some nonnegative integer, $\lambda_n \in [0, 1]$ with $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\limsup_{n\to\infty}(b_n/\lambda_n) \le 0$, and $\sum_{n=1}^{\infty} c_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.

3. Implicit iteration scheme

Theorem 3.1. *Let E be a real reflexive and strictly convex Banach space with a uniformly* Gâteaux *differentiable norm, K a nonempty closed convex subset of E, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r generalized asymptotically nonexpansive semigroup from K into itself with sequences $\{k_h^{(1)}\}\{k_h^{(2)}\}, h \geq 0$, such that $F \neq \varnothing$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. If $\{x_n\}$ is given by* (1.2), *where* $\lim_{n\to\infty} t_n = \infty$, $\alpha_n \in (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\lim_{n\to\infty}(k_{t_n}^{(i)}/\alpha_n) = 0$ $(i = 1,2)$, *then $\{x_n\}$ converges strongly to some common fixed point p of F such that p is the unique solution in F to variational inequality:*

$$\langle f(p) - p, j(y-p)\rangle \leq 0 \quad \forall y \in F. \tag{3.1}$$

Proof. For any fixed $y \in F$,

$$\begin{aligned}
\|x_n - y\|^2 &= \langle \alpha_n(f(x_n) - y) + (1-\alpha_n)(T(t_n)x_n - y), j(x_n - y)\rangle \\
&= \alpha_n\langle f(x_n) - f(y), j(x_n - y)\rangle + \alpha_n\langle f(y) - y, j(x_n - y)\rangle \\
&\quad + (1-\alpha_n)\langle T(t_n)x_n - T(t_n)y, j(x_n - y)\rangle \\
&\leq \alpha_n\beta\|x_n - y\|^2 + \alpha_n\langle f(y) - y, j(x_n - y)\rangle \\
&\quad + (1-\alpha_n)\|x_n - y\|\left[\left(1 + k_{t_n}^{(1)}\right)\|x_n - y\| + k_{t_n}^{(2)}\right] \\
&= \left(1 - \alpha_n(1-\beta) + (1-\alpha_n)k_{t_n}^{(1)}\right)\|x_n - y\|^2 + \alpha_n\langle f(y) - y, j(x_n - y)\rangle \\
&\quad + (1-\alpha_n)k_{t_n}^{(2)}\|x_n - y\|.
\end{aligned} \tag{3.2}$$

Let $d_n^{(i)} = (k_{t_n}^{(i)}/\alpha_n)$ $(i = 1,2)$. Since $\lim_{n\to\infty}(k_{t_n}^{(i)}/\alpha_n) = 0$ for all $\varepsilon \in (0, 1-\beta)$, there exists $N \in \mathbb{N}$ such that $k_{t_n}^{(i)}/\alpha_n < \varepsilon < 1-\beta < (1-\beta)/(1-\alpha_n)$ for all $n \geq N$.

Furthermore,

$$\|x_n - y\|^2 \leq \frac{\langle f(y) - y, j(x_n - y)\rangle}{1-\beta-(1-\alpha_n)d_n^{(1)}} + \frac{(1-\alpha_n)d_n^{(2)}\|x_n - y\|}{1-\beta-(1-\alpha_n)d_n^{(1)}} \tag{3.3}$$

for all $n \geq N$. That is, $\|x_n - y\| \leq (\|f(y) - y\| + (1-\alpha_n)d_n^{(2)})/(1-\beta-(1-\alpha_n)d_n^{(1)})$ for all $n \geq N$. Thus $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. This imply that

$$\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = \lim_{n\to\infty}\alpha_n\|T(t_n)x_n - f(x_n)\| = 0. \tag{3.4}$$

Since $\{T(h)\}$ is u.a.r and $\lim_{n\to\infty} t_n = \infty$, then for all $h \geq 0$,

$$\lim_{n\to\infty}\|T(h)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n\to\infty}\sup_{x\in C}\|T(h)T(t_n)x - T(t_n)x\| = 0, \tag{3.5}$$

where C is any bounded subset of K containing $\{x_n\}$. Since $\{T(h)\}$ is continuous, hence

$$\begin{aligned}
\|x_n - T(h)x_n\| &\leq \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| \\
&\quad + \|T(h)(T(t_n)x_n) - T(h)x_n\| \longrightarrow 0.
\end{aligned} \tag{3.6}$$

That is, for all $h \geq 0$, $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$. We claim that the set $\{x_n\}$ is sequentially compact. Indeed, define the set

$$K^* = \left\{x \in K : \mu_n\|x_n - x\|^2 = \min_{y\in K} \mu_n\|x_n - y\|^2\right\}. \tag{3.7}$$

By Lemma 2.9, we can found $p \in K^* \cap F$. Using Lemma 2.10, we get that

$$\mu_n\langle y - p, j(x_n - p)\rangle \leq 0 \quad \forall y \in K. \tag{3.8}$$

It follows from (3.3) that

$$\mu_n\|x_n - p\|^2 \leq \mu_n \frac{\langle f(p) - p, j(x_n - p)\rangle}{1 - \beta - (1 - \alpha_n)d_n^{(1)}} + \mu_n \frac{(1 - \alpha_n)d_n^{(2)}\|x_n - p\|}{1 - \beta - (1 - \alpha_n)d_n^{(1)}} \longrightarrow 0. \tag{3.9}$$

Then we have $\mu_n\|x_n - p\| = 0$.

Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which strongly converges to $p \in F$ as $k \to \infty$.

Next we show that p is a solution in F to the variational inequality (3.1). In fact, for any fixed $y \in F$, there exists a constant $Q > 0$ such that $\|x_n - y\| \leq Q$, then

$$\begin{aligned}
\|x_n - y\|^2 &= \langle \alpha_n(f(x_n) - y) + (1 - \alpha_n)(T(t_n)x_n - y), j(x_n - y)\rangle \\
&= \alpha_n\langle f(x_n) - f(p) + p - x_n, j(x_n - y)\rangle + \alpha_n\langle f(p) - p, j(x_n - y)\rangle \\
&\quad + \alpha_n\langle x_n - y, j(x_n - y)\rangle + (1 - \alpha_n)\langle T(t_n)x_n - T(t_n)y, j(x_n - y)\rangle \\
&\leq \alpha_n(\beta + 1)\|x_n - p\|Q + \alpha_n\langle f(p) - p, j(x_n - y)\rangle + \|x_n - y\|^2 \\
&\quad + (1 - \alpha_n)k_{t_n}^{(1)}Q^2 + (1 - \alpha_n)k_{t_n}^{(2)}Q.
\end{aligned} \tag{3.10}$$

Therefore,

$$\langle f(p) - p, j(y - x_n)\rangle \leq (\beta + 1)\|x_n - p\|Q + (1 - \alpha_n)d_n^{(1)}Q^2 + (1 - \alpha_n)d_n^{(2)}Q. \tag{3.11}$$

Taking limit as $n_k \to \infty$ in two sides of (3.11), by Lemma 2.8 and $\{x_{n_k}\} \to p$ as $k \to \infty$, we obtain

$$\langle f(p) - p, j(y - p)\rangle \leq 0 \quad \forall y \in F. \tag{3.12}$$

That is, $p \in F$ is a solution of variational inequality (3.1).

Suppose that $p, q \in F$ satisfy (3.1), we have

$$\langle f(p) - p, j(q - p)\rangle \leq 0, \tag{3.13}$$

$$\langle f(q) - q, j(p - q)\rangle \leq 0. \tag{3.14}$$

Combining (3.13) and (3.14), it follows that

$$(1 - \beta)\|p - q\|^2 \leq \langle (p - q) - f(p) + f(q), j(p - q)\rangle \leq 0. \tag{3.15}$$

Hence $p = q$, that is, $p \in F$ is the unique solution of variational inequality (3.1), so each cluster point of sequence $\{x_n\}$ is equal to p. Therefore, $\{x_n\}$ converges to p and the proof is complete. □

Remark 3.2. Let E, K, F, f, $\{\alpha_n\}$, and $\{t_n\}$ be as in Theorem 3.1, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r nonexpansive semigroup from K into itself, then our result coincides with Theorem 3.2 in [3].

4. Modified implicit iteration scheme

Theorem 4.1. *Let E be a real reflexive and strictly convex Banach space with a uniformly* Gâteaux *differentiable norm, K a nonempty closed convex subset of E, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r generalized asymptotically nonexpansive semigroup from K into itself with sequences $\{k_h^{(1)}\}\{k_h^{(2)}\}$, $h \geq 0$, such that $F \neq \varnothing$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. If $\{x_n\}$ is given by* (1.4)*, where $\lim_{n\to\infty} t_n = \infty$, $\alpha_n, \beta_n \in (0,1]$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\sum_{n=1}^{\infty}(k_{t_n}^{(1)}/\alpha_n) < \infty$, $\sum_{n=1}^{\infty}(k_{t_n}^{(2)}/\alpha_n) < \infty$, then $\{x_n\}$ converges strongly to some common fixed point p of F such that p is the unique solution in F to variational inequality* (3.1)*.*

Proof. For any fixed $y \in F$,

$$\begin{aligned}\|x_n - y\| &= \|\alpha_n(y_n - y) + (1-\alpha_n)(T(t_n)x_n - y)\| \\ &\leq (1-\alpha_n)\|T(t_n)x_n - y\| + \alpha_n\|y_n - y\| \\ &\leq (1-\alpha_n)\left[\left(1 + k_{t_n}^{(1)}\right)\|x_n - y\| + k_{t_n}^{(2)}\right] + \alpha_n\|y_n - y\|.\end{aligned} \tag{4.1}$$

Let $d_n^{(i)} = (k_{t_n}^{(i)}/\alpha_n)$ $(i = 1,2)$. Hence,

$$\begin{aligned}\|x_n - y\| &\leq \frac{(1-\alpha_n)d_n^{(2)}}{1-(1-\alpha_n)d_n^{(1)}} + \frac{\|y_n - y\|}{1-(1-\alpha_n)d_n^{(1)}} \\ &\leq \frac{d_n^{(2)}}{1-d_n^{(1)}} + \frac{\beta_n\|f(x_{n-1}) - y\| + (1-\beta_n)\|x_{n-1} - y\|}{1-d_n^{(1)}} \\ &\leq \frac{d_n^{(2)}}{1-d_n^{(1)}} + \frac{\beta_n\|f(x_{n-1}) - f(y)\|}{1-d_n^{(1)}} + \frac{\beta_n\|f(y) - y\|}{1-d_n^{(1)}} + (1-\beta_n)\frac{\|x_{n-1} - y\|}{1-d_n^{(1)}} \\ &\leq \frac{(1-\beta_n(1-\beta))\left(\|x_{n-1} - y\| + d_n^{(2)}\right)}{1-d_n^{(1)}} + \frac{\beta_n\left(\|f(y) - y\| + d_n^{(2)}\right)}{1-d_n^{(1)}} \\ &\leq \frac{1}{1-d_n^{(1)}}\max\left\{\|x_{n-1} - y\| + d_n^{(2)}, \frac{\|f(y) - y\| + d_n^{(2)}}{1-\beta}\right\}.\end{aligned} \tag{4.2}$$

By induction, we get that

$$\begin{aligned}\|x_n - y\| &\leq \frac{1}{1-d_n^{(1)}}\max\left\{\frac{\|x_{n-2} - y\| + d_{n-1}^{(2)}}{1-d_{n-1}^{(1)}} + d_n^{(2)}, \frac{\|f(y) - y\| + d_{n-1}^{(2)}}{\left(1-d_{n-1}^{(1)}\right)(1-\beta)}, \frac{\|f(y) - y\| + d_n^{(2)}}{1-\beta}\right\} \\ &\cdots \\ &\leq \frac{1}{\left(1-d_n^{(1)}\right)\cdots\left(1-d_1^{(1)}\right)} \\ &\quad\times\max\left\{\|x_1 - y\| + \sum_{i=1}^{n} d_i^{(2)}, \frac{\|f(y) - y\| + d_1^{(2)}}{1-\beta}, \ldots, \frac{\|f(y) - y\| + d_n^{(2)}}{1-\beta}\right\}.\end{aligned} \tag{4.3}$$

Since $\sum_{n=1}^{\infty} d_n^{(i)} < \infty$ $(i = 1, 2)$, we know from Abel–Dini theorem that there exists $r > 0$ such that $\lim_{n\to\infty}(1 - d_n^{(1)}) \cdots (1 - d_1^{(1)}) = r$. Thus $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$, $\{f(x_n)\}$ and $\{y_n\}$. This imply that

$$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = \lim_{n\to\infty} \alpha_n \|y_n - T(t_n)x_n\| = 0. \tag{4.4}$$

Since $\{T(h) : h \geq 0\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$ for all $h \geq 0$,

$$\lim_{n\to\infty} \|T(h)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n\to\infty} \sup_{x\in C} \|T(h)T(t_n)x - T(t_n)x\| = 0, \tag{4.5}$$

where C is any bounded subset of K containing $\{x_n\}$. Since $T(h)$ is continuous, hence

$$\|x_n - T(h)x_n\| \leq \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| + \|T(h)(T(t_n)x_n) - T(h)x_n\| \longrightarrow 0. \tag{4.6}$$

That is, for all $h \geq 0$,

$$\lim_{n\to\infty} \|x_n - T(h)x_n\| = 0. \tag{4.7}$$

From Theorem 3.1, there exists the unique solution $p \in F$ to the variational inequality (3.1). Since $p = T(h)p$ for all $h \geq 0$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_{n+1}\langle y_{n+1} - p, j(x_{n+1} - p)\rangle + (1 - \alpha_{n+1})\langle T(t_{n+1})x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&= \alpha_{n+1}\langle \beta_{n+1} f(x_n) + (1 - \beta_{n+1})x_n - p, j(x_{n+1} - p)\rangle \\
&\quad + (1 - \alpha_{n+1})\langle T(t_{n+1})x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\leq \alpha_{n+1}\beta_{n+1}\langle f(x_n) - f(p), j(x_{n+1} - p)\rangle + \alpha_{n+1}\beta_{n+1}\langle f(p) - p, j(x_{n+1} - p)\rangle \\
&\quad + \alpha_{n+1}(1 - \beta_{n+1})\langle x_n - p, j(x_{n+1} - p)\rangle \\
&\quad + (1 - \alpha_{n+1})\|x_{n+1} - p\|\left[(1 + k_{t_{n+1}}^{(1)})\|x_{n+1} - p\| + k_{t_{n+1}}^{(2)}\right] \\
&\leq \alpha_{n+1}\beta_{n+1}\frac{\beta^2\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} + \alpha_{n+1}(1 - \beta_{n+1})\frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} \\
&\quad + (1 - \alpha_{n+1})\|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})k_{t_{n+1}}^{(1)}\|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})k_{t_{n+1}}^{(2)}\|x_{n+1} - p\| \\
&\quad + \alpha_{n+1}\beta_{n+1}\langle f(p) - p, j(x_{n+1} - p)\rangle \\
&= \frac{\alpha_{n+1}}{2}\|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})\|x_{n+1} - p\|^2 + \frac{\alpha_{n+1}}{2}(1 + \beta^2\beta_{n+1} - \beta_{n+1})\|x_n - p\|^2 \\
&\quad + (1 - \alpha_{n+1})k_{t_{n+1}}^{(1)}\|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})k_{t_{n+1}}^{(2)}\|x_{n+1} - p\| \\
&\quad + \alpha_{n+1}\beta_{n+1}\langle f(p) - p, j(x_{n+1} - p)\rangle.
\end{aligned} \tag{4.8}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq{}& (1 - (1 - \beta^2)\beta_{n+1})\|x_n - p\|^2 + 2\beta_{n+1}\langle f(p) - p, j(x_{n+1} - p)\rangle \\
&+ 2(1 - \alpha_{n+1})\left(d_{n+1}^{(1)}\|x_{n+1} - p\| + d_{n+1}^{(2)}\right)\|x_{n+1} - p\|.
\end{aligned} \tag{4.9}$$

That is,

$$\|x_{n+1}-p\|^2 \le (1-\lambda_n)\|x_n-p\|^2 + b_n + c_n, \tag{4.10}$$

where $\lambda_n = (1-\beta^2)\beta_{n+1}$, $b_n = 2\beta_{n+1}\langle f(p)-p, j(x_{n+1}-p)\rangle$ and $c_n = 2(1-\alpha_{n+1})(d_{n+1}^{(1)}\|x_{n+1}-p\| + d_{n+1}^{(2)})\|x_{n+1}-p\|$. Since $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\sum_{n=1}^{\infty} d_n^{(i)} < \infty$ $(i = 1,2)$, $\|x_{n+1}-p\|$ is bounded, we have $\sum_{n=1}^{\infty} c_n < \infty$. So we only need to show that $\limsup_{n\to\infty}(b_n/\lambda_n) \le 0$, that is,

$$\overline{\lim_{n\to\infty}}\langle f(p)-p, j(x_{n+1}-p)\rangle \le 0. \tag{4.11}$$

Let $z_m = a_m f(z_m) + (1-a_m)T(t_m)z_m$, where t_m and α_m satisfy the condition of Theorem 3.1. Then it follows from Theorem 3.1 that $p = \lim_{m\to\infty} z_m$.

Since

$$\begin{aligned}
\|x_{n+1}-z_m\|^2 &= (1-\alpha_m)\langle T(t_m)z_m - x_{n+1}, j(z_m-x_{n+1})\rangle + \alpha_m\langle f(z_m)-x_{n+1}, j(z_m-x_{n+1})\rangle \\
&= (1-\alpha_m)(\langle T(t_m)z_m - T(t_m)x_{n+1}, j(z_m-x_{n+1})\rangle + \langle T(t_m)x_{n+1}-x_{n+1}, j(z_m-x_{n+1})\rangle) \\
&\quad + \alpha_m\langle f(z_m)-z_m-(f(p)-p), j(z_m-x_{n+1})\rangle + \alpha_m\langle f(p)-p, j(z_m-x_{n+1})\rangle \\
&\quad + \alpha_m\langle z_m-x_{n+1}, j(z_m-x_{n+1})\rangle \\
&\le \|x_{n+1}-z_m\|^2 + (1-\alpha_m)\left(k_{t_m}^{(1)}Q + k_{t_m}^{(2)}\right)Q + (1-\alpha_m)\|T(t_m)x_{n+1}-x_{n+1}\|Q \\
&\quad + \alpha_m\langle f(p)-p, j(z_m-x_{n+1})\rangle + \alpha_m(1+\beta)\|z_m-p\|Q.
\end{aligned} \tag{4.12}$$

Furthermore,

$$\langle f(p)-p, j(x_{n+1}-z_m)\rangle \le \frac{1-\alpha_m}{\alpha_m}\left(k_{t_m}^{(1)}Q + k_{t_m}^{(2)}\right)Q + \frac{\|T(t_m)x_{n+1}-x_{n+1}\|}{\alpha_m}Q + (1+\beta)\|z_m-p\|Q, \tag{4.13}$$

where Q is a constant such that $Q \ge \|z_m - x_{n+1}\|$. Hence, taking upper limit as $n\to\infty$ firstly, and then as $m\to\infty$ in (4.13), we have

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\langle f(p)-p, j(x_{n+1}-z_m)\rangle \le 0. \tag{4.14}$$

On the other hand, since $p = \lim_{m\to\infty} z_m$ and by Lemma 2.8, we have

$$\langle f(p)-p, j(x_{n+1}-z_m)\rangle \longrightarrow \langle f(p)-p, j(x_{n+1}-p)\rangle \text{ uniformly.} \tag{4.15}$$

Thus given $\varepsilon > 0$, there exists $N \ge 1$ such that if $m > N$ for all n we have

$$\langle f(p)-p, j(x_{n+1}-p)\rangle < \langle f(p)-p, j(x_{n+1}-z_m)\rangle + \varepsilon. \tag{4.16}$$

Hence, taking upper limit as $n\to\infty$ firstly, and then as $m\to\infty$ in two sides of (4.16), we get that

$$\overline{\lim_{n\to\infty}}\langle f(p)-p, j(x_{n+1}-p)\rangle \le \overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\langle f(p)-p, j(x_{n+1}-z_m)\rangle + \varepsilon \le \varepsilon. \tag{4.17}$$

For the arbitrariness of ε, (4.11) holds. By Lemma 2.11, $x_n \to p$ and the proof is complete. □

Acknowledgments

This work was supported by National Natural Science Foundation of China (10771173) and Natural Science Foundation Project of Henan (2008B110012).

References

[1] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, no. 1, pp. 82–90, 1967.
[2] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
[3] Y. Song and S. Xu, "Strong convergence theorems for nonexpansive semigroup in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 152–161, 2008.
[4] Y. Song and R. Chen, "Convergence theorems of iterative algorithms for continuous pseudocontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 2, pp. 486–497, 2007.
[5] C. E. Chidume and E. U. Ofoedu, "Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 128–141, 2007.
[6] S. Reich, "On the asymptotic behavior of nonlinear semigroups and the range of accretive operators," *Journal of Mathematical Analysis and Applications*, vol. 79, no. 1, pp. 113–126, 1981.
[7] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama, Yokohama, Japan, 2000.
[8] W. Takahashi and Y. Ueda, "On Reich's strong convergence theorems for resolvents of accretive operators," *Journal of Mathematical Analysis and Applications*, vol. 104, no. 2, pp. 546–553, 1984.
[9] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.