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*Research Article*

# Properties of Matrix Variate Beta Type 3 Distribution

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We study several properties of matrix variate beta type 3 distribution. We also derive probability density functions of the product of two independent random matrices when one of them is beta type 3. These densities are expressed in terms of Appell's first hypergeometric function $F_1$ and Humbert's confluent hypergeometric function $\Phi_1$ of matrix arguments. Further, a bimatrix variate generalization of the beta type 3 distribution is also defined and studied.



## 1. Introduction

The beta families of distributions are defined by the density functions

$$\frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < u < 1, \tag{1.1}$$

$$\frac{v^{\alpha-1}(1+v)^{-(\alpha+\beta)}}{B(\alpha,\beta)}, \quad v > 0, \tag{1.2}$$

respectively, where $\alpha > 0$, $\beta > 0$, and

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.3}$$

The beta type 1 and beta type 2 are very flexible distributions for positive random variables and have wide applications in statistical analysis, for example, see Johnson et al. [1].

Recently, Cardeño et al. [2] have defined and studied family of beta type 3 distributions. A random variable $w$ is said to follow a beta type 3 distribution if its density function is given by

$$\frac{2^{\alpha} w^{\alpha-1}(1-w)^{\beta-1}}{B(\alpha,\beta)(1+w)^{\alpha+\beta}}, \quad 0 < w < 1. \tag{1.4}$$

If a random variable $u$ has the p.d.f (1.1), then we will write $u \sim B1(\alpha,\beta)$, and if the p.d.f. of a random variable $v$ is given by (1.2), then $v \sim B2(\alpha,\beta)$. The density (1.4) will be designated by $w \sim B3(\alpha,\beta)$. The matrix variate generalizations of (1.1) and (1.2) have been studied extensively in the literature, for example, see Gupta and Nagar [3]. The matrix variate beta type 3 distribution has been defined, and some of its properties have been studied by Gupta and Nagar [4].

In this paper, we study several properties of matrix variate beta type 3 distribution. We also derive probability density functions of the product of two independent random matrices when one of them is beta type 3. We also define bimatrix beta type 3 distribution and study some of its properties.

## 2. Some Known Results and Definitions

We begin with a brief review of some definitions and notations. We adhere to standard notations (cf. Gupta and Nagar [3]). Let $A = (a_{ij})$ be an $m \times m$ matrix. Then, $A'$ denotes the transpose of $A$; $\operatorname{tr}(A) = a_{11} + \cdots + a_{mm}$; $\operatorname{etr}(A) = \exp(\operatorname{tr}(A))$; $\det(A)$ = determinant of $A$; $\|A\|$ = norm of $A$; $A > 0$ means that $A$ is symmetric positive definite and $A^{1/2}$ denotes the unique symmetric positive definite square root of $A > 0$. The multivariate gamma function which is frequently used in multivariate statistical analysis is defined by

$$\begin{aligned}\Gamma_m(a) &= \int_{X>0} \operatorname{etr}(-X)\det(X)^{a-(m+1)/2} dX \\ &= \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{i-1}{2}\right), \quad \operatorname{Re}(a) > \frac{m-1}{2}.\end{aligned} \tag{2.1}$$

The multivariate generalization of the beta function is given by

$$\begin{aligned}B_m(a,b) &= \int_0^{I_m} \det(X)^{a-(m+1)/2} \det(I_m - X)^{b-(m+1)/2} dX \\ &= \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} = B_m(b,a),\end{aligned} \tag{2.2}$$

where $\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{Re}(b) > (m-1)/2$.

The generalized hypergeometric coefficient $(a)_\rho$ is defined by

$$(a)_\rho = \prod_{i=1}^{m} \left(a - \frac{i-1}{2}\right)_{r_i}, \tag{2.3}$$

where $\rho = (r_1, \ldots, r_m)$, $r_1 \geq \cdots \geq r_m \geq 0$, $r_1 + \cdots + r_m = r$, and $(a)_k = a(a+1)\cdots(a+k-1)$, $k = 1, 2, \ldots$ with $(a)_0 = 1$. The generalized hypergeometric function of one matrix is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X)}{k!}, \tag{2.4} $$

where $a_i$, $i = 1, \ldots, p$, $b_j$, $j = 1, \ldots, q$ are arbitrary complex numbers, $X$ $(m \times m)$ is a complex symmetric matrx, and $\sum_{\kappa \vdash k}$ denotes summation over all partitions $\kappa$. Conditions for convergence of the series in (2.4) are available in the literature. From (2.4) it follows that

$$ {}_0F_0(X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{C_\kappa(X)}{k!} = \sum_{k=0}^{\infty} \frac{(\operatorname{tr} X)^k}{k!} = \operatorname{etr}(X), \tag{2.5} $$

$$ {}_1F_0(a; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_\kappa C_\kappa(X)}{k!} = \det(I_m - X)^{-a}, \quad \|X\| < 1, \tag{2.6} $$

$$ {}_1F_1(a; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_\kappa}{(c)_\kappa} \frac{C_\kappa(X)}{k!}, \tag{2.7} $$

$$ {}_2F_1(a, b; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(X)}{k!}, \quad \|X\| < 1. \tag{2.8} $$

The integral representations of the confluent hypergeometric function ${}_1F_1$ and the Gauss hypergeometric function ${}_2F_1$ are given by

$$ {}_1F_1(a; c; X) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_0^{I_m} \operatorname{etr}(RX) \det(R)^{a-(m+1)/2} \det(I_m - R)^{c-a-(m+1)/2} dR, \tag{2.9} $$

$$ {}_2F_1(a, b; c; X) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m - R)^{c-a-(m+1)/2} \det(I_m - XR)^{-b} dR, \tag{2.10} $$

where $\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$. For properties and further results on these functions the reader is referred to Constantine [5] and Gupta and Nagar [3].

Davis [6, 7] introduced a class of polynomials $C_\phi^{\kappa,\lambda}(X, Y)$ of $m \times m$ symmetric matrix arguments $X$ and $Y$, which are invariant under the transformation $(X, Y) \to (HXH', HYH')$, $H \in O(m)$. For properties and applications of invariant polynomials we refer to Davis [6, 7], Chikuse [8], and Nagar and Gupta [9]. Let $\kappa$, $\lambda$, $\phi$, and $\rho$ be ordered partitions of the nonnegative integers $k$, $\ell$, $f = k + \ell$ and $r$, respectively, into not more than $m$

parts. Then

$$C_{\phi}^{\kappa,\lambda}(X,X) = \theta_{\phi}^{\kappa,\lambda} C_{\phi}(X), \quad \theta_{\phi}^{\kappa,\lambda} = \frac{C_{\phi}^{\kappa,\lambda}(I_m, I_m)}{C_{\phi}(I_m)}, \tag{2.11}$$

$$C_{\phi}^{\kappa,\lambda}(X, I_m) = \theta_{\phi}^{\kappa,\lambda} \frac{C_{\phi}(I_m) C_{\kappa}(X)}{C_{\kappa}(I_m)}, \tag{2.12}$$

$$C_{\kappa}^{\kappa,0}(X,Y) \equiv C_{\kappa}(X), \qquad C_{\lambda}^{0,\lambda}(X,Y) \equiv C_{\lambda}(Y), \tag{2.13}$$

$$C_{\kappa}(X) C_{\lambda}(Y) = \sum_{\phi \in \kappa \cdot \lambda} \theta_{\phi}^{\kappa,\lambda} C_{\phi}^{\kappa,\lambda}(X,Y), \tag{2.14}$$

where $\phi \in \kappa \cdot \lambda$ signifies that irreducible representation of $Gl(m,R)$ indexed by $2\phi$ occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\lambda$ of the irreducible representations indexed by $2\kappa$ and $2\lambda$. Further

$$\int_0^{I_m} \det(R)^{t-(m+1)/2} \det(I_m - R)^{u-(m+1)/2} C_{\phi}^{\kappa,\lambda}(R, I_m - R) \mathrm{d}R = \frac{\Gamma_m(t,\kappa)\Gamma_m(u,\lambda)}{\Gamma_m(t+u,\phi)} \theta_{\phi}^{\kappa,\lambda} C_{\phi}(I_m), \tag{2.15}$$

$$\int_0^{I_m} \det(R)^{t-(m+1)/2} \det(I_m - R)^{u-(m+1)/2} C_{\phi}^{\kappa,\lambda}(AR, BR) \mathrm{d}R = \frac{\Gamma_m(t,\phi)\Gamma_m(u)}{\Gamma_m(t+u,\phi)} C_{\phi}^{\kappa,\lambda}(A,B). \tag{2.16}$$

In expressions (2.15) and (2.16), $\Gamma_m(a,\rho)$ is defined by

$$\Gamma_m(a,\rho) = (a)_{\rho} \Gamma_m(a). \tag{2.17}$$

Note that $\Gamma_m(a,0) = \Gamma_m(a)$, which is the multivariate gamma function.

The matrix variate generalizations of (1.1), (1.2), and (1.4) are given as follows (Gupta and Nagar [3, 4]).

*Definition 2.1.* An $m \times m$ random symmetric positive definite matrix $U$ is said to have a matrix variate beta type 1 distribution with parameters $(\alpha,\beta)$, denoted as $U \sim B1(m,\alpha,\beta)$, if its p.d.f. is given by

$$\frac{\det(U)^{\alpha-(m+1)/2} \det(I_m - U)^{\beta-(m+1)/2}}{B_m(\alpha,\beta)}, \quad 0 < U < I_m, \tag{2.18}$$

where $\alpha > (m-1)/2$ and $\beta > (m-1)/2$.

If $U \sim B1(m,\alpha,\beta)$, then the cumulative distribution function $F(\Lambda) = P(U < \Lambda)$ is given by

$$F(\Lambda) = \frac{\Gamma_m(\alpha+\beta)\Gamma_m[(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha+(m+1)/2]}\det(\Lambda)^{\alpha} \times {}_2F_1\left(\alpha, -\beta+\frac{m+1}{2}; \alpha+\frac{m+1}{2}; \Lambda\right), \quad 0 < \Lambda < I_m, \tag{2.19}$$

$$E\left[\det(U)^{r_1}\det(I_m-U)^{r_2}\right] = \frac{\Gamma_m(\alpha+r_1)\Gamma_m(\beta+r_2)\Gamma_m(\alpha+\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta)\Gamma_m(\alpha+\beta+r_1+r_2)}. \tag{2.20}$$

*Definition 2.2.* An $m\times m$ random symmetric positive definite matrix $V$ is said to have a matrix variate beta type 2 distribution with parameters $(\alpha,\beta)$, denoted as $V \sim B2(m,\alpha,\beta)$, if its p.d.f. is given by

$$\frac{\det(V)^{\alpha-(m+1)/2}\det(I_m+V)^{-(\alpha+\beta)}}{B_m(\alpha,\beta)}, \quad V > 0, \tag{2.21}$$

where $\alpha > (m-1)/2$ and $\beta > (m-1)/2$.

*Definition 2.3.* An $m\times m$ random symmetric positive definite matrix $W$ is said to have a matrix variate beta type 3 distribution with parameters $(\alpha,\beta)$, denoted as $W \sim B3(m,\alpha,\beta)$, if its p.d.f. is given by

$$\frac{2^{m\alpha}\det(W)^{\alpha-(m+1)/2}\det(I_m-W)^{\beta-(m+1)/2}}{B_m(\alpha,\beta)\det(I_m+W)^{\alpha+\beta}}, \quad 0 < W < I_m, \tag{2.22}$$

where $\alpha > (m-1)/2$ and $\beta > (m-1)/2$.

## 3. Hypergeometric Functions of Two Matrices

In this section we define Appell's first hypergeometric function $F_1$ and Humbert's confluent hypergeometric function $\mathbf{\Phi}_1$ of $m \times m$ symmetric matrices $Z_1$ and $Z_2$ and give their series expansions involving invariant polynomials. Following Prudnikov et al. [10, equations 7.2.4(43), (48)], $F_1$ and $\mathbf{\Phi}_1$ are defined as

$$F_1(a,b_1,b_2;c;Z_1,Z_2) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)}\int_0^{I_m}\frac{\det(V)^{a-(m+1)/2}\det(I_m-V)^{c-a-(m+1)/2}dV}{\det(I_m-VZ_1)^{b_1}\det(I_m-VZ_2)^{b_2}}, \tag{3.1}$$

$$\mathbf{\Phi}_1[a,b_1;c;Z_1,Z_2] = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)}\int_0^{I_m}\frac{\det(V)^{a-(m+1)/2}\det(I_m-V)^{c-a-(m+1)/2}dV}{\det(I_m-VZ_1)^{b_1}\operatorname{etr}(-VZ_2)}, \tag{3.2}$$

respectively, where $\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$. Note that for $b_1 = 0$, $F_1$ and $\mathbf{\Phi}_1$ reduce to ${}_2F_1$ and ${}_1F_1$ functions, respectively. Expanding $\det(I_m-VZ_1)^{-b_1}$, $\|VZ_1\| < 1$,

$\det(I_m - VZ_2)^{-b_2}$, $\|VZ_2\| < 1$ and $\operatorname{etr}(VZ_2)$ using (2.6) and (2.5), and applying (2.14), one can write

$$\begin{aligned}
&\det(I_m - VZ_1)^{-b_1}\det(I_m - VZ_2)^{-b_2} \\
&\quad = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{\kappa\vdash k}\sum_{\lambda\vdash \ell}\sum_{\phi\in\kappa\cdot\lambda}\frac{(b_1)_\kappa (b_2)_\lambda}{k!\ \ell!}C_\phi^{\kappa,\lambda}(VZ_1, VZ_2), \quad \|Z_1\| < 1,\ \|Z_2\| < 1,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\det(I_m - VZ_1)^{-b_1}\operatorname{etr}(VZ_2) \\
&\quad = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{\kappa\vdash k}\sum_{\lambda\vdash \ell}\sum_{\phi\in\kappa\cdot\lambda}\frac{(b_1)_\kappa}{k!\ \ell!}C_\phi^{\kappa,\lambda}(VZ_1, VZ_2), \quad \|Z_1\| < 1.
\end{aligned} \tag{3.4}$$

Now, substituting (3.3) and (3.4) in (3.1) and (3.2), respectively, and integrating $V$ using (2.16), the series expansions for $F_1$ and $\mathbf{\Phi}_1$ are derived as

$$\begin{aligned}
F_1(a, b_1, b_2; c; Z_1, Z_2) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{\kappa\vdash k}\sum_{\lambda\vdash \ell}\sum_{\phi\in\kappa\cdot\lambda}\frac{(b_1)_\kappa (b_2)_\lambda}{k!\ell!}\frac{(a)_\phi}{(c)_\phi}C_\phi^{\kappa,\lambda}(Z_1, Z_2), \\
\mathbf{\Phi}_1[a, b_1; c; Z_1, Z_2] &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{\kappa\vdash k}\sum_{\lambda\vdash \ell}\sum_{\phi\in\kappa\cdot\lambda}\frac{(b_1)_\kappa}{k!\ell!}\frac{(a)_\phi}{(c)_\phi}C_\phi^{\kappa,\lambda}(Z_1, Z_2).
\end{aligned} \tag{3.5}$$

## 4. Properties

In this section we derive several properties of the matrix variate beta type 3 distribution. For the sake of completeness we first state the following results established in Gupta and Nagar [4].

(1) Let $W \sim B3(m, \alpha, \beta)$ and $A$ $(m \times m)$ be a constant nonsingular matrix. Then, the density of $X = AWA'$ is

$$\frac{2^{m\alpha}\det(X)^{\alpha-(m+1)/2}\det(AA' - X)^{\beta-(m+1)/2}}{\det(AA')^{-(m+1)/2}B_m(\alpha, \beta)\det(AA' + X)^{\alpha+\beta}}, \quad 0 < X < AA'. \tag{4.1}$$

(2) Let $W \sim B3(m, \alpha, \beta)$ and $H$ $(m \times m)$ be an orthogonal matrix, whose elements are either constants or random variables distributed independent of $W$. Then, the distribution of $W$ is invariant under the transformation $W \to HWH'$, and is independent of $H$ in the latter case.

(3) Let $W \sim B3(m, \alpha, \beta)$. Then, the density of $Y = W^{-1}$ is

$$\frac{2^{m\alpha}\det(Y - I_m)^{\beta-(m+1)/2}}{B_m(\alpha, \beta)\det(I_m + Y)^{\alpha+\beta}}, \quad Y > I_m. \tag{4.2}$$

(4) If $U \sim B1(m, \alpha, \beta)$, then $(I_m + U)^{-1}(I_m - U) \sim B3(m, \beta, \alpha)$ and $(2I_m - U)^{-1}U \sim B3(m, \alpha, \beta)$.

(5) If $V \sim B2(m,\alpha,\beta)$, then $(2I_m + V)^{-1}V \sim B3(m,\alpha,\beta)$ and $(I_m + 2V)^{-1} \sim B3(m,\beta,\alpha)$.

(6) If $W \sim B3(m,\alpha,\beta)$, then $2(I_m + W)^{-1}W \sim B1(m,\alpha,\beta)$, $(I_m + W)^{-1}(I_m - W) \sim B1(m,\beta,\alpha)$, $2(I_m - W)^{-1}W \sim B2(m,\alpha,\beta)$, and $(1/2)(I_m - W)W^{-1} \sim B2(m,\beta,\alpha)$.

(7) Let $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $W_{11}(q \times q)$. Define $W_{11\cdot 2} = W_{11} - W_{12}W_{22}^{-1}W_{21}$ and $W_{22\cdot 1} = W_{22} - W_{21}W_{11}^{-1}W_{12}$. If $W \sim B3(m,\alpha,\beta)$, then $W_{22\cdot 1} \sim B3(m-q,\alpha-q/2,\beta)$ and $W_{11\cdot 2} \sim B3(q,\alpha-(m-q)/2,\beta)$.

(8) Let $A$ $(q \times m)$ be a constant matrix of rank $q$ $(\leq m)$. If $W \sim B3(m,\alpha,\beta)$, then $[(AA')^{-1/2}AW^{-1}A'(AA')^{-1/2}]^{-1} \sim B3(q,\alpha-(m-q)/2,\beta)$.

(9) Let $W \sim B3(m,\alpha,\beta)$ and $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{a} \neq 0$, then $\mathbf{a}'\mathbf{a}(\mathbf{a}'W^{-1}\mathbf{a})^{-1} \sim B3(\alpha-(m-1)/2,\beta)$. Further, if $\mathbf{y}$ $(m \times 1)$ is a random vector, independent of $W$, and $P(\mathbf{y} \neq 0) = 1$, then it follows that $\mathbf{y}'\mathbf{y}(\mathbf{y}'W^{-1}\mathbf{y})^{-1} \sim B3(\alpha-(m-1)/2,\beta)$.

From the above results it is straightforward to show that, if $\mathbf{c}$ $(m \times 1)$ is a nonzero constant vector or a random vector independent of $W$ with $P(\mathbf{c} \neq 0) = 1$, then

$$\begin{aligned}
&\frac{\mathbf{c}'(W^{-1} - I_m)\mathbf{c}}{\mathbf{c}'(W^{-1} + I_m)\mathbf{c}} \sim B1\left(\beta, \alpha - \frac{m-1}{2}\right), \\
&\frac{2\mathbf{c}'\mathbf{c}}{\mathbf{c}'(W^{-1} + I_m)\mathbf{c}} \sim B1\left(\alpha - \frac{m-1}{2}, \beta\right), \\
&\frac{2\mathbf{c}'\mathbf{c}}{\mathbf{c}'(W^{-1} - I_m)\mathbf{c}} \sim B2\left(\alpha - \frac{m-1}{2}, \beta\right), \\
&\frac{\mathbf{c}'(W^{-1} - I_m)\mathbf{c}}{2\mathbf{c}'\mathbf{c}} \sim B2\left(\beta, \alpha - \frac{m-1}{2}\right).
\end{aligned} \tag{4.3}$$

The expectation of $W^{-1}$, $E(W^{-1})$, can easily be obtained from the above results. For any fixed $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c} \neq 0$,

$$E\left[\frac{\mathbf{c}'(W^{-1} - I_m)\mathbf{c}}{2\mathbf{c}'\mathbf{c}}\right] = E(v), \tag{4.4}$$

where $v \sim B2(\beta, \alpha-(m-1)/2)$. Hence, for all $\mathbf{c} \in \mathbb{R}^m$,

$$\mathbf{c}'E\left(W^{-1} - I_m\right)\mathbf{c} = 2\mathbf{c}'\mathbf{c}E(v) = \frac{2\beta}{\alpha-(m+1)/2}\mathbf{c}'\mathbf{c}, \quad \alpha > \frac{m+1}{2}, \tag{4.5}$$

which implies that

$$E\left(W^{-1}\right) = \frac{2\beta + \alpha - (m+1)/2}{\alpha - (m+1)/2} I_m, \quad \alpha > \frac{m+1}{2}. \tag{4.6}$$

The matrix variate beta type 3 distribution can be derived by using independent gamma matrices. An $m \times m$ random symmetric positive definite matrix $Y$ is said to have a matrix variate gamma distribution with parameters $\Psi$ ($> 0$), and $\kappa$ ($> (m-1)/2$), denoted by $Y \sim Ga(m, \kappa, \Psi)$, if its p.d.f. is given by

$$\frac{\operatorname{etr}(-\Psi^{-1}Y)\det(Y)^{\kappa-(m+1)/2}}{\Gamma_m(\kappa)\det(\Psi)^{\kappa}}, \quad Y > 0. \tag{4.7}$$

It is well known that if $Y_1$ and $Y_2$ are independent, $Y_i \sim Ga(m, \kappa_i, I_m)$, $i = 1, 2$, then (i) $(Y_1 + Y_2)^{-1/2} Y_1 (Y_1 + Y_2)^{-1/2}$ and $Y_1 + Y_2$ are independent and (ii) $Y_2^{-1/2} Y_1 Y_2^{-1/2}$ and $Y_1 + Y_2$ are independent. Further, $(Y_1 + Y_2)^{-1/2} Y_1 (Y_1 + Y_2)^{-1/2} \sim B1(m, \kappa_1, \kappa_2)$, $Y_2^{-1/2} Y_1 Y_2^{-1/2} \sim B2(m, \kappa_1, \kappa_2)$ and $Y_1 + Y_2 \sim Ga(m, \kappa_1 + \kappa_2, I_m)$. In the following theorem we derive similar result for matrix variate beta type 3 distribution.

**Theorem 4.1.** *Let the $m \times m$ random matrices $Y_1$ and $Y_2$ be independent, $Y_i \sim Ga\,(m, \kappa_i, I_m)$, $i = 1, 2$. Then, $(Y_1 + 2Y_2)^{-1/2} Y_1 (Y_1 + 2Y_2)^{-1/2} \sim B3(m, \kappa_1, \kappa_2)$.*

*Proof.* The joint density function of $Y_1$ and $Y_2$ is given by

$$\frac{\operatorname{etr}[-(Y_1 + Y_2)]\det(Y_1)^{\kappa_1-(m+1)/2}\det(Y_2)^{\kappa_2-(m+1)/2}}{\Gamma_m(\kappa_1)\Gamma_m(\kappa_2)}, \quad Y_1 > 0,\ Y_2 > 0. \tag{4.8}$$

Making the transformation $W = Y^{-1/2} Y_1 Y^{-1/2}$ and $Y = Y_1 + 2Y_2$ with the Jacobian $J(Y_1, Y_2 \to W, Y) = 2^{-m(m+1)/2}\det(Y)^{(m+1)/2}$ in the joint density of $Y_1$ and $Y_2$, we obtain the joint density of $W$ and $Y$ as

$$\begin{aligned} &\frac{\det(W)^{\kappa_1-(m+1)/2}\det(I_m - W)^{\kappa_2-(m+1)/2}}{2^{m\kappa_2}\Gamma_m(\kappa_1)\Gamma_m(\kappa_2)} \\ &\quad \times \operatorname{etr}\left[-\frac{1}{2}(I_m + W)Y\right]\det(Y)^{\kappa_1+\kappa_2-(m+1)/2}, \quad 0 < W < I_m,\ Y > 0. \end{aligned} \tag{4.9}$$

Now, the desired result is obtained by integrating $Y$ using (2.1). □

Next, we derive the cumulative distribution function (cdf) and several expected values of functions of beta type 3 matrix.

If $W \sim B3(m, \alpha, \beta)$, then the cdf of $W$, denoted by $G(\Omega)$, is given by

$$\begin{aligned} G(\Omega) &= P(W < \Omega) \\ &= P\Big(U < (I_m + \Omega)^{-1}(I_m - \Omega)\Big), \end{aligned} \tag{4.10}$$

where $U \sim B1(m,\beta,\alpha)$. Now, using (2.19), the cdf $G(\Omega)$ is obtained as

$$G(\Omega) = \frac{\Gamma_m(\alpha+\beta)\Gamma_m[(m+1)/2]}{\Gamma_m(\alpha)\Gamma_m[\beta+(m+1)/2]}\det\left((I_m+\Omega)^{-1}(I_m-\Omega)\right)^{\beta} \times {}_2F_1\left(\beta,-\alpha+\frac{m+1}{2};\beta+\frac{m+1}{2};(I_m+\Omega)^{-1}(I_m-\Omega)\right), \tag{4.11}$$

where $0 < \Omega < I_m$.

**Theorem 4.2.** *Let $W \sim B3(m,\alpha,\beta)$, then*

$$E\left[\frac{\det(W)^r\det(I_m-W)^s}{\det(I_m+W)^t}\right] = 2^{-m(\beta+t)}\frac{\Gamma_m(\alpha+r)\Gamma_m(\beta+s)\Gamma_m(\alpha+\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta)\Gamma_m(\alpha+\beta+r+s)} \times {}_2F_1\left(\beta+s,\alpha+\beta+t;\alpha+\beta+r+s;\frac{I_m}{2}\right), \tag{4.12}$$

*where* $\mathrm{Re}(\alpha+r) > (m-1)/2$ *and* $\mathrm{Re}(\beta+s) > (m-1)/2$.

*Proof.* By definition

$$E\left[\frac{\det(W)^r\det(I_m-W)^s}{\det(I_m+W)^t}\right] = \frac{2^{m\alpha}}{B_m(\alpha,\beta)}\int_0^{I_m}\frac{\det(W)^{\alpha+r-(m+1)/2}\det(I_m-W)^{\beta+s-(m+1)/2}dW}{\det(I_m+W)^{\alpha+\beta+t}}. \tag{4.13}$$

Writing

$$\det(I_m+W)^{-(\alpha+\beta+t)} = 2^{-m(\alpha+\beta+t)}\det\left(I_m-\frac{1}{2}(I_m-W)\right)^{-(\alpha+\beta+t)} \tag{4.14}$$

and substituting $Z = I_m - W$, we have

$$\begin{aligned} E\left[\frac{\det(W)^r\det(I_m-W)^s}{\det(I_m+W)^t}\right] &= \frac{1}{2^{m(\beta+t)}B_m(\alpha,\beta)}\int_0^{I_m}\frac{\det(Z)^{\beta+s-(m+1)/2}\det(I_m-Z)^{\alpha+r-(m+1)/2}dZ}{\det(I_m-Z/2)^{\alpha+\beta+t}} \\ &= \frac{B_m(\alpha+r,\beta+s)}{2^{m(\beta+t)}B_m(\alpha,\beta)}\,{}_2F_1\left(\beta+s,\alpha+\beta+t;\alpha+\beta+r+s;\frac{I_m}{2}\right), \end{aligned} \tag{4.15}$$

where the integral has been evaluated using integral representation of the Gauss hypergeometric function given in (2.10). □

**Corollary 4.3.** *Let $W \sim B3(m,\alpha,\beta)$, then for* $\mathrm{Re}(h) > -\alpha + (m-1)/2$, *one has*

$$E\left[\frac{\det(W)^h}{\det(I_m+W)^h}\right] = \frac{\Gamma_m(\alpha+\beta)\Gamma_m(\alpha+h)}{2^{mh}\Gamma_m(\alpha)\Gamma_m(\alpha+\beta+h)},$$
$$E\left[\det(W)^h\right] = \frac{\Gamma_m(\alpha+\beta)\Gamma_m(\alpha+h)}{2^{m\beta}\Gamma_m(\alpha)\Gamma_m(\alpha+\beta+h)}{}_2F_1\left(\beta,\alpha+\beta;\alpha+\beta+h;\frac{I_m}{2}\right). \tag{4.16}$$

*Further, for* $\mathrm{Re}(h) > -\beta + (m-1)/2$,

$$E\left[\det(I_m+W)^h\right] = \frac{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+h)}{2^{m\beta}\Gamma_m(\beta)\Gamma_m(\alpha+\beta+h)} \times {}_2F_1\left(\beta+h,\alpha+\beta;\alpha+\beta+h,\frac{I_m}{2}\right). \tag{4.17}$$

From the density of $W$, we have

$$E[C_\kappa(W)] = \frac{2^{m\alpha}}{B_m(\alpha,\beta)} \times \int_0^{I_m} \frac{C_\kappa(W)\det(W)^{\alpha-(m+1)/2}\det(I_m-W)^{\beta-(m+1)/2}dW}{(I_m+W)^{\alpha+\beta}}. \tag{4.18}$$

Now, expanding $(I_m+W)^{-(\alpha+\beta)}$ in series involving zonal polynomials using (2.6), the above expression is rewritten as

$$E[C_\kappa(W)] = \frac{1}{2^{m\beta}B_m(\alpha,\beta)}\sum_{\ell=0}^{\infty}\sum_{\lambda\vdash\ell}\frac{(\alpha+\beta)_\lambda}{2^\ell \ell!} \times \int_0^{I_m} C_\kappa(W)\det(W)^{\alpha-(m+1)/2}\det(I_m-W)^{\beta-(m+1)/2}C_\lambda(I_m-W)dW. \tag{4.19}$$

Further, writing

$$C_\kappa(W)C_\lambda(I_m-W) = \sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(W,I_m-W) \tag{4.20}$$

and integrating $W$ using (2.15), we get

$$\begin{aligned} E[C_\kappa(W)] &= \frac{1}{2^{m\beta}B_m(\alpha,\beta)}\sum_{\ell=0}^{\infty}\sum_{\lambda\vdash\ell}\frac{(\alpha+\beta)_\lambda}{2^\ell \ell!}\sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda} \\ &\quad\times\int_0^{I_m}\det(W)^{\alpha-(m+1)/2}\det(I_m-W)^{\beta-(m+1)/2}C_\phi^{\kappa,\lambda}(W,I_m-W)dW \\ &= \frac{1}{2^{m\beta}}\sum_{\ell=0}^{\infty}\sum_{\lambda\vdash\ell}\frac{(\alpha+\beta)_\lambda}{2^\ell \ell!}\sum_{\phi\in\kappa\cdot\lambda}\left(\theta_\phi^{\kappa,\lambda}\right)^2\frac{(\alpha)_\kappa(\beta)_\lambda}{(\alpha+\beta)_\phi}C_\phi(I_m). \end{aligned} \tag{4.21}$$

## 5. Distributions of Random Quadratic Forms

In this section we obtain distributional results for the product of two independent random matrices involving beta type 3 distribution.

**Theorem 5.1.** *Let $X_1 \sim B1(m,\alpha_1,\beta_1)$ and $X_2 \sim B3(m,\alpha_2,\beta_2)$ be independent. Then, the p.d.f. of $Z = X_2^{1/2}X_1X_2^{1/2}$ is*

$$\begin{aligned} &\frac{2^{-m\beta_2}\Gamma_m(\alpha_1+\beta_1)\Gamma_m(\alpha_2+\beta_2)}{\Gamma_m(\alpha_1)\Gamma_m(\alpha_2)\Gamma_m(\beta_1+\beta_2)}\det(Z)^{\alpha_1-(m+1)/2}\det(I_m-Z)^{\beta_1+\beta_2-(m+1)/2} \\ &\quad\times F_1\left(\beta_2,\alpha_1+\beta_1-\alpha_2,\alpha_2+\beta_2,\beta_1+\beta_2;I_m-Z,\frac{I_m-Z}{2}\right),\quad 0<Z<I_m. \end{aligned} \tag{5.1}$$

*Proof.* Using the independence, the joint p.d.f. of $X_1$ and $X_2$ is given by

$$\begin{aligned} &K_1\det(X_1)^{\alpha_1-(m+1)/2}\det(I_m-X_1)^{\beta_1-(m+1)/2} \\ &\quad\times\frac{\det(X_2)^{\alpha_2-(m+1)/2}\det(I_m-X_2)^{\beta_2-(m+1)/2}}{\det(I_m+X_2)^{\alpha_2+\beta_2}}, \end{aligned} \tag{5.2}$$

where $0<X_i<I_m$, $i=1,2$, and

$$K_1 = 2^{\alpha_2 m}\left\{B_m(\alpha_1,\beta_1)B_m(\alpha_2,\beta_2)\right\}^{-1}. \tag{5.3}$$

Transforming $Z = X_2^{1/2}X_1X_2^{1/2}$, $X_2 = X_2$ with the Jacobian $J(X_1,X_2 \to Z,X_2) = \det(X_2)^{-(m+1)/2}$ we obtain the joint p.d.f. of $Z$ and $X_2$ as

$$K_1\det(Z)^{\alpha_1-(m+1)/2}\frac{\det(X_2-Z)^{\beta_1-(m+1)/2}\det(I_m-X_2)^{\beta_2-(m+1)/2}}{\det(X_2)^{\alpha_1+\beta_1-\alpha_2}\det(I_m+X_2)^{\alpha_2+\beta_2}}, \tag{5.4}$$

where $0 < Z < X_2 < I_m$. To find the marginal p.d.f. of $Z$, we integrate (5.4) with respect to $X_2$ to get

$$K_1 \det(Z)^{\alpha_1-(m+1)/2} \times \int_Z^{I_m} \frac{\det(X_2 - Z)^{\beta_1-(m+1)/2}\det(I_m - X_2)^{\beta_2-(m+1)/2} dX_2}{\det(X_2)^{\alpha_1+\beta_1-\alpha_2}\det(I_m + X_2)^{\alpha_2+\beta_2}}. \tag{5.5}$$

In (5.5) change of variable $V = (I_m - Z)^{-1/2}(I_m - X_2)(I_m - Z)^{-1/2}$ with the Jacobian $J(X_2 \to V) = \det(I_m - Z)^{(m+1)/2}$ yields

$$\begin{aligned} &K_1 2^{-m(\alpha_2+\beta_2)}\det(Z)^{\alpha_1-(m+1)/2}\det(I_m - Z)^{\beta_1+\beta_2-(m+1)/2} \\ &\quad\times \int_0^{I_m} \frac{\det(V)^{\beta_2-(m+1)/2}\det(I_m - V)^{\beta_1-(m+1)/2} dV}{\det(I_m - (I_m - Z)V)^{\alpha_1+\beta_1-\alpha_2}\det(I_m - (I_m - Z)V/2)^{\alpha_2+\beta_2}} \\ &= K_1 2^{-m(\alpha_2+\beta_2)}\det(Z)^{\alpha_1-(m+1)/2}\det(I_m - Z)^{\beta_1+\beta_2-(m+1)/2} \\ &\quad\times \frac{\Gamma_m(\beta_1)\Gamma_m(\beta_2)}{\Gamma_m(\beta_1+\beta_2)} F_1\left(\beta_2, \alpha_1+\beta_1-\alpha_2, \alpha_2+\beta_2, \beta_1+\beta_2; I_m - Z, \frac{I_m - Z}{2}\right), \end{aligned} \tag{5.6}$$

where the last step has been obtained by using the definition of $F_1$. Finally, substituting for $K_1$ we obtain the desired result. □

**Corollary 5.2.** *Let $X_1$ and $X_2$ be independent random matrices, $X_1 \sim B1(m, \alpha_1, \beta_1)$ and $X_2 \sim B3(m, \alpha_2, \beta_2)$. If $\alpha_2 = \alpha_1 + \beta_1$, then the p.d.f. of $Z = X_2^{1/2}X_1X_2^{1/2}$ is given by*

$$\begin{aligned} &\frac{2^{-m\beta_2}\Gamma_m(\alpha_1+\beta_1+\beta_2)}{\Gamma_m(\alpha_1)\Gamma_m(\beta_1+\beta_2)}\det(Z)^{\alpha_1-(m+1)/2}\det(I_m - Z)^{\beta_1+\beta_2-(m+1)/2} \\ &\quad\times {}_2F_1\left(\beta_2, \alpha_1+\beta_1+\beta_2; \beta_1+\beta_2; \frac{I_m - Z}{2}\right), \quad 0 < Z < I_m. \end{aligned} \tag{5.7}$$

**Theorem 5.3.** *Let $X_1$ and $X_2$ be independent random matrices, $X_1 \sim B3(m, \alpha_1, \beta_1)$ and $X_2 \sim B2(m, \alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1^{1/2}X_2X_1^{1/2}$ is given by*

$$\begin{aligned} &\frac{2^{-m\beta_1}B_m(\beta_1, \alpha_1+\beta_2)}{B_m(\alpha_1, \beta_1)B_m(\alpha_2, \beta_2)}\frac{\det(Z)^{\alpha_2-(m+1)/2}}{\det(I_m + Z)^{\alpha_2+\beta_2}} \\ &\quad\times F_1\left(\beta_1, \alpha_1+\beta_1, \alpha_2+\beta_2; \alpha_1+\beta_1+\beta_2; \frac{I_m}{2}, (I_m + Z)^{-1}\right), \quad Z > 0. \end{aligned} \tag{5.8}$$

*Proof.* Since $X_1$ and $X_2$ are independent, their joint p.d.f. is given by

$$K_2 \frac{\det(X_1)^{\alpha_1-(m+1)/2}\det(I_m - X_1)^{\beta_1-(m+1)/2}\det(X_2)^{\alpha_2-(m+1)/2}}{\det(I_m + X_1)^{\alpha_1+\beta_1}\det(I_m + X_2)^{\alpha_2+\beta_2}}, \tag{5.9}$$

where $0 < X_1 < I_m$, $X_2 > 0$, and

$$K_2 = 2^{m\alpha_1}\left\{B_m(\alpha_1,\beta_1)B_m(\alpha_2,\beta_2)\right\}^{-1}. \tag{5.10}$$

Now consider the transformation $Z = X_1^{1/2}X_2X_1^{1/2}$ and $V = I_m - X_1$ whose Jacobian is $J(X_1, X_2 \to V, Z) = \det(I_m - V)^{-(m+1)/2}$. Thus, we obtain the joint p.d.f. of $V$ and $Z$ as

$$\frac{K_2\det(Z)^{\alpha_2-(m+1)/2}}{2^{m(\alpha_1+\beta_1)}\det(I_m+Z)^{\alpha_2+\beta_2}}\,\frac{\det(V)^{\beta_1-(m+1)/2}\det(I_m-V)^{\alpha_1+\beta_2-(m+1)/2}}{\det(I_m-V/2)^{\alpha_1+\beta_1}\det\left(I_m-(I_m+Z)^{-1}V\right)^{\alpha_2+\beta_2}}, \tag{5.11}$$

where $Z > 0$ and $0 < V < I_m$. Finally, integrating $V$ using (3.1) and substituting for $K_2$, we obtain the desired result. □

In the next theorem we derive the density of $Z_1 = X^{-1/2}YX^{-1/2}$, where the random matrices $X$ and $Y$ are independent, $X \sim B3(m,\alpha,\beta)$, and the distribution of $Y$ is matrix variate gamma.

**Theorem 5.4.** *Let the $m \times m$ random matrices $X$ and $Y$ be independent, $X \sim B3(m,\alpha,\beta)$ and $Y \sim Ga\ (m,\kappa,I_m)$. Then, the p.d.f. of $Z_1 = X^{-1/2}YX^{-1/2}$ is given by*

$$\frac{\Gamma_m(\alpha+\kappa)\Gamma_m(\alpha+\beta)\det(Z_1)^{\kappa-(m+1)/2}\operatorname{etr}(-Z_1)}{2^{m\beta}\Gamma_m(\kappa)\Gamma_m(\alpha)\Gamma_m(\alpha+\beta+\kappa)}\mathbf{\Phi}_1\left(\beta,\alpha+\beta;\alpha+\beta+\kappa;\frac{I_m}{2},Z_1\right), \tag{5.12}$$

*where $Z_1 > 0$.*

*Proof.* The joint p.d.f. of $X$ and $Y$ is given by

$$\frac{\det(X)^{\alpha-(m+1)/2}\det(I_m-X)^{\beta-(m+1)/2}\det(Y)^{\kappa-(m+1)/2}}{2^{-m\alpha}\Gamma(\kappa)B(\alpha,\beta)\det(I_m+X)^{\alpha+\beta}\operatorname{etr}(Y)}, \tag{5.13}$$

where $0 < X < I_m$ and $Y > 0$. Now, transforming $Z_1 = X^{-1/2}YX^{-1/2}$ and $W = I_m - X$, with the Jacobian $J(X, Y \to W, Z_1) = \det(I_m - W)^{(m+1)/2}$, we obtain the joint p.d.f. of $Z_1$ and $W$ as

$$\frac{\operatorname{etr}(-Z_1)\det(Z_1)^{\kappa-(m+1)/2}}{2^{m\beta}\Gamma(\kappa)B(\alpha,\beta)}\,\frac{\det(W)^{\beta-(m+1)/2}\det(I_m-W)^{\alpha+\kappa-(m+1)/2}}{\det(I_m-W/2)^{\alpha+\beta}\operatorname{etr}(-WZ_1)}, \tag{5.14}$$

where $0 < W < I_m$ and $Z_1 > 0$. Now, integrating $W$ using (3.2), we get the marginal density of $Z_1$. □

## 6. Bimatrix Beta Type 3 Distribution

The bimatrix generalization of the beta type 1 density is defined by

$$\frac{\det(U_1)^{\alpha_1-(m+1)/2}\det(U_2)^{\alpha_2-(m+1)/2}\det(I_m-U_1-U_2)^{\beta-(m+1)/2}}{B_m(\alpha_1,\alpha_2,\beta)}, \quad U_1>0,\ U_2>0,\ U_1+U_2<I_m, \tag{6.1}$$

where $\alpha_1>(m-1)/2$, $\alpha_2>(m-1)/2$, $\beta>(m-1)/2$, and

$$B_m(\alpha_1,\alpha_2,\beta)=\frac{\Gamma_m(\alpha_1)\Gamma_m(\alpha_2)\Gamma_m(\beta)}{\Gamma_m(\alpha_1+\alpha_2+\beta)}. \tag{6.2}$$

This distribution, denoted by $(U_1,U_2)\sim D1(m,\alpha_1,\alpha_2;\beta)$, is a special case of the matrix variate Dirichlet type 1 distribution. The $m\times m$ random symmetric positive definite matrices $V_1$ and $V_2$ are said to have a bimatrix variate generalization of the beta type 2 distribution, denoted as $(V_1,V_2)\sim D2(m,\alpha_1,\alpha_2;\beta)$, if their joint p.d.f. is given by

$$\frac{\det(V_1)^{\alpha_1-(m+1)/2}\det(V_2)^{\alpha_2-(m+1)/2}}{B_m(\alpha_1,\alpha_2,\beta)\det(I_m+V_1+V_2)^{\alpha_1+\alpha_2+\beta}}, \quad V_1>0,\ V_2>0, \tag{6.3}$$

where $\alpha_1>(m-1)/2$, $\alpha_2>(m-1)/2$, and $\beta>(m-1)/2$.

A natural bimatrix generalization of the beta type 3 distribution can be given as follows.

*Definition 6.1.* The $m\times m$ symmetric positive definite random matrices $W_1$ and $W_2$ are said to have a bimatrix beta type 3 distribution, denoted as $(W_1,W_2)\sim D3(m,\alpha_1,\alpha_2;\beta)$, if their joint p.d.f. is given by

$$\frac{\det(W_1)^{\alpha_1-(m+1)/2}\det(W_2)^{\alpha_2-(m+1)/2}\det(I_m-W_1-W_2)^{\beta-(m+1)/2}}{2^{-m(\alpha_1+\alpha_2)}B_m(\alpha_1,\alpha_2,\beta)\det(I_m+W_1+W_2)^{\alpha_1+\alpha_2+\beta}}, \quad W_1>0,\ W_2>0,\ W_1+W_2<I_m, \tag{6.4}$$

where $\alpha_1>(m-1)/2$, $\alpha_2>(m-1)/2$, and $\beta>(m-1)/2$.

The bimatrix beta type 3 distribution belongs to the Liouville family of distributions and can be obtained using independent gamma matrices as shown in the following theorem.

**Theorem 6.2.** *Let $Y_1$, $Y_2$, and $Y_3$ be independent, $Y_i\sim Ga\,(m,\kappa_i,I_m)$, $i=1,2,3$. Define $W_i=(Y_1+Y_2+2Y_3)^{-1/2}Y_i(Y_1+Y_2+2Y_3)^{-1/2}$, $i=1,2$. Then, $(W_1,W_2)\sim D3(m,\kappa_1,\kappa_2;\kappa_3)$.*

*Proof.* Similar to the proof of Theorem 4.1. □

The next two theorems derive the bimatrix beta type 3 distribution from the bimatrix beta type 1 and type 2 distributions.

**Theorem 6.3.** *Let* $(U_1, U_2) \sim D1(m, \alpha_1, \alpha_2; \beta)$ *and define*

$$W_i = (2I_m - U_1 - U_2)^{-1/2} U_i (2I_m - U_1 - U_2)^{-1/2}, \quad i = 1, 2. \tag{6.5}$$

*Then,* $(W_1, W_2) \sim D3(m, \alpha_1, \alpha_2; \beta)$.

*Proof.* Let $Z = 2I_m - U_1 - U_2$ and $W_1 = Z^{-1/2} U_1 Z^{-1/2}$. Then, $W_2 = 2Z^{-1} - (I_m + W_1)$. The Jacobian of the transformation (6.5) is given by

$$\begin{aligned} J(U_1, U_2 \longrightarrow W_1, W_2) &= J(U_1, U_2 \longrightarrow W_1, Z) J(W_1, Z \longrightarrow W_1, W_2) \\ &= \det(Z)^{(m+1)/2} 2^{-m(m+1)/2} \det(Z)^{m+1} \\ &= 2^{m(m+1)} \det(I_m + W_1 + W_2)^{-3(m+1)/2}. \end{aligned} \tag{6.6}$$

Now, substituting $U_i = 2(I_m + W_1 + W_2)^{-1/2} W_i (I_m + W_1 + W_2)^{-1/2}$, $i = 1, 2$ and the Jacobian in the joint density of $U_1$ and $U_2$ given in (6.1), we get the desired result. □

**Theorem 6.4.** *Let* $(V_1, V_2) \sim D2(m, \alpha_1, \alpha_2; \beta)$ *and define*

$$W_i = (2I_m + V_1 + V_2)^{-1/2} V_i (2I_m + V_1 + V_2)^{-1/2}, \quad i = 1, 2. \tag{6.7}$$

*Then,* $(W_1, W_2) \sim D3(m, \alpha_1, \alpha_2; \beta)$.

*Proof.* Let $Z = 2I_m + V_1 + V_2$ and $W_1 = Z^{-1/2} V_1 Z^{-1/2}$. Then, $W_2 = I_m - W_1 - 2Z^{-1}$. The Jacobian of the transformation (6.7) is given by

$$\begin{aligned} J(V_1, V_2 \longrightarrow W_1, W_2) &= J(V_1, V_2 \longrightarrow W_1, Z) J(W_1, Z \longrightarrow W_1, W_2) \\ &= \det(Z)^{(m+1)/2} 2^{-m(m+1)/2} \det(Z)^{m+1} \\ &= 2^{m(m+1)} \det(I_m - W_1 - W_2)^{-3(m+1)/2}. \end{aligned} \tag{6.8}$$

Now, substitution of $V_i = 2(I_m - W_1 - W_2)^{-1/2} W_i (I_m - W_1 - W_2)^{-1/2}$, $i = 1, 2$, along with the Jacobian in the joint density of $V_1$ and $V_2$ given in (6.3) yields the desired result. □

The marginal distribution of $W_1$, when the random matrices $W_1$ and $W_2$ follow a bimatrix beta type 3 distribution, is given next.

**Theorem 6.5.** *Let* $(W_1, W_2) \sim D3(m, \alpha_1, \alpha_2; \beta)$. *Then, the marginal p.d.f. of* $W_1$ *is given by*

$$\begin{aligned} &\frac{\det(W_1)^{\alpha_1 - (m+1)/2} \det(I_m - W_1)^{\alpha_2 + \beta - (m+1)/2}}{2^{-m(\alpha_1 + \alpha_2)} B_m(\alpha_1, \alpha_2 + \beta) \det(I_m + W_1)^{\alpha_1 + \alpha_2 + \beta}} \\ &\quad \times {}_2F_1\Big(\alpha_2, \alpha_1 + \alpha_2 + \beta; \alpha_2 + \beta; -(I_m + W_1)^{-1}(I_m - W_1)\Big), \end{aligned} \tag{6.9}$$

*where* $0 < W_1 < I_m$. *Further,* $(I_m - W_1)^{-1/2} W_2 (I_m - W_1)^{-1/2} \sim B3(m, \alpha_2, \beta)$.

*Proof.* Substituting $X_2 = (I_m - W_1)^{-1/2} W_2 (I_m - W_1)^{-1/2}$ with the Jacobian $J(W_2 \to X_2) = \det(I_m - W_1)^{(m+1)/2}$ in (6.4), the joint density of $W_1$ and $X_2$ is derived as

$$\begin{aligned} &\frac{2^{m(\alpha_1+\alpha_2)} \det(W_1)^{\alpha_1-(m+1)/2} \det(I_m - W_1)^{\alpha_2+\beta-(m+1)/2}}{B_m(\alpha_1,\alpha_2,\beta)\det(I_m + W_1)^{\alpha_1+\alpha_2+\beta}} \\ &\quad \times \frac{\det(X_2)^{\alpha_2-(m+1)/2}\det(I_m - X_2)^{\beta-(m+1)/2}}{\det(I_m + (I_m + W_1)^{-1}(I_m - W_1)X_2)^{\alpha_1+\alpha_2+\beta}}, \quad 0 < W_1 < I_m,\ 0 < X_2 < I_m. \end{aligned} \tag{6.10}$$

Now, integration of the above expression with respect to $X_2$ yields the marginal density of $W_1$. Further, by integrating (6.10) with respect to $W_1$ we find the marginal density of $X_2$ as

$$\begin{aligned} &\frac{2^{m(\alpha_1+\alpha_2)} \det(X_2)^{\alpha_2-(m+1)/2} \det(I_m - X_2)^{\beta-(m+1)/2}}{B_m(\alpha_1,\alpha_2,\beta)\det(I_m + X_2)^{\alpha_1+\alpha_2+\beta}} \\ &\quad \times \int_0^{I_m} \frac{\det(W_1)^{\alpha_1-(m+1)/2}\det(I_m - W_1)^{\alpha_2+\beta-(m+1)/2} dW_1}{\det(I_m + (I_m + X_2)^{-1}(I_m - X_2)W_1)^{\alpha_1+\alpha_2+\beta}}, \quad 0 < X_2 < I_m. \end{aligned} \tag{6.11}$$

Now, by evaluating the above integral using results on Gauss hypergeometric function, we obtain

$$\begin{aligned} &\int_0^{I_m} \frac{\det(W_1)^{\alpha_1-(m+1)/2}\det(I_m - W_1)^{\alpha_2+\beta-(m+1)/2}\, dW_1}{\det(I_m + (I_m + X_2)^{-1}(I_m - X_2)W_1)^{\alpha_1+\alpha_2+\beta}} \\ &\quad = \frac{\Gamma_m(\alpha_1)\Gamma_m(\alpha_2+\beta)}{\Gamma_m(\alpha_1+\alpha_2+\beta)} {}_2F_1\left(\alpha_1, \alpha_1+\alpha_2+\beta; \alpha_1+\alpha_2+\beta; -(I_m + X_2)^{-1}(I_m - X_1)\right) \\ &\quad = \frac{\Gamma_m(\alpha_1)\Gamma_m(\alpha_2+\beta)}{\Gamma_m(\alpha_1+\alpha_2+\beta)} {}_1F_0\left(\alpha_1; -(I_m + X_2)^{-1}(I_m - X_1)\right) \\ &\quad = \frac{\Gamma_m(\alpha_1)\Gamma_m(\alpha_2+\beta)}{\Gamma_m(\alpha_1+\alpha_2+\beta)} 2^{-m\alpha_1}\det(I_m + X_2)^{\alpha_1}. \end{aligned} \tag{6.12}$$

Finally, substituting (6.12) in (6.11) and simplifying the resulting expression we obtain the desired result. □

Using the result

$${}_2F_1(a,b;c;X) = \det(I_m - X)^{-b} {}_2F_1\left(c-a, b; c; -X(I_m - X)^{-1}\right), \tag{6.13}$$

the Gauss hypergeometric function given in (6.9) can be rewritten as

$$\begin{aligned} &{}_2F_1\Big(\alpha_2, \alpha_1+\alpha_2+\beta; \alpha_2+\beta; -(I_m+W_1)^{-1}(I_m-W_1)\Big) \\ &\quad = \frac{\det(I_m+W_1)^{\alpha_1+\alpha_2+\beta}}{2^{m(\alpha_1+\alpha_2+\beta)}} {}_2F_1\left(\beta, \alpha_1+\alpha_2+\beta; \alpha_2+\beta; \frac{I_m-W_1}{2}\right). \end{aligned} \tag{6.14}$$

Hence, the density of $W_1$ can also be written as

$$\begin{aligned} &\frac{\det(W_1)^{\alpha_1-(m+1)/2}\det(I_m-W_1)^{\alpha_2+\beta-(m+1)/2}}{2^{m\beta}B_m(\alpha_1, \alpha_2+\beta)} \\ &\quad \times {}_2F_1\left(\beta, \alpha_1+\alpha_2+\beta; \alpha_2+\beta; \frac{I_m-W_1}{2}\right), \quad 0 < W_1 < I_m. \end{aligned} \tag{6.15}$$

It can clearly be observed that the p.d.f. in (6.9) is not a beta type 3 density and differs by a factor involving ${}_2F_1$. In the next theorem we give distribution of sum of random matrices distributed jointly as bimatrix beta type 3.

**Theorem 6.6.** *Let $(W_1, W_2) \sim D3(m, \alpha_1, \alpha_2; \beta)$. Define $U = W^{-1/2}W_1W^{-1/2}$ and $W = W_1 + W_2$. Then, (i) $U$ and $W$ are independently distributed, (ii) $U \sim B1(m, \alpha_1, \alpha_2)$, and (iii) $W \sim B3(m, \alpha_1 + \alpha_2, \beta)$.*

*Proof.* Making the transformation $U = W^{-1/2}W_1W^{-1/2}$ and $W = W_1 + W_2$ with the Jacobian $J(W_1, W_2 \to U, W) = \det(W)^{(m+1)/2}$ in the joint density of $(W_1, W_2)$ given by (6.4), we get the joint density of $U$ and $W$ as

$$\begin{aligned} &\frac{\det(U)^{\alpha_1-(m+1)/2}\det(I_m-U)^{\alpha_2-(m+1)/2}}{B_m(\alpha_1, \alpha_2)} \\ &\quad \times \frac{\det(W)^{\alpha_1+\alpha_2-(m+1)/2}\det(I_m-W)^{\beta-(m+1)/2}}{2^{-m(\alpha_1+\alpha_2)}B_m(\alpha_1+\alpha_2, \beta)\det(I_m+W)^{\alpha_1+\alpha_2+\beta}}, \end{aligned} \tag{6.16}$$

where $0 < U < I_m$ and $0 < W < I_m$. From the above factorization, it is easy to see that $U$ and $W$ are independently distributed. Further, $U \sim B1(m, \alpha_1, \alpha_2)$ and $W \sim B3(m, \alpha_1 + \alpha_2, \beta)$. □

Using Theorem 6.6, the joint moments of $\det(W_1)$ and $\det(W_2)$ are given by

$$E\big[\det(W_1)^{r_1}\det(W_2)^{r_2}\big] = E\big[\det(U)^{r_1}\det(I_m-U)^{r_2}\big]E\big[\det(W)^{r_1+r_2}\big], \tag{6.17}$$

where $U \sim B1(m, \alpha_1, \alpha_2)$ and $W \sim B3(m, \alpha_1 + \alpha_2, \beta)$. Now, computing $E[\det(W)^{r_1+r_2}]$ and $E[\det(U)^{r_1}\det(I_m - U)^{r_2}]$ using Corollary 4.3 and (2.20) and simplifying the resulting

expression, we obtain

$$E\left[\det(W_1)^{r_1}\det(W_2)^{r_2}\right] = \frac{\Gamma_m(\alpha_1+r_1)\Gamma_m(\alpha_2+r_2)\Gamma_m(\alpha_1+\alpha_2+\beta)}{2^{m\beta}\Gamma_m(\alpha_1)\Gamma_m(\alpha_2)\Gamma_m(\alpha_1+\alpha_2+\beta+r_1+r_2)} \times {}_2F_1\left(\beta,\alpha_1+\alpha_2+\beta;\alpha_1+\alpha_2+\beta+r_1+r_2;\frac{I_m}{2}\right). \tag{6.18}$$

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