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Research Article

Approximation and Shape Preserving Properties of the Bernstein Operator of Max-Product Kind

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Starting from the study of the *Shepard nonlinear operator of max-prod type* by Bede et al. (2006, 2008), in the book by Gal (2008), Open Problem 5.5.4, pages 324–326, the *Bernstein max-prod-type operator* is introduced and the question of the approximation order by this operator is raised. In recent paper, Bede and Gal by using a very complicated method to this open question an answer is given by obtaining an upper estimate of the approximation error of the form $C\omega_1(f;1/\sqrt{n})$ (with an unexplicit absolute constant $C > 0$) and the question of improving the order of approximation $\omega_1(f;1/\sqrt{n})$ is raised. The first aim of this note is to obtain this order of approximation but by a simpler method, which in addition presents, at least, two advantages: it produces an explicit constant in front of $\omega_1(f;1/\sqrt{n})$ and it can easily be extended to other max-prod operators of Bernstein type. However, for subclasses of functions f including, for example, that of concave functions, we find the order of approximation $\omega_1(f;1/n)$, which for many functions f is essentially better than the order of approximation obtained by the linear Bernstein operators. Finally, some shape-preserving properties are obtained.

1. Introduction

Starting from the study of the *Shepard nonlinear operator of max-prod-type* in [1, 2], by the Open Problem in a recent monograph [3, pages 324–326, 5.5.4], the following *nonlinear Bernstein operator of max-prod type* is introduced (here $\bigvee$ means maximum):

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f(k/n)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, for which by a very complicated method in [4, Theorem 6], an upper estimate of the approximation error of the form $C\omega_1(f;1/\sqrt{n})$ (with $C > 0$ unexplicit absolute constant) is obtained. Also, by Remark 7,2 in the same paper [4], the question if this order of approximation could be improved is raised.

The first aim of this note is to obtain the same order of approximation but by a simpler method, which in addition presents, at least, two advantages: it produces an explicit constant in front of $\omega_1(f;1/\sqrt{n})$, and it can easily be extended to other max-prod operators of Bernstein type. Then, one proves by a counterexample that in a sense, for arbitrary f, this order of approximation with respect to $\omega_1(f;\cdot)$ cannot be improved, giving thus a negative answer to a question raised in [4, Remark 7, 2]. However, for subclasses of functions f including, for example, that of concave functions, we find the order of approximation $\omega_1(f;1/n)$, which for many functions, f is essentially better than the order of approximation obtained by the linear Bernstein operators. Finally, some shape-preserving properties are presented.

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, Section 4 contains the approximation results, while in Section 5 we present some shape-preserving properties. The paper ends with Section 6 containing some conclusions concerning the comparisons between the max-product and the linear Bernstein operators.

2. Preliminaries

For the proof of the main results, we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, $\mathbb{R}_+$, we consider the operations $\vee$ (maximum) and "$\cdot$" product. Then $(\mathbb{R}_+, \vee, \cdot)$ has a semiring structure and we call it as Max-Product algebra.

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I) = \{f : I \longrightarrow \mathbb{R}_+; f \text{ continuous and bounded on } I\}. \tag{2.1}$$

The general form of $L_n : CB_+(I) \to CB_+(I)$, (called here a discrete max-product-type approximation operator) studied in the paper will be

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) \cdot f(x_i), \tag{2.2}$$

or

$$L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) \cdot f(x_i), \tag{2.3}$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(\cdot, x_i) \in CB_+(I)$, and $x_i \in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy a pseudolinearity condition of the form

$$L_n(\alpha \cdot f \vee \beta \cdot g)(x) = \alpha \cdot L_n(f)(x) \vee \beta \cdot L_n(g)(x), \quad \forall \alpha, \beta \in \mathbb{R}_+, \; f, g : I \longrightarrow \mathbb{R}_+. \tag{2.4}$$

In this section we present some general results on these kinds of operators which will be useful later in the study of the Bernstein max-product-type operator considered in Section 1.

Lemma 2.1 (see [4]). *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval:*

$$CB_+(I) = \{f : I \longrightarrow \mathbb{R}_+; f \text{ continuous and bounded on } I\}, \tag{2.5}$$

and let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$, be a sequence of operators satisfying the following properties:

(i) *if $f, g \in CB_+(I)$ satisfy $f \leq g$, then $L_n(f) \leq L_n(g)$ for all $n \in N$;*

(ii) *$L_n(f + g) \leq L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.*

Then for all $f, g \in CB_+(I)$, $n \in N$, and $x \in I$, we have

$$\left|L_n(f)(x) - L_n(g)(x)\right| \leq L_n(\left|f - g\right|)(x). \tag{2.6}$$

Proof. Since it is very simple, we reproduce here the proof in [4]. Let $f, g \in CB_+(I)$. We have $f = f - g + g \leq |f - g| + g$, which by conditions (i)-(ii) successively implies $L_n(f)(x) \leq L_n(|f - g|)(x) + L_n(g)(x)$, that is, $L_n(f)(x) - L_n(g)(x) \leq L_n(|f - g|)(x)$.

Writing now $g = g - f + f \leq |f - g| + f$ and applying the above reasonings, it follows that $L_n(g)(x) - L_n(f)(x) \leq L_n(|f - g|)(x)$, which, combined with the above inequality, gives $|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|)(x)$. □

Remark 2.2. (1) It is easy to see that the Bernstein max-product operator satisfies the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i), it satisfies the stronger condition:

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x), \quad f, g \in CB_+(I). \tag{2.7}$$

Indeed, taking in the above equality $f \leq g$, $f, g \in CB_+(I)$, it easily follows that $L_n(f)(x) \leq L_n(g)(x)$.

(2) In addition, it is immediate that the Bernstein max-product operator is positive homogenous, that is, $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$.

Corollary 2.3 (see [4]). *Let $L_n : CB_+(I) \to CB_+(I)$, $n \in N$, be a sequence of operators satisfying conditions* (i)-(ii) *in Lemma 2.1 and in addition being positive homogenous. Then for all $f \in CB_+(I)$, $n \in N$, and $x \in I$, one has*

$$\left|f(x) - L_n(f)(x)\right| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f;\delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|, \tag{2.8}$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$ for all $t \in I$, $x \in I$, $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x - y| \leq \delta\}$, and if I is unbounded, then we suppose that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup\{+\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

Proof. The proof is identical with that for positive linear operators and because of its simplicity, we reproduce it what follows. Indeed, from the identity

$$L_n(f)(x) - f(x) = \left[L_n(f)(x) - f(x) \cdot L_n(e_0)(x)\right] + f(x)[L_n(e_0)(x) - 1], \tag{2.9}$$

it follows (by the positive homogeneity and by Lemma 2.1) that

$$\begin{aligned} \left|f(x) - L_n(f)(x)\right| &\le \left|L_n(f(x))(x) - L_n(f(t))(x)\right| + \left|f(x)\right| \cdot |L_n(e_0)(x) - 1| \\ &\le L_n(\left|f(t) - f(x)\right|)(x) + \left|f(x)\right| \cdot |L_n(e_0)(x) - 1|. \end{aligned} \tag{2.10}$$

Now, since for all $t, x \in I$, we have

$$\left|f(t) - f(x)\right| \le \omega_1(f; |t - x|)_I \le \left[\frac{1}{\delta}|t - x| + 1\right]\omega_1(f; \delta)_I, \tag{2.11}$$

replacing the above, we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.3 is as follows.

Corollary 2.4 (see [4]). *Suppose that in addition to the conditions in Corollary 2.3, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in N$. Then for all $f \in CB_+(I)$, $n \in N$, and $x \in I$, one has*

$$\left|f(x) - L_n(f)(x)\right| \le \left[1 + \frac{1}{\delta}L_n(\varphi_x)(x)\right]\omega_1(f; \delta)_I. \tag{2.12}$$

3. Auxiliary Results

Since it is easy to check that $B_n^{(M)}(f)(0) - f(0) = B_n^{(M)}(f)(1) - f(1) = 0$ for all n, notice that in the notations, proofs and statements of the all approximation results, that is, in Lemmas 3.1–3.3, Theorem 4.1, Lemmas 4.2–4.4, Corollaries 4.6, 4.7, in fact we always may suppose that $0 < x < 1$. For the proofs of the main results, we need some notations and auxiliary results, as follows.

For each $k, j \in \{0, 1, 2, \ldots n\}$ and $x \in [j/(n+1), (j+1)/(n+1)]$, let us denote

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)|k/n - x|}{p_{n,j}(x)}, \qquad m_{k,n,j}(x) = \frac{p_{n,k}(x)}{p_{n,j}(x)}. \tag{3.1}$$

It is clear that if $k \ge j + 1$, then

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)(k/n - x)}{p_{n,j}(x)}, \tag{3.2}$$

and if $k \leq j-1$, then

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)(x-k/n)}{p_{n,j}(x)}. \tag{3.3}$$

Also, for each $k, j \in \{0,1,2,\ldots n\}$, $k \geq j+2$, and $x \in [j/(n+1),(j+1)/(n+1)]$ let us denote

$$\overline{M}_{k,n,j}(x) = \frac{p_{n,k}(x)(k/(n+1)-x)}{p_{n,j}(x)}, \tag{3.4}$$

and for each $k, j \in \{0,1,2,\ldots n\}$, $k \leq j-2$, and $x \in [j/(n+1),(j+1)/(n+1)]$ let us denote

$$\underline{M}_{k,n,j}(x) = \frac{p_{n,k}(x)(x-k/(n+1))}{p_{n,j}(x)}. \tag{3.5}$$

Lemma 3.1. *Let $x \in [j/(n+1),(j+1)/(n+1)]$:*

(i) *for all $k, j \in \{0,1,2,\ldots n\}$, $k \geq j+2$, one has*

$$\overline{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 3\overline{M}_{k,n,j}(x); \tag{3.6}$$

(ii) *for all $k, j \in \{0,1,2,\ldots n\}$, $k \leq j-2$, one has*

$$M_{k,n,j}(x) \leq \underline{M}_{k,n,j}(x) \leq 6M_{k,n,j}(x). \tag{3.7}$$

Proof. (i) The inequality $\overline{M}_{k,n,j}(x) \leq M_{k,n,j}(x)$ is immediate.

On the other hand,

$$\begin{aligned}\frac{M_{k,n,j}(x)}{\overline{M}_{k,n,j}(x)} &= \frac{k/n-x}{k/(n+1)-x} \leq \frac{k/n-j/(n+1)}{k/(n+1)-(j+1)/(n+1)} \\ &\leq \frac{kn+k-nj}{n(k-j-1)} = \frac{k-j}{k-j-1} + \frac{k}{n(k-j-1)} \leq 3,\end{aligned} \tag{3.8}$$

which proves (i).

(ii) The inequality $M_{k,n,j}(x) \le \underline{M}_{k,n,j}(x)$ is immediate.
On the other hand,

$$\frac{\underline{M}_{k,n,j}(x)}{M_{k,n,j}(x)} = \frac{x-k/(n+1)}{x-k/n} \le \frac{(j+1)/(n+1)-k/(n+1)}{j/(n+1)-k/n}$$
$$= \frac{(n+1)(j+1-k)}{nj-nk-k} \le \frac{(n+1)(j+1-k)}{nj-nk-n} = \frac{n+1}{n}\cdot\frac{j+1-k}{j-k-1} \tag{3.9}$$
$$\le 2\cdot\frac{j+1-k}{j-k-1} = 2\left(1+\frac{2}{j-k-1}\right) \le 6,$$

which proves (ii) and the lemma. □

Lemma 3.2. *For all $k, j \in \{0,1,2,\ldots,n\}$ and $x \in [j/(n+1),(j+1)/(n+1)]$, one has*

$$m_{k,n,j}(x) \le 1. \tag{3.10}$$

Proof. We have two cases: 1) $k \ge j$, 2) $k \le j$.

Case 1. Since clearly the function $h(x) = (1-x)/x$ is nonincreasing on $[j/(n+1),(j+1)/(n+1)]$, it follows that

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{k+1}{n-k}\cdot\frac{1-x}{x} \ge \frac{k+1}{n-k}\cdot\frac{1-(j+1)/(n+1)}{(j+1)/(n+1)} = \frac{k+1}{n-k}\cdot\frac{n-j}{j+1} \ge 1, \tag{3.11}$$

which implies $m_{j,n,j}(x) \ge m_{j+1,n,j}(x) \ge m_{j+2,n,j}(x) \ge \cdots \ge m_{n,n,j}(x)$.

Case 2. We get

$$\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{n-k+1}{k}\cdot\frac{x}{1-x} \ge \frac{n-k+1}{k}\cdot\frac{j/(n+1)}{1-j/(n+1)}$$
$$= \frac{n-k+1}{k}\cdot\frac{j}{n+1-j} \ge 1, \tag{3.12}$$

which immediately implies that

$$m_{j,n,j}(x) \ge m_{j-1,n,j}(x) \ge m_{j-2,n,j}(x) \ge \cdots \ge m_{0,n,j}(x). \tag{3.13}$$

Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. □

Lemma 3.3. *Let $x \in [j/(n+1),(j+1)/(n+1)]$.*

(i) *If $k \in \{j+2, j+3,\ldots,n-1\}$ is such that $k-\sqrt{k+1} \ge j$, then $\overline{M}_{k,n,j}(x) \ge \overline{M}_{k+1,n,j}(x)$.*

(ii) *If $k \in \{1,2,\ldots j-2\}$ is such that $k+\sqrt{k} \le j$, then $\underline{M}_{k,n,j}(x) \ge \underline{M}_{k-1,n,j}(x)$.*

Proof. (i) We observe that

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} = \frac{k+1}{n-k} \cdot \frac{1-x}{x} \cdot \frac{k/(n+1)-x}{(k+1)/(n+1)-x}. \tag{3.14}$$

Since the function $g(x) = ((1-x)/x)(k/(n+1)-x)/((k+1)/(n+1)-x)$ clearly is nonincreasing, it follows that $g(x) \geq g((j+1)/(n+1)) = ((n-j)/(j+1)) \cdot ((k-j-1)/(k-j))$ for all $x \in [j/(n+1), (j+1)/(n+1)]$. Then, since the condition $k-\sqrt{k+1} \geq j$ implies $(k+1)(k-j-1) \geq (j+1)(k-j)$, we obtain

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} \cdot \frac{k-j-1}{k-j} \geq 1. \tag{3.15}$$

(ii) We observe that

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} = \frac{n-k+1}{k} \cdot \frac{x}{1-x} \cdot \frac{x-k/(n+1)}{x-(k-1)/(n+1)}. \tag{3.16}$$

Since the function $h(x) = (x/(1-x)) \cdot (x-k/(n+1))/(x-(k-1)/(n+1))$ is nondecreasing, it follows that $h(x) \geq h(j/(n+1)) = (j/(n+1-j)) \cdot ((j-k)/(j-k+1))$ for all $x \in [j/(n+1), (j+1)/(n+1)]$. Then, since the condition $k+\sqrt{k} \leq j$ implies $j(j-k) \geq k(j-k+1)$, we obtain

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n-k+1}{k} \cdot \frac{j}{n+1-j} \cdot \frac{j-k}{j-k+1} \geq 1, \tag{3.17}$$

which proves the lemma. □

Also, a key result in the proof of the main result is the following.

Lemma 3.4. *One has*

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right],\ j = 0, 1, \ldots, n, \tag{3.18}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Proof. First we show that for fixed $n \in \mathbb{N}$ and $0 \leq k < k+1 \leq n$, we have

$$0 \leq p_{n,k+1}(x) \leq p_{n,k}(x), \quad \text{iff } x \in \left[0, \frac{k+1}{n+1}\right]. \tag{3.19}$$

Indeed, the inequality one reduces to

$$0 \le \binom{n}{k+1} x^{k+1}(1-x)^{n-(k+1)} \le \binom{n}{k} x^{k}(1-x)^{n-k} \tag{3.20}$$

after simplifications is equivalent to

$$0 \le x\left[\binom{n}{k+1} + \binom{n}{k}\right] \le \binom{n}{k}. \tag{3.21}$$

However, since $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$, the above inequality immediately becomes equivalent to

$$0 \le x \le \frac{k+1}{n+1}. \tag{3.22}$$

By taking $k = 0, 1, \dots,$ in the inequality just proved above, we get

$$\begin{aligned} p_{n,1}(x) &\le p_{n,0}(x), \quad \text{iff } x \in \left[0, \frac{1}{n+1}\right], \\ p_{n,2}(x) &\le p_{n,1}(x), \quad \text{iff } x \in \left[0, \frac{2}{n+1}\right], \\ p_{n,3}(x) &\le p_{n,2}(x), \quad \text{iff } x \in \left[0, \frac{3}{n+1}\right], \end{aligned} \tag{3.23}$$

and so on,

$$p_{n,k+1}(x) \le p_{n,k}(x), \quad \text{iff } x \in \left[0, \frac{k+1}{n+1}\right], \tag{3.24}$$

and so on,

$$\begin{aligned} p_{n,n-2}(x) &\le p_{n,n-3}(x), \quad \text{iff } x \in \left[0, \frac{n-2}{n+1}\right], \\ p_{n,n-1}(x) &\le p_{n,n-2}(x), \quad \text{iff } x \in \left[0, \frac{n-1}{n+1}\right], \\ p_{n,n}(x) &\le p_{n,n-1}(x), \quad \text{iff } x \in \left[0, \frac{n}{n+1}\right]. \end{aligned} \tag{3.25}$$

From all these inequalities, reasoning by recurrence we easily obtain

$$\begin{aligned}
&\text{if } x \in \left[0, \frac{1}{n+1}\right], \text{ then } p_{n,k}(x) \le p_{n,0}(x), \quad \forall k = 0, 1, \dots, n, \\
&\text{if } x \in \left[\frac{1}{n+1}, \frac{2}{n+1}\right], \text{ then } p_{n,k}(x) \le p_{n,1}(x), \quad \forall k = 0, 1, \dots, n, \\
&\text{if } x \in \left[\frac{2}{n+1}, \frac{3}{n+1}\right], \text{ then } p_{n,k}(x) \le p_{n,2}(x), \quad \forall k = 0, 1, \dots, n,
\end{aligned} \tag{3.26}$$

and so on, finally

$$\text{if } x \in \left[\frac{n}{n+1}, 1\right], \text{ then } p_{n,k}(x) \le p_{n,n}(x), \quad \forall k = 0, 1, \dots, n, \tag{3.27}$$

which proves the lemma. □

4. Approximation Results

If $B_n^{(M)}(f)(x)$ represents the nonlinear Bernstein operator of max-product type defined in Section 1, then the first main result of this section is the following.

Theorem 4.1. *If $f : [0,1] \to \mathbb{R}_+$ is continuous, then one has the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 12\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right), \quad \forall n \in \mathbb{N}, \ x \in [0,1], \tag{4.1}$$

where

$$\omega_1(f;\delta) = \sup\{\left|f(x) - f(y)\right|; x, y \in [0,1], \ \left|x - y\right| \le \delta\}. \tag{4.2}$$

Proof. It is easy to check that the max-product Bernstein operators fulfill the conditions in Corollary 2.4 and we have

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \left(1 + \frac{1}{\delta_n} B_n^{(M)}(\varphi_x)(x)\right)\omega_1(f;\delta_n), \tag{4.3}$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := B_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x)|k/n - x|}{\bigvee_{k=0}^{n} p_{n,k}(x)}. \tag{4.4}$$

Let $x \in [j/(n+1),(j+1)/(n+1)]$, where $j \in \{0,\dots,n\}$ is fixed, arbitrary. By Lemma 3.4 we easily obtain

$$E_n(x) = \max_{k=0,\dots,n} \{M_{k,n,j}(x)\}, \quad x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]. \tag{4.5}$$

In all what follows we may suppose that $j \in \{1,\dots,n\}$, because for $j = 0$, simple calculation shows that in this case, we get $E_n(x) \le 1/n$, for all $x \in [0,1/(n+1)]$. So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j = 1,\dots,n$ is fixed, $x \in [j/(n+1),(j+1)/(n+1)]$ and $k = 0,\dots,n$. In fact, we will prove that

$$M_{k,n,j}(x) \le \frac{6}{\sqrt{n+1}}, \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right],\ k = 0,\dots,n, \tag{4.6}$$

which immediately will implies that

$$E_n(x) \le \frac{6}{\sqrt{n+1}}, \quad \forall x \in [0,1],\ n \in \mathbb{N}, \tag{4.7}$$

and taking $\delta_n = 6/\sqrt{n+1}$ in (4.3), we immediately obtain the estimate in the statement.

In order to prove (4.6) we distinguish the following cases: 1) $k \in \{j-1, j, j+1\}$, 2) $k \ge j+2$ and, 3) $k \le j-2$.

Case 1. If $k = j$, then $M_{j,n,j}(x) = |j/n - x|$. Since $x \in [j/(n+1),(j+1)/(n+1)]$, it easily follows that $M_{j,n,j}(x) \le 1/(n+1)$.

If $k = j+1$, then $M_{j+1,n,j}(x) = m_{j+1,n,j}(x)((j+1)/n - x)$. Since by Lemma 3.2 we have $m_{j+1,n,j}(x) \le 1$, we obtain $M_{j+1,n,j}(x) \le (j+1)/n - x \le (j+1)/n - j/(n+1) = (n+j+1)/n(n+1) \le 3/(n+1)$.

If $k = j-1$, then $M_{j-1,n,j}(x) = m_{j-1,n,j}(x)(x - (j-1)/n) \le (j+1)/(n+1) - (j-1)/n = (2n - (j+1))/n(n+1) \le 2/(n+1)$.

Case 2. Subcase a

Suppose first that $k - \sqrt{k+1} < j$. We get

$$\begin{aligned}\overline{M}_{k,n,j}(x) &= m_{k,n,j}(x)\left(\frac{k}{n+1} - x\right) \le \frac{k}{n+1} - x \le \frac{k}{n+1} - \frac{j}{n+1} \\ &\le \frac{k}{n+1} - \frac{k-\sqrt{k+1}}{n+1} = \frac{\sqrt{k+1}}{n+1} \le \frac{1}{\sqrt{n+1}}.\end{aligned} \tag{4.8}$$

Subcase b

Suppose now that $k - \sqrt{k+1} \geq j$. Since the function $g(x) = x - \sqrt{x+1}$ is nondecreasing on the interval $[0, \infty)$, it follows that there exists $\overline{k} \in \{0, 1, 2, \ldots n\}$, of maximum value, such that $\overline{k} - \sqrt{\overline{k}+1} < j$. Then, for $k_1 = \overline{k} + 1$, we get $k_1 - \sqrt{k_1 + 1} \geq j$ and

$$\begin{aligned} \overline{M}_{\overline{k}+1,n,j}(x) = m_{\overline{k}+1,n,j}(x)\left(\frac{\overline{k}+1}{n+1} - x\right) &\leq \frac{\overline{k}+1}{n+1} - x \\ &\leq \frac{\overline{k}+1}{n+1} - \frac{j}{n+1} \leq \frac{\overline{k}+1}{n+1} - \frac{\overline{k} - \sqrt{\overline{k}+1}}{n+1} \\ &= \frac{\sqrt{\overline{k}+1}+1}{n+1} \leq \frac{2}{\sqrt{n+1}}. \end{aligned} \tag{4.9}$$

Also, we have $k_1 \geq j + 2$. Indeed, this is a consequence of the fact that g is nondecreasing on the interval $[0, \infty)$ and because it is easy to see that $g(j+1) < j$. By Lemma 3.3, (i) it follows that $\overline{M}_{\overline{k}+1,n,j}(x) \geq \overline{M}_{\overline{k}+2,n,j}(x) \geq \cdots \geq \overline{M}_{n,n,j}(x)$. We thus obtain $\overline{M}_{k,n,j}(x) \leq 2/\sqrt{n+1}$ for any $k \in \{\overline{k}+1, \overline{k}+2, \ldots n\}$.

Therefore, in both subcases, by Lemma 3.1, (i) too, we get $M_{k,n,j}(x) \leq 6/\sqrt{n+1}$.

Case 3. Subcase a

Suppose first that $k + \sqrt{k} \geq j$. Then we obtain

$$\begin{aligned} \underline{M}_{k,n,j}(x) = m_{k,n,j}(x)\left(x - \frac{k}{n+1}\right) &\leq \frac{j+1}{n+1} - \frac{k}{n+1} \\ &\leq \frac{k + \sqrt{k} + 1}{n+1} - \frac{k}{n+1} = \frac{\sqrt{k}+1}{n+1} \leq \frac{\sqrt{n}+1}{n+1} \leq \frac{2}{\sqrt{n+1}}. \end{aligned} \tag{4.10}$$

Subcase b

Suppose now that $k + \sqrt{k} < j$. Let $\widetilde{k} \in \{0, 1, 2, \ldots n\}$ be the minimum value such that $\widetilde{k} + \sqrt{\widetilde{k}} \geq j$. Then $k_2 = \widetilde{k} - 1$ satisfies $k_2 + \sqrt{k_2} < j$ and

$$\begin{aligned} \underline{M}_{\widetilde{k}-1,n,j}(x) = m_{\widetilde{k}-1,n,j}(x)\left(x - \frac{\widetilde{k}-1}{n+1}\right) &\leq \frac{j+1}{n+1} - \frac{\widetilde{k}-1}{n+1} \\ &\leq \frac{\widetilde{k} + \sqrt{\widetilde{k}} + 1}{n+1} - \frac{\widetilde{k}-1}{n+1} = \frac{\sqrt{\widetilde{k}}+2}{n+1} \leq \frac{3}{\sqrt{n+1}}. \end{aligned} \tag{4.11}$$

Also, because in this case we have $j \geq 2$, it is immediate that $k_2 \leq j-2$. By Lemma 3.3, (ii), it follows that $\underline{M}_{k-1,n,j}(x) \geq \underline{M}_{k-2,n,j}(x) \geq \cdots \geq \underline{M}_{0,n,j}(x)$. We obtain $\underline{M}_{k,n,j}(x) \leq 3/\sqrt{n+1}$ for any $k \leq j-2$ and $x \in [j/(n+1), (j+1)/(n+1)]$.

In both subcases, by Lemma 3.1, (ii) too, we get $M_{k,n,j}(x) \leq 3/\sqrt{n+1}$.

In conclusion, collecting all the estimates in the above cases and subcases we easily get the relationship (4.6), which completes the proof. □

Remarks. (1) The order of approximation in terms of $\omega_1(f;\sqrt{n})$ in Theorem 4.1 cannot be improved, in the sense that the order of $\max_{x\in[0,1]}\{E_n(x)\}$ is exactly $1/\sqrt{n}$ (here $E_n(x)$ is defined in the proof of Theorem 4.1). Indeed, for $n \in \mathbb{N}$, let us take $j_n = [n/2]$, $k_n = j_n + [\sqrt{n}]$, $x_n = (j_n+1)/(n+1)$ and denote $\tilde{n} = n - [n/2]$. Then, we can write

$$\begin{aligned}\overline{M}_{k_n,n,j_n}(x_n) &= \frac{\binom{n}{k_n} x_n^{k_n}(1-x_n)^{n-k_n}}{\binom{n}{j_n} x_n^{j_n}(1-x_n)^{n-j_n}}\left(\frac{k_n}{n+1} - x_n\right) \\ &= \frac{(\tilde{n}-[\sqrt{n}]+1)(\tilde{n}-[\sqrt{n}]+2)\cdots\tilde{n}}{([n/2]+1)([n/2]+2)\cdots([n/2]+[\sqrt{n}])}\left(\frac{[n/2]+1}{\tilde{n}}\right)^{[\sqrt{n}]} \cdot \frac{[\sqrt{n}]-1}{n+1}.\end{aligned} \tag{4.12}$$

Since $2[n/2] \geq n-1$, we easily get $[n/2]+1 \geq \tilde{n}$, which implies $(([n/2]+1)/\tilde{n}\,)^{[\sqrt{n}]} \geq 1$ for all $n \in \mathbb{N}$. On the other hand,

$$\frac{(\tilde{n}-[\sqrt{n}]+1)(\tilde{n}-[\sqrt{n}]+2)\cdots\tilde{n}}{([n/2]+1)([n/2]+2)\cdots([n/2]+[\sqrt{n}])} \geq \left(\frac{\tilde{n}-[\sqrt{n}]+1}{[n/2]+[\sqrt{n}]}\right)^{[\sqrt{n}]} \geq \left(\frac{n/2-\sqrt{n}+1}{n/2+\sqrt{n}}\right)^{\sqrt{n}}. \tag{4.13}$$

Because $\lim_{n\to\infty}((n/2-\sqrt{n}+1)/(n/2+\sqrt{n}))^{\sqrt{n}} = e^{-4}$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(\tilde{n}-[\sqrt{n}]+1)(\tilde{n}-[\sqrt{n}]+2)\cdots\tilde{n}}{([n/2]+1)([n/2]+2)\cdots([n/2]+[\sqrt{n}])} \geq e^{-5} \tag{4.14}$$

for all $n \geq n_0$. It follows

$$\overline{M}_{k_n,n,j_n}(x_n) \geq \frac{e^{-5}([\sqrt{n}]-1)}{n+1} \geq \frac{e^{-5}}{6\sqrt{n}}, \tag{4.15}$$

for all $n \geq \max\{n_0, 4\}$. Taking into account Lemma 3.1, (i) too, it follows that for all $n \geq \max\{n_0, 4\}$, we have $M_{k_n,n,j_n}(x_n) \geq e^{-5}/6\sqrt{n}$, which implies the desired conclusion.

(2) With respect to the method of the proof in [4], the method in this paper presents, at least, two advantages: it produces the explicit constant 12 in front of $\omega_1(f;1/\sqrt{n+1})$ and its ideas can be easily used for other max-prod Bernstein operators too, which will be done in several forthcoming papers.

In what follows, we will prove that for large subclasses of functions f, the order of approximation $\omega_1(f;1/\sqrt{n+1})$ in Theorem 4.1 can essentially be improved to $\omega_1(f;1/n)$.

For this purpose, for any $k, j \in \{0, 1, \ldots, n\}$, let us define the functions $f_{k,n,j} : [j/(n+1), (j+1)/(n+1)] \to \mathbb{R}$:

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j} f\left(\frac{k}{n}\right). \tag{4.16}$$

Then it is clear that for any $j \in \{0, 1, \ldots, n\}$ and $x \in [j/(n+1), (j+1)/(n+1)]$, we can write

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x). \tag{4.17}$$

Also we need the following four auxiliary lemmas.

Lemma 4.2. *Let $f : [0,1] \to [0, \infty)$ be such that*

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\} \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]. \tag{4.18}$$

Then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right), \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], \tag{4.19}$$

where $\omega_1(f;\delta) = \max\{|f(x) - f(y)|; x, y \in [0,1],\ |x-y| \le \delta\}$.

Proof. We distinguish the two following cases.

Case (i). Let $x \in [j/(n+1), (j+1)/(n+1)]$ be fixed such that $B_n^{(M)}(f)(x) = f_{j,n,j}(x)$. Because by simple calculation we have $-1/(n+1) \le x - j/n \le 1/(n+1)$ and $f_{j,n,j}(x) = f(j/n)$, it follows that

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f; \frac{1}{n+1}\right). \tag{4.20}$$

Case (ii). Let $x \in [j/(n+1), (j+1)/(n+1)]$ be such that $B_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$. We have two subcases:

(a) $B_n^{(M)}(f)(x) \le f(x)$, when evidently $f_{j,n,j}(x) \le f_{j+1,n,j}(x) \le f(x)$ and we immediately get

$$\begin{aligned}\left|B_n^{(M)}(f)(x) - f(x)\right| &= \left|f_{j+1,n,j}(x) - f(x)\right| \\ &= f(x) - f_{j+1,n,j}(x) \le f(x) - f\left(\frac{j}{n}\right) \le \omega_1\left(f; \frac{1}{n+1}\right);\end{aligned} \tag{4.21}$$

(b) $B_n^{(M)}(f)(x) > f(x)$, when

$$\left|B_n^{(M)}(f)(x) - f(x)\right| = f_{j+1,n,j}(x) - f(x) = m_{j+1,n,j}(x) f\left(\frac{j+1}{n}\right) - f(x) \\ \leq f\left(\frac{j+1}{n}\right) - f(x). \tag{4.22}$$

Because $0 \leq (j+1)/n - x \leq (j+1)/n - j/(n+1) = j/n(n+1) + 1/n < 2/n$, it follows $f((j+1)/n) - f(x) \leq 2\omega_1(f; 1/n)$, which proves the lemma. □

Lemma 4.3. *Let $f : [0,1] \to [0, \infty)$ be such that*

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\} \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]. \tag{4.23}$$

Then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]. \tag{4.24}$$

Proof. We distinguish the two following cases:

Case (i). $B_n^{(M)}(f)(x) = f_{j,n,j}(x)$, when as in Lemma 4.2 we get

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n+1}\right), \tag{4.25}$$

Case (ii). $B_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$, when we have two subcases:

(a) $B_n^{(M)}(f)(x) \leq f(x)$, when as in the case of Lemma 4.2 we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n+1}\right). \tag{4.26}$$

(b) $B_n^{(M)}(f)(x) > f(x)$, when by using the same idea as in the subcase (b) of Lemma 4.2 and taking into account that

$$0 \leq x - \frac{j-1}{n} \leq \frac{j+1}{n+1} - \frac{j-1}{n} = \frac{-j}{n(n+1)} + \frac{1}{n+1} + \frac{1}{n} < \frac{2}{n}, \tag{4.27}$$

we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \tag{4.28}$$

which proves the lemma. □

Lemma 4.4. *Let $f : [0,1] \to [0,\infty)$ be such that*

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \tag{4.29}$$

for all $x \in [j/(n+1), (j+1)/(n+1)]$. Then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right), \quad \forall x \in \left[\frac{j}{(n+1)}, \frac{(j+1)}{(n+1)}\right]. \tag{4.30}$$

Proof. Let $x \in [j/(n+1), (j+1)/(n+1)]$. If $B_n^{(M)}(f)(x) = f_{j,n,j}(x)$ or $B_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$, then $B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$ and from Lemma 4.2, it follows that

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right). \tag{4.31}$$

If $B_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$, then $B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\}$ and from Lemma 4.3, we get

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right), \tag{4.32}$$

which ends the proof. □

Lemma 4.5. *Let $f : [0,1] \to [0,\infty)$ be concave. Then the following two properties hold:*

(i) *the function $g : (0,1] \to [0,\infty)$, $g(x) = f(x)/x$ is nonincreasing;*

(ii) *the function $h : [0,1) \to [0,\infty)$, $h(x) = f(x)/(1-x)$ is nondecreasing.*

Proof. (i) Let $x, y \in (0,1]$ be with $x \le y$. Then

$$f(x) = f\left(\frac{x}{y}y + \frac{y-x}{y}0\right) \ge \frac{x}{y}f(y) + \frac{y-x}{y}f(0) \ge \frac{x}{y}f(y), \tag{4.33}$$

which implies that $f(x)/x \ge f(y)/y$.

(ii) Let $x, y \in [0,1)$ be with $x \ge y$. Then

$$f(x) = f\left(\frac{1-x}{1-y}y + \frac{x-y}{1-y}1\right) \ge \frac{1-x}{1-y}f(y) + \frac{x-y}{1-y}f(1) \ge \frac{1-x}{1-y}f(y), \tag{4.34}$$

which implies $f(x)/(1-x) \ge f(y)/(1-y)$. □

Corollary 4.6. *Let $f : [0,1] \to [0,\infty)$ be a concave function. Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right), \quad \forall x \in [0,1]. \tag{4.35}$$

Proof. Let $x \in [0,1]$ and $j \in \{0,1,\ldots n\}$ such that $x \in [j/(n+1),(j+1)/(n+1)]$. Let $k \in \{0,1,\ldots n\}$ be with $k \geq j$. Then

$$\begin{aligned} f_{k+1,n,j}(x) &= \frac{\binom{n}{k+1}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k+1-j} f\left(\frac{k+1}{n}\right) \\ &= \frac{\binom{n}{k}}{\binom{n}{j}} \frac{n-k}{k+1} \left(\frac{x}{1-x}\right)^{k-j} \frac{x}{1-x} f\left(\frac{k+1}{n}\right). \end{aligned} \tag{4.36}$$

From Lemma 4.5, (i), we get $f((k+1)/n)/(k+1)/n \leq f(k/n)/k/n$, that is, $f((k+1)/n) \leq ((k+1)/k)(f(k/n))$. Since $x/(1-x) \leq (j+1)/(n-j)$, we get

$$\begin{aligned} f_{k+1,n,j}(x) &\leq \frac{\binom{n}{k}}{\binom{n}{j}} \frac{n-k}{k+1} \left(\frac{x}{1-x}\right)^{k-j} \frac{j+1}{n-j} \cdot \frac{k+1}{k} f\left(\frac{k}{n}\right) \\ &= f_{k,n,j}(x) \frac{j+1}{k} \cdot \frac{n-k}{n-j}. \end{aligned} \tag{4.37}$$

It is immediate that for $k \geq j+1$, it follows that $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Thus we obtain

$$f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \cdots \geq f_{n,j,n}(x). \tag{4.38}$$

Now let $k \in \{0,1,\ldots n\}$ be with $k \leq j$. Then

$$\begin{aligned} f_{k-1,n,j}(x) &= \frac{\binom{n}{k-1}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-1-j} f\left(\frac{k-1}{n}\right) \\ &= \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \frac{k}{n-k+1} \left(\frac{x}{1-x}\right)^{k-j} \frac{1-x}{x} f\left(\frac{k-1}{n}\right). \end{aligned} \tag{4.39}$$

From Lemma 4.5, (ii), we get $f(k/n)/(1-k/n) \geq f((k-1)/n)/(1-(k-1)/n)$, that is, $f(k/n) \geq ((n-k)/(n-k+1))(f((k-1)/n))$. Because $(1-x)/x \leq (n+1-j)/j$, we get

$$\begin{aligned} f_{k-1,n,j}(x) &\leq \frac{\binom{n}{k}}{\binom{n}{j}} \frac{k}{n-k+1} \left(\frac{x}{1-x}\right)^{k-j} \frac{n+1-j}{j} \cdot \frac{n-k+1}{n-k} f\left(\frac{k}{n}\right) \\ &= f_{k,n,j}(x) \frac{k}{j} \cdot \frac{n+1-j}{n-k}. \end{aligned} \tag{4.40}$$

For $k \leq j-1$ it is immediate that $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$, which implies

$$f_{j-1,n.j}(x) \geq f_{j-2,n.j}(x) \geq \cdots \geq f_{0,n,j}(x). \tag{4.41}$$

From (4.38) and (4.42), we obtain

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \tag{4.42}$$

which combined with Lemma 4.4 implies

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \tag{4.43}$$

and proves the corollary. □

Corollary 4.7. (i) *If $f : [0,1] \to [0,\infty)$ is nondecreasing and such that the function $g : (0,1] \to [0,\infty)$, $g(x) = f(x)/x$ is nonincreasing, then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \quad \forall x \in [0,1]. \tag{4.44}$$

(ii) *If $f : [0,1] \to [0,\infty)$ is nonincreasing and such that the function $h : [0,1) \to [0,\infty)$, $h(x) = f(x)/(1-x)$ is nondecreasing, then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \quad \forall x \in [0,1]. \tag{4.45}$$

Proof. (i) Since f is nondecreasing it follows (see the proof of Theorem 5.5 in Section 5) that

$$B_n^{(M)}(f)(x) = \bigvee_{k \geq j}^{n} f_{k,n,j}(x), \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]. \tag{4.46}$$

Following the proof of Corollary 4.6, we get

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], \tag{4.47}$$

and from Lemma 4.2, we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right). \tag{4.48}$$

(ii) Since f is nonincreasing, it follows (see the proof of Corollary 5.6 in Section 5) that

$$B_n^{(M)}(f)(x) = \bigvee_{k\geq 0}^{j} f_{k,n,j}(x), \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]. \tag{4.49}$$

Following the proof of Corollary 4.6, we get

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x)\}, \tag{4.50}$$

and from Lemma 4.3 we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right). \tag{4.51}$$

□

Remark 4.8. By simple reasonings, it follows that if $f : [0,1] \to [0,\infty)$ is a convex, nondecreasing function satisfying $f(x)/x \geq f(1)$ for all $x \in [0,1]$, then the function $g : (0,1] \to [0,\infty)$, $g(x) = f(x)/x$ is nonincreasing and as a consequence for f is valid the conclusion of Corollary 4.7, (i). Indeed, for simplicity, let us suppose that $f \in C^1[0,1]$ and denote $F(x) = xf'(x) - f(x)$, $x \in [0,1]$. Then $g'(x) = F(x)/x^2$, for all $x \in (0,1]$. Since the inequality $f(x)/x \geq f(1)$ can be written as $(f(1) - f(x))/(1-x) \leq f(1)$, for all $x \in [0,1)$, passing to limit with $x \to 1$, it follows $f'(1) \leq f(1)$, which implies (since f' is nondecreasing)

$$F(x) \leq xf'(1) - f(x) \leq xf'(1) - xf(1) = x\left[f'(1) - f(1)\right] \leq 0, \quad \forall x \in (0,1], \tag{4.52}$$

which means that $g(x)$ is nonincreasing.

An example of function satisfying the above conditions is $f(x) = e^x$, $x \in [0,1]$.

Analogously, if $f : [0,1] \to [0,\infty)$ is a convex, nonincreasing function satisfying $f(x)/(1-x) \geq f(0)$, then for f is valid the conclusion of Corollary 4.7, (ii). An example of function satisfying these conditions is $f(x) = e^{-x}$, $x \in [0,1]$.

5. Shape-Preserving Properties

In this section, we will present some shape preserving properties, by proving that the max-product Bernstein operator preserves the monotonicity and the quasiconvexity. First, we have the following simple result.

Lemma 5.1. *For any arbitrary function $f : [0,1] \to \mathbb{R}_+$, $B_n^{(M)}(f)(x)$ is positive, continuous on $[0,1]$, and satisfies $B_n^{(M)}(f)(0) = f(0)$, $B_n^{(M)}(f)(1) = f(1)$.*

Proof. Since $p_{n,k}(x) > 0$ for all $x \in (0,1)$, $n \in \mathbb{N}$, $k \in \{0,\ldots,n\}$, it follows that the denominator $\bigvee_{k=0}^{n} p_{n,k}(x) > 0$ for all $x \in (0,1)$ and $n \in \mathbb{N}$. However, the numerator is a maximum of continuous functions on $[0,1]$, so it is a continuous function on $[0,1]$, and this implies that $B_n^{(M)}(f)(x)$ is continuous on $(0,1)$. To prove now the continuity of $B_n^{(M)}(f)(x)$ at $x = 0$ and $x = 1$, we observe that $p_{n,k}(0) = 0$ for all $k \in \{1,2,\ldots,n\}$, $p_{n,k}(0) = 1$ for $k = 0$ and $p_{n,k}(1) = 0$

for all $k \in \{0,1,\ldots,n-1\}$, $p_{n,k}(1) = 1$ for $k = n$, which implies that $\bigvee_{k=0}^{n} p_{n,k}(x) = 1$ in the case of $x = 0$ and $x = 1$. The fact that $B_n^{(M)}(f)(x)$ coincides with $f(x)$ at $x = 0$ and $x = 1$ immediately follows from the above considerations, which proves the theorem. □

Remark 5.2. Note that because of the continuity of $B_n^{(M)}(f)(x)$ on $[0,1]$, it will suffice to prove the shape properties of $B_n^{(M)}(f)(x)$ on $(0,1)$ only. As a consequence, in the notations and proofs below, we always may suppose that $0 < x < 1$.

As in Section 4, for any $k, j \in \{0,1,\ldots,n\}$, let us consider the functions $f_{k,n,j} : [j/(n+1),(j+1)/(n+1)] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j} f\left(\frac{k}{n}\right). \tag{5.1}$$

For any $j \in \{0,1,\ldots,n\}$ and $x \in [j/(n+1),(j+1)/(n+1)]$, we can write

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x). \tag{5.2}$$

Lemma 5.3. *If $f : [0,1] \to \mathbb{R}_+$ is a nondecreasing function, then for any $k, j \in \{0,1,\ldots n\}$, $k \le j$ and $x \in [j/(n+1),(j+1)/(n+1)]$, one has $f_{k,n,j}(x) \ge f_{k-1,n,j}(x)$.*

Proof. Because $k \le j$, by the proof of Lemma 3.2, Case 2, it follows that $m_{k,n,j}(x) \ge m_{k-1,n,j}(x)$. From the monotonicity of f, we get $f(k/n) \ge f((k-1)/n)$. Thus, we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \ge m_{k-1,n,j}(x) f\left(\frac{k-1}{n}\right), \tag{5.3}$$

which proves the lemma. □

Corollary 5.4. *If $f : [0,1] \to \mathbb{R}_+$ is nonincreasing, then $f_{k,n,j}(x) \ge f_{k+1,n,j}(x)$ for any $k, j \in \{0,1,\ldots n\}$, $k \ge j$, and $x \in [j/(n+1),(j+1)/(n+1)]$.*

Proof. Because $k \ge j$, by the proof of Lemma 3.2, Case 1, it follows that $m_{k,n,j}(x) \ge m_{k+1,n,j}(x)$. From the monotonicity of f, we get $f(k/n) \ge f((k+1)/n)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \ge m_{k+1,n,j}(x) f\left(\frac{k+1}{n}\right), \tag{5.4}$$

which proves the corollary. □

Theorem 5.5. *If $f : [0,1] \to \mathbb{R}_+$ is nondecreasing, then $B_n^{(M)}(f)$ is nondecreasing.*

Proof. Because $B_n^{(M)}(f)$ is continuous on $[0,1]$, it suffices to prove that on each subinterval of the form $[j/(n+1),(j+1)/(n+1)]$, with $j \in \{0,1,\dots n\}$, $B_n^{(M)}(f)$ is nondecreasing.

So let $j \in \{0,1,\dots n\}$ and $x \in [j/(n+1),(j+1)/(n+1)]$. Because f is nondecreasing, from Lemma 5.3 it follows that

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq \cdots \geq f_{0,n,j}(x), \tag{5.5}$$

but then it is immediate that

$$B_n^{(M)}(f)(x) = \bigvee_{k \geq j}^{n} f_{k,n,j}(x), \tag{5.6}$$

for all $x \in [j/(n+1),(j+1)/(n+1)]$. Clearly that for $k \geq j$, the function $f_{k,n,j}$ is nondecreasing and since $B_n^{(M)}(f)$ is defined as the maximum of nondecreasing functions, it follows that it is nondecreasing. □

Corollary 5.6. *If $f : [0,1] \to \mathbb{R}_+$ is nonincreasing, then $B_n^{(M)}(f)$ is nonincreasing.*

Proof. Because $B_n^{(M)}(f)$ is continuous on $[0,1]$, it suffices to prove that on each subinterval of the form $[j/(n+1),(j+1)/(n+1)]$, with $j \in \{0,1,\dots n\}$, $B_n^{(M)}(f)$ is nonincreasing.

So let $j \in \{0,1,\dots n\}$ and $x \in [j/(n+1),(j+1)/(n+1)]$. Because f is nonincreasing, from Corollary 5.4, it follows that

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \cdots \geq f_{n,n,j}(x), \tag{5.7}$$

but then it is immediate that

$$B_n^{(M)}(f)(x) = \bigvee_{k \geq 0}^{j} f_{k,n,j}(x), \tag{5.8}$$

for all $x \in [j/(n+1),(j+1)/(n+1)]$. Clearly that for $k \leq j$ the function $f_{k,n,j}$ is nonincreasing and since $B_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. □

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 5.7. Let $f : [0,1] \to \mathbb{R}$ be continuous on $[0,1]$. One says that the function $f : [0,1] \to \mathbb{R}$ is quasiconvex on $[0,1]$ if it satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad \forall x, y, \lambda \in [0,1], \tag{5.9}$$

(see, e.g., [3, page 4, (iv)]).

Remark 5.8. By [5], the continuous function f is quasiconvex on $[0,1]$ equivalently means that there exists a point $c \in [0,1]$ such that f is nonincreasing on $[0,c]$ and nondecreasing on $[c,1]$. The class of quasiconvex functions includes the class of nondecreasing functions and the class of nonincreasing functions. Also, it obviously includes the class of convex functions on $[0,1]$.

Corollary 5.9. *If $f : [0,1] \to \mathbb{R}_+$ is continuous and quasiconvex on $[0,1]$, then for all $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is quasiconvex on $[0,1]$.*

Proof. If f is nonincreasing (or nondecreasing) on $[0,1]$ (i.e., the point $c = 1$ (or $c = 0$) in Remark 5.8), then by the Corollary 5.6 (or Theorem 5.5, resp.), it follows that for all $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0,1]$.

Suppose now that there exists $c \in (0,1)$, such that f is nonincreasing on $[0,c]$ and nondecreasing on $[c,1]$. Define the functions $F, G : [0,1] \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0,c]$, $F(x) = f(c)$ for all $x \in [c,1]$ and $G(x) = f(c)$ for all $x \in [0,c]$, $G(x) = f(x)$ for all $x \in [c,1]$.

It is clear that F is nonincreasing and continuous on $[0,1]$, G is nondecreasing and continuous on $[0,1]$, and $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0,1]$.

However, it is easy to show (see also Remark 2.2 after the proof of Lemma 2.1) that

$$B_n^{(M)}(f)(x) = \max\left\{B_n^{(M)}(F)(x), B_n^{(M)}(G)(x)\right\}, \quad \forall x \in [0,1], \tag{5.10}$$

where by Corollary 5.6 and Theorem 5.5, $B_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0,1]$ and $B_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0,1]$. We have two cases: 1) $B_n^{(M)}(F)(x)$ and $B_n^{(M)}(G)(x)$ do not intersect each other; 2) $B_n^{(M)}(F)(x)$ and $B_n^{(M)}(G)(x)$ intersect each other.

Case 1. We have $\max\{B_n^{(M)}(F)(x), B_n^{(M)}(G)(x)\} = B_n^{(M)}(F)(x)$ for all $x \in [0,1]$ or $\max\{B_n^{(M)}(F)(x), B_n^{(M)}(G)(x)\} = B_n^{(M)}(G)(x)$ for all $x \in [0,1]$, which obviously proves that $B_n^{(M)}(f)(x)$ is quasiconvex on $[0,1]$.

Case 2. In this case, it is clear that there exists a point $c' \in [0,1]$ such that $B_n^{(M)}(f)(x)$ is nonincreasing on $[0,c']$ and nondecreasing on $[c',1]$, which by the result in [5] implies that $B_n^{(M)}(f)(x)$ is quasiconvex on $[0,1]$ and proves the corollary. □

Remark 5.10. The preservation of the quasiconvexity by the linear Bernstein operators was proved in [6].

It is of interest to exactly calculate $B_n^{(M)}(f)(x)$ for $f(x) = e_0(x) = 1$ and for $f(x) = e_1(x) = x$. In this sense, we can state the following.

Lemma 5.11. *For all $x \in [0,1]$ and $n \in \mathbb{N}$, one has $B_n^{(M)}(e_0)(x) = 1$ and*

$$
\begin{aligned}
B_n^{(M)}(e_1)(x) &= x \cdot \frac{p_{n-1,0}(x)}{p_{n,0}(x)} = \frac{x}{1-x}, \quad \text{if } x \in \left[0, \frac{1}{n+1}\right], \\
B_n^{(M)}(e_1)(x) &= x \cdot \frac{p_{n-1,0}(x)}{p_{n,1}(x)} = \frac{1}{n}, \quad \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right], \\
B_n^{(M)}(e_1)(x) &= x \cdot \frac{p_{n-1,1}(x)}{p_{n,1}(x)} = \frac{x}{1-x} \cdot \frac{n-1}{n}, \quad \text{if } x \in \left[\frac{1}{n}, \frac{2}{n+1}\right], \\
B_n^{(M)}(e_1)(x) &= x \cdot \frac{p_{n-1,1}(x)}{p_{n,2}(x)} = \frac{2}{n}, \quad \text{if } x \in \left[\frac{2}{n+1}, \frac{2}{n}\right], \\
B_n^{(M)}(e_1)(x) &= x \cdot \frac{p_{n-1,2}(x)}{p_{n,2}(x)} = \frac{x}{1-x} \cdot \frac{n-2}{n}, \quad \text{if } x \in \left[\frac{2}{n}, \frac{3}{n+1}\right], \\
B_n^{(M)}(e_1)(x) &= x \cdot \frac{p_{n-1,2}(x)}{p_{n,3}(x)} = \frac{3}{n}, \quad \text{if } x \in \left[\frac{3}{n+1}, \frac{3}{n}\right],
\end{aligned}
\tag{5.11}
$$

and so on, in general one has

$$
\begin{aligned}
B_n^{(M)}(e_1)(x) &= \frac{x}{1-x} \cdot \frac{n-j}{n}, \quad \text{if } x \in \left[\frac{j}{n}, \frac{j+1}{n+1}\right], \\
B_n^{(M)}(e_1)(x) &= \frac{j+1}{n}, \quad \text{if } x \in \left[\frac{j+1}{n+1}, \frac{(j+1)}{n}\right],
\end{aligned}
\tag{5.12}
$$

for $j \in \{0,1,\ldots,n-1\}$.

Proof. The formula $B_n^{(M)}(e_0)(x) = 1$ is immediate by the definition of $B_n^{(M)}(f)(x)$.

To find the formula for $B_n^{(M)}(e_1)(x)$, we will use the explicit formula in Lemma 3.4 which says that

$$
\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), \quad \forall x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], \ j = 0,1,\ldots,n, \tag{5.13}
$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Indeed, since

$$
\max_{k=0,\ldots,n}\left\{p_{n,k}(x)\frac{k}{n}\right\} = \max_{k=1,\ldots,n}\left\{p_{n,k}(x)\frac{k}{n}\right\} = x \cdot \max_{k=0,\ldots,n-1}\{p_{n-1,k}(x)\}, \tag{5.14}
$$

this follows by applying Lemma 3.4 to both expressions $\max_{k=0,\ldots,n}\{p_{n,k}(x)\}$, $\max_{k=0,\ldots,n-1}\{p_{n-1,k}(x)\}$, taking into account that we get the following division of the interval $[0,1]$

$$0 < \frac{1}{n+1} \leq \frac{1}{n} \leq \frac{2}{n+1} \leq \frac{2}{n} \leq \frac{3}{n+1} \leq \frac{3}{n} \leq \frac{4}{n+1} \leq \frac{4}{n} \cdots . \tag{5.15}$$

□

Remarks. (1) The convexity of f on $[0,1]$ is not preserved by $B_n^{(M)}(f)$ as can be seen from Lemma 5.11. Indeed, while $f(x) = e_1(x) = x$ is obviously convex on $[0,1]$, it is easy to see that $B_n^{(M)}(e_1)$ is not convex on $[0,1]$.

(2) Also, if f is supposed to be starshaped on $[0,1]$ (i.e., $f(\lambda x) \leq \lambda f(x)$ for all $x, \lambda \in [0,1]$), then again by Lemma 5.11, it follows that $B_n^{(M)}(f)$ for $f(x) = e_1(x)$ is not starshaped on $[0,1]$, although $e_1(x)$ obviously is starshaped on $[0,1]$.

Despite of the absence of the preservation of the convexity, we can prove the interesting property that for any arbitrary function f, the max-product Bernstein operator $B_n^{(M)}(f)$ is piecewise convex on $[0,1]$. We present the following.

Theorem 5.12. *For any function $f : [0,1] \to [0,\infty)$, $B_n^{(M)}(f)$ is convex on any interval of the form $[j/(n+1),(j+1)/(n+1)]$, $j = 0,1,\ldots,n$.*

Proof. For any $k, j \in \{0,1,\ldots,n\}$, let us consider the functions $f_{k,n,j} : [j/(n+1),(j+1)/(n+1)] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j} f\left(\frac{k}{n}\right). \tag{5.16}$$

Clearly, we have

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x), \tag{5.17}$$

for any $j \in \{0,1,\ldots,n\}$ and $x \in [j/(n+1),(j+1)/(n+1)]$.

We will prove that for any fixed j, each function $f_{k,n,j}(x)$ is convex on $[j/(n+1),(j+1)/(n+1)]$, which will imply that $B_n^{(M)}(f)$ can be written as a maximum of some convex functions on $[j/(n+1),(j+1)/(n+1)]$.

Since $f \geq 0$, it suffices to prove that the functions $g_{k,j} : [0,1] \to \mathbb{R}$, $g_{k,j}(x) = (x/(1-x))^{k-j}$ are convex on $[j/(n+1),(j+1)/(n+1)]$.

For $k = j$, $g_{j,j}$ is constant so is convex.

For $k = j+1$, we get $g_{j+1,j}(x) = x/(1-x)$ for any $x \in [j/(n+1),(j+1)/(n+1)]$. Then $g''_{j+1,j}(x) = 2/(1-x)^3 > 0$ for any $x \in [j/(n+1),(j+1)/(n+1)]$.

For $k = j-1$, it follows that $g_{j-1,j}(x) = (1-x)/x$ for any $x \in [j/(n+1),(j+1)/(n+1)]$. Then $g''_{j-1,j}(x) = 2/x^3 > 0$ for any $x \in [j/(n+1),(j+1)/(n+1)]$.

If $k \geq j+2$, then $g''_{k,j}(x) = ((k-j)/(1-x)^4)(x/(1-x))^{k-j-2}(k-j-1+2x) > 0$ for any $x \in [j/(n+1),(j+1)/(n+1)]$.

If $k \leq j-2$, then $g''_{k,j}(x) = ((k-j)/(1-x)^4)(x/(1-x))^{k-j-2}(k-j-1+2x)$. Sice $(k-j-1+2x) \leq k-j+1 \leq -1$ for any $x \in [j/(n+1),(j+1)/(n+1)]$, it follows that $(k-j)(k-j-1+2x) > 0$, which implies $g''_{k,j}(x) > 0$ for any $x \in [j/(n+1),(j+1)/(n+1)]$.

Since all the functions $g_{k,j}$ are convex on $[j/(n+1),(j+1)/(n+1)]$, we get that $B_n^{(M)}(f)$ is convex on $[j/(n+1),(j+1)/(n+1)]$ as maximum of these functions, which proves the theorem. □

At the end of this section, let us note that although $B_n^{(M)}(f)$ does not preserve the convexity too, by using $B_n^{(M)}(f)$ it easily can be constructed new nonlinear operators which converge to the function and preserve the convexity too.

Indeed, in this sense, for example, we present the following.

Theorem 5.13. *For f belonging to the set*

$$S[0,1] = \left\{f : [0,1] \longrightarrow \mathbb{R}; f \in C^1[0,1],\ f(0) = 0,\ f \text{ is nondecreasing on } [0,1]\right\}, \tag{5.18}$$

let us define the following subadditive and positive homogenous operators (as function of f):

$$L_n(f)(x) = \int_0^x B_n^{(M)}(f')(t)dt, \quad x \in [0,1],\ n \in \mathbb{N}. \tag{5.19}$$

If $f \in S[0,1]$ is convex, then $L_n(f)(x)$ is nondecreasing and convex on $[0,1]$. In addition, if f' is concave on $[0,1]$, then the order of approximation of f through $L_n(f)$ is $\omega_1(f';1/n)$.

Proof. Indeed, since f is convex, it follows that $f'(x)$ is nondecreasing on $[0,1]$, which by Theorem 5.5 implies that $B_n^{(M)}(f')(x)$ is nondecreasing and, therefore, we get the convexity of $L_n(f)(x)$ on $[0,1]$. The monotonicity of $L_n(f)(x)$ is immediate by $f' \geq 0$ on $[0,1]$ and by the relationship $L'_n(f)(x) = B_n^{(M)}(f')(x) \geq 0$ for all $x \in [0,1]$.

Also, writing $f(x) = \int_0^x f'(t)dt$ and supposing that f' is concave, by Corollary 4.6, we get that the order of approximation of f by $L_n(f)$ is $\omega_1(f';1/n)$. In addition, $L_n(f)(x)$ obviously is of C^1-class (which is not the case of original operator $B_n^{(M)}(f)(x)$) and $L'_n(f)(x)$ converges uniformly to f' on $[0,1]$ with the same order of approximation $\omega_1(f';1/n)$. □

Remarks. (1) A simple example of function f verifying the statement of Theorem 5.13 is $f(x) = 1 - \cos x$, because in this case, we easily get that $f(0) = 0$, $f'(x) = \sin x \geq 0$, $f''(x) = \cos x \geq 0$ and $f'''(x) = -\sin x \leq 0$, for all $x \in [0,1]$.

(2) In the definition of $L_n(f)(x)$ in the above Theorem 5.13, obviously that the values $f'(k/n)$ are involved. To involve values of f only but without to loose the properties mentioned in Theorem 5.13, we can replace there $f'(k/n)$ by, for example, $(f((k+1)/n) - f(k/n))/((k+1)/n - k/n) = n[f((k+1)/n) - f(k/n)]$ or by $(f((k+1)/(n+1)) - f(k/n))/((k+1)/(n+1) - k/n)$.

6. Comparisons with the Linear Bernstein Operator

In this section, we compare the max-product Bernstein operator $B_n^{(M)}(f)$ with the linear Bernstein operator $B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n)$. First, it is known that for the linear Bernstein operator, the best possible uniform approximation result is given by the equivalence: (see [7, 8])

$$\|B_n(f) - f\| \sim \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right), \tag{6.1}$$

where $\|f\| = \sup\{|f(x)|; x \in [0,1]\}$ and $\omega_2^{\varphi}(f;\delta)$ is the Ditzian-Totik second-order modulus of smoothness given by

$$\omega_2^{\varphi}(f;\delta) = \sup\{\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \in I_h\},\ h \in [0,\delta]\}, \tag{6.2}$$

with $\varphi(x) = \sqrt{x(1-x)}$, $\delta \leq 1$ and $I_h = [h^2/(1+h^2), 1/(1+h^2)]$.

Now, if f is, for example, a nondecreasing concave polygonal line on $[0,1]$, then by simple reasonings we get that $\omega_2^{\varphi}(f;\delta) \sim \delta$ for $\delta \leq 1$, which shows that the order of approximation obtained in this case by the linear Bernstein operator is exactly $1/\sqrt{n}$. On the other hand, since such of function f obviously is a Lipschitz function on $[0,1]$ (as having bounded all the derivative numbers) by Corollary 4.6, we get that the order of approximation by the max-product Bernstein operator is less than $1/n$, which is essentially better than $1/\sqrt{n}$. In a similar manner, by Corollary 4.7 and by the Remark 4.8 after this corollary, we can produce many subclasses of functions for which the order of approximation given by the max-product Bernstein operator is essentially better than the order of approximation given by the linear Bernstein operator. In fact, the Corollaries 4.6 and 4.7 have no corespondent in the case of linear Bernstein operator. All these prove the advantages we may have in some cases, by using the max-product Bernstein operator. Intuitively, the max-product Bernstein operator has better approximation properties than its linear counterpart, for nondifferentiable functions in a finite number of points (with the graphs having some "corners"), as an example for functions defined as a maximum of a finite number of continuous functions on $[0,1]$.

On the other hand, in other cases (e.g., for differentiable functions), the linear Bernstein operator has better approximation properties than the max-product Bernstein operator, as can be seen from the formula for $B_n^{(M)}(e_1)(x)$ in Lemma 5.11. Indeed, by direct calculation can be easily proved that $\|B_n^{(M)}(e_1) - e_1\| \sim 1/n$, while it is well known that $\|B_n(e_1) - e_1\| = 0$.

Concerning now the shape-preserving properties, it is clear from Section 5 that the linear Bernstein operator has better properties. However, for some particular classes of functions, the type of construction in Theorem 5.13, combined with Corollaries 4.6 and 4.7, can produce max-product Bernstein-type operators with good preservation properties (e.g., preserving monotonicity and convexity) and giving in some cases (supposing, e.g., that f' is a concave polygonal line), the same order of approximation as the linear Bernstein operator.

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