Research Article

Generalized Alpha-Close-to-Convex Functions

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We define the classes $G_\beta(\alpha, k, \gamma)$ as follows: $f \in G_\beta(\alpha, k, \gamma)$ if and only if, for $z \in E = \{z \in \mathbb{C} : |z| < 1\}$, $|\arg\{(1 - \alpha^2 z^2)f'(z)/e^{-i\beta}\phi'(z)\}| \leq \gamma\pi/2$, $0 < \gamma \leq 1$; $\alpha \in [0, 1]$; $\beta \in (-\pi/2, \pi/2)$, where ϕ is a function of bounded boundary rotation. Coefficient estimates, an inclusion result, arclength problem, and some other properties of these classes are studied.

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1. Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. By S , K , S^* , and C denote the subclasses of A which are univalent, close-to-convex, starlike, and convex in E , respectively. Let V_k be the class of functions of bounded boundary rotation. Paate [1] showed that a function f , defined by (1.1) and $f'(z) \neq 0$, is in V_k if and only if, for $z = re^{i\theta}$,

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \leq k\pi. \quad (1.2)$$

It is geometrically obvious that $k \geq 2$ and $V_2 \equiv C$.

A class T_k of analytic functions related with the class V_k was introduced and studied in [2]. A function $f \in A$ is in T_k , $k \geq 2$, if and only if there exists a function $g \in V_k$ such that, for $z \in E$, $\operatorname{Re}\{f'(z)/g'(z)\} > 0$. It is clear that $T_2 \equiv K$.

Let P denote the class of analytic functions p defined by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (1.3)$$

with $\operatorname{Re} p(z) > 0$ for $z \in E$.

We denote $K(\gamma)$ as the class of strongly close-to-convex functions of order γ in the sense of Pommerenke [3]. A function $f \in A$ belongs to $K(\gamma)$ if and only if there exists $g \in S^*$ such that $|\operatorname{Arg}(zf'(z)/g(z))| \leq \pi\gamma/2$, for $z \in E$ and $\gamma \geq 0$.

Clearly $K(0) = C$, $K(1) = K$, and when $0 \leq \gamma < 1$, $K(\gamma)$ is a subset of K and hence contains only univalent functions. For $\gamma > 1$, $f \in K(\gamma)$ can be of infinite valence; see [4].

We now define the following.

Definition 1.1. A function $f \in A$ is said to belong to $G_\beta(\alpha, k, \gamma)$, where β is a real number, $\alpha \in \mathbb{C} : |\alpha| \leq 1$, $k \geq 2$, and $\gamma \in (0, 1]$ is called generalized alpha-close-to-convex with argument β if and only if there exists $\phi \in V_k$ such that

$$\left| \operatorname{Arg} \left\{ \frac{(1 - \alpha^2 z^2) f'(z)}{e^{-i\beta} \phi'(z)} \right\} \right| \leq \frac{\gamma\pi}{2}, \quad z \in E. \quad (1.4)$$

In (1.4), we choose this branch of argument which equals β , $|\beta| < \pi\gamma/2$, $\gamma \in (0, 1]$, when $z = 0$. We note that the condition $|\alpha| \leq 1$ implies that $G_\beta(\alpha, k, \gamma)$ is nonempty. From the normalization conditions $f'(0) = \phi'(0) = 1$, it follows from Definition 1.1 that $\operatorname{Re} e^{-i\beta} > 0$ and therefore $|\beta| < \gamma\pi/2$. Also, it follows from (1.4) that if $f \in G_\beta(\alpha, k, \gamma)$, then $f'(z) \neq 0$ for $z \in E$. Condition (1.4) is equivalent to the following $f \in G_\beta(\alpha, k, \gamma)$ if and only if there exists $p \in P$ such that

$$\frac{(1 - \alpha^2 z^2) f'(z)}{e^{-i\beta} \phi'(z)} = \left(p(z) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^\gamma, \quad \phi \in V_k. \quad (1.5)$$

We define $G(\alpha, k, \gamma)$ the class of generalized α -close-to-convex functions as

$$G(\alpha, k, \gamma) = \bigcup_{|\beta| < \pi/2} G_\beta(\alpha, k, \gamma). \quad (1.6)$$

If $\alpha = 0$ in (1.6), then the class $G(0, k, 1)$ is identical with the class T_k and $G(\alpha, 2, 1)$ is the class K of close-to-convex functions. Also $G_\beta(\alpha, 2, 1)$ in the class of close-to-convex function with argument β was defined by Goodman and Saff [5]. For details of special cases of $G_\beta(\alpha, 2, 1)$ with $\phi(z) = z$ in (1.4), we refer to [6]. The special case with $\gamma = 1 = \alpha$, $k = 2$, and $\phi(z) = z$ in (1.4) leads to the class of functions convex in the direction of the imaginary axis having special normalization; see [7].

2. Main Results

We now prove the main results as follows.

Theorem 2.1. *Let $\alpha \in [0, 1]$. Then $G(\alpha, k, \gamma) \subset G(0, k, \gamma_1)$, where*

$$\gamma_1(\gamma, \alpha) = \gamma + \frac{2}{\pi} \arcsin(\alpha^2). \quad (2.1)$$

The constant $\gamma_1(\gamma, \alpha)$ cannot be smaller.

Proof. We will use an extended version of the method given in [8] to prove this result.

For $\alpha = 0$, the result is obvious. Let $f \in G(\alpha, k, \gamma)$. By (1.4), (1.5), and (1.6), then there exists a function $\phi \in V_k$ and a function $p \in P$, $|\beta| < \pi/2$ such that

$$\frac{f'(z)}{e^{-i\beta}\phi'(z)} = \frac{(p(z) \cos(\beta/\gamma) - i \sin(\beta/\gamma))^\gamma}{1 - \alpha^2 z^2}, \quad z \in E. \quad (2.2)$$

Let $q(z) = (p(z) \cos(\beta/\gamma) - i \sin(\beta/\gamma))^\gamma$, $z \in E$. Then we have

$$\left| \operatorname{Arg} \frac{f'(z)}{e^{-i\beta}\phi'(z)} \right| = \left| \operatorname{Arg} q(z) - \operatorname{Arg}(1 - \alpha^2 z^2) \right| < \frac{\pi}{2} \left[\gamma + \frac{2}{\pi} \left| \operatorname{Arg}(1 - \alpha^2 z^2) \right| \right]. \quad (2.3)$$

We choose in (2.3) this branch of argument which is equal $-\beta$ when $z = 0$.

Since $|\operatorname{Arg}(1 - \alpha^2 z^2)| < \arcsin(\alpha^2)$, $z \in E$, we have from (2.3) $f \in G(0, k, \gamma_1)$, where γ_1 is given by (2.1). The constant $\gamma_1(\gamma, \alpha)$ cannot be smaller. Let $\alpha \in (0, 1)$ be fixed. Let us consider the point $z_0 \in \mathbb{C}$ with $|z_0| = 1$ and $\operatorname{Arg}(1 - \alpha^2 z^2) = -\arcsin(\alpha^2)$. Let $\phi_0 \in V_k$ be such that $\phi_0(z_0)$ is finite. Then, let

$$f'_0(z) = \frac{e^{-i\beta}\phi'_0(z)}{1 - \alpha^2 z^2} \left[\left(p(z) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^\gamma \right], \quad z \in E, \quad |\beta| < \frac{\pi}{2}, \quad (2.4)$$

where

$$\begin{aligned} P_0(z) &= \frac{1 + \varepsilon z}{1 - \varepsilon z}, \quad \varepsilon \in \mathbb{C}, \quad |\varepsilon| < 1, \\ \phi'_0(z) &= \frac{(1 + \delta_1 z)^{k/2-1}}{(1 + \delta_2 z)^{k/2+1}}, \quad |\delta_1| = |\delta_2| = 1. \end{aligned} \quad (2.5)$$

Now, for $z \in E$,

$$\left| \operatorname{Arg} \frac{f'_0(z)}{e^{-i\beta}\phi'_0(z)} \right| = \left| \operatorname{Arg} \left(p_0(z) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^\gamma - \operatorname{Arg}(1 - \alpha^2 z^2) \right|, \quad (2.6)$$

and $\text{Arg } e^{-i\beta} = -\beta$. Since p_0 maps the unit circle $|z| = 1$ onto imaginary axis, we may choose $\varepsilon_0, |\varepsilon_0| = 1$ such that $\varepsilon_0 \neq 1/z_0, P_0(z_0) = (1 + \varepsilon_0 z_0)/(1 - \varepsilon_0 z_0) \neq i \tan \beta, p_0(z_0) = ai, a > 0$. This means that $p_0(z_0)$ is finite and $\text{Arg } p_0(z_0) = \pi/2$. Hence

$$\text{Arg} \left[\left(p_0(z) \cos \frac{\beta}{\gamma} i \sin \frac{\beta}{\gamma} \right)^\gamma \right] = \frac{\gamma\pi}{2}. \quad (2.7)$$

Thus, from (2.4) and (2.6), we have

$$\left| \text{Arg} \frac{f'_0(z)}{e^{-i\beta} \phi'_0(z)} \right| = \frac{\pi}{2} \left[\gamma + \frac{2}{\pi} \arcsin(\alpha^2) \right] = \gamma_1 \frac{\pi}{2}. \quad (2.8)$$

Therefore γ_1 cannot be smaller.

For $\alpha = 1$, consider the sequence $\{z_n\}$, $z_n = e^{i\theta_n}$, $\theta_n \in (0, \pi/4)$, $n \in \mathbb{N} = 1$ such that $\lim_{n \rightarrow \infty} z_n = 1$. So

$$\lim_{n \rightarrow \infty} \text{Arg}(1 - z_n^2) = -\frac{\pi}{2}. \quad (2.9)$$

Let $\phi \in V_k$ with $\phi(e^{i\theta})$ finite and $\theta \in (0, \pi/2)$. The function f_1 defined as

$$f'_1(z) = \frac{e^{-i\beta} \phi'(z)}{1 - z^2} \left[\left(\left(\frac{1+z}{1-z} \right) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^\gamma \right], \quad z \in E, \quad |\beta| < \frac{\pi}{2} \quad (2.10)$$

belongs to $G(1, k, \gamma)$. Thus, from (2.9), it follows that

$$\lim_{n \rightarrow \infty} \left| \text{Arg} \frac{f'_1(z)}{e^{-i\beta} \phi'_1(z)} \right| = \lim_{n \rightarrow \infty} \left| \text{Arg} \left\{ \left(\left(\frac{1+z_n}{1-z_n} \right) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^\gamma \right\} - \text{Arg}(1 - z_n^2) \right| = (1 + \gamma) \frac{\pi}{2}. \quad (2.11)$$

This means that $\gamma_1(1, \gamma) = 1 + \gamma$ is best possible.

We note that, for $\gamma = 1, k \geq 2$, we obtain a result proved in [8]. $\square$

Theorem 2.2. Let $f \in G(\alpha, k, \gamma), \alpha \in [0, 1]$. Then, for every $\gamma \in (0, 1)$ and θ_1, θ_2 with $0 \leq \theta_2 - \theta_1 \leq 2\pi$, one has

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ 1 + r e^{i\theta} \frac{f''(r e^{i\theta})}{f'(r e^{i\theta})} \right\} d\theta > - \left(\gamma + \frac{k}{2} - 1 - \Re \right) \pi, \quad (2.12)$$

where

$$\begin{aligned} \Re &= \frac{1}{\pi} \{ \psi(r, \theta_2) - \psi(r, \theta_1) \}, \\ \psi(r, \theta) &= -\text{Arg} \left(1 - \alpha^2 \gamma^2 e^{2i\theta} \right) = \arctan \frac{\alpha^2 r^2 \sin 2\theta}{1 - \alpha^2 r^2 \cos 2\theta}. \end{aligned} \quad (2.13)$$

Proof. To prove this result, we shall essentially use the similar method given by Kaplan [9].

Let $f \in G(\alpha, k, \gamma)$ for fixed $\alpha \in [0, 1]$. Then f satisfies the inequality (1.4) for some β , $|\beta| < \pi/2$ and $\phi \in V_k$. Let $\phi_1(z) = \phi(z)e^{i\beta}$, $z \in E$. Since $f'(z) \neq 0$, $\phi_1'(z) \neq 0$ for $z \in E$, we can define, for $z = re^{i\theta}$, $r \in (0, 1)$, θ is a real number, the following:

$$\wp(r, \theta) = \text{Arg} \left\{ \left(1 - \alpha^2 r^2 e^{2i\theta} \right) f'(re^{i\theta}) \right\}, \quad (2.14)$$

$$V(r, \theta) = \text{Arg} \phi_1'(re^{i\theta}), \quad (2.15)$$

$$\psi(r, \theta) = \text{Arg} \left\{ \left(1 - \alpha^2 r^2 e^{2i\theta} \right) re^{i\theta} f'(re^{i\theta}) \right\} = \wp(r, \theta) + \theta, \quad (2.16)$$

$$V(r, \theta) = \text{Arg} \left\{ re^{i\theta} \phi_1'(re^{i\theta}) \right\} = \tau(r, \theta) + \theta. \quad (2.17)$$

The functions $\wp$, τ , ψ , and V are continuous and periodic with period 2π . From (1.4), we can choose the branches of argument of $\wp(z)$ and $\tau(z)$ as

$$|\wp(r, \theta) - \tau(r, \theta)| < \frac{\gamma\pi}{2}, \quad \gamma \in [0, 1]. \quad (2.18)$$

Now, for $\phi_1 \in V_k$, it is known [10] that, for $\theta_1 < \theta_2$, $z = re^{i\theta}$,

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{(z\phi_1'(z))'}{\phi_1'(z)} \right\} d\theta > -\left(\frac{k}{2} - 1\right)\pi. \quad (2.19)$$

From (2.16), (2.17), and (2.19), we have

$$\begin{aligned} \psi(r, \theta_2) - \psi(r, \theta_1) &= \wp(r, \theta_2) + \theta_2 - \wp(r, \theta_1) - \theta_1 \\ &= [\wp(r, \theta_2) - \tau(r, \theta_2)] + [\tau(r, \theta_2) + \theta_2 - \tau(r, \theta_1) - \theta_1] - [\wp(r, \theta_1) - \tau(r, \theta_1)] \\ &> \gamma\pi - \left(\frac{k}{2} - 1\right)\pi = -\left(r + \frac{k}{2} - 1\right)\pi. \end{aligned} \quad (2.20)$$

Moreover, by (2.16), we have

$$\frac{d}{d\theta} \psi(r, \theta) = \frac{d}{d\theta} \text{Arg} \left(1 - \alpha^2 r^2 e^{2i\theta} \right) + \text{Re} \left\{ 1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right\}, \quad (2.21)$$

and therefore, from (2.20)

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ 1 + r e^{i\theta} \frac{f''(r e^{i\theta})}{f'(r e^{i\theta})} \right\} d\theta &= \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} \psi(r, \theta) d\theta - \int_{\theta_1}^{\theta_2} \operatorname{Arg} (1 + \alpha^2 r^2 e^{2i\theta}) d\theta \\ &> - \left(\gamma + \frac{k}{2} - 1 \right) \pi - [\psi(r, \theta_1) - \psi(r, \theta_2)] \\ &= - \left(\gamma + \frac{k}{2} - 1 - \Re \right) \pi, \end{aligned} \quad (2.22)$$

where $\psi(r, \theta)$ and $\Re$ are defined by (2.13). This completes the proof. $\square$

We note that, for $\gamma = 1$, $k = 2$, $\alpha = 0$, we obtain the necessary condition for f to be close-to-convex in E , proved in [9].

Remark 2.3. From Theorem 2.2, we can interpret some geometrical meaning for the functions in $G(\alpha, k, \gamma)$. For simplicity, let us suppose that the image domain is bounded by an analytic curve Γ . At a point on Γ , the outward drawn normal turns back at most $-(\gamma + k/2 - 1 - \Re)\pi$, where A is given by (2.13). This is a necessary condition for a function f to belong to $G(\alpha, k, \gamma)$. Goodman [4] showed that if $f \in K(\sigma)$, $\sigma \geq 0$, then, for $z = r e^{i\theta}$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $\int_{\theta_1}^{\theta_2} \operatorname{Re}((zf'(z))' / f'(z)) d\theta > -\sigma\pi$.

We note that $f \in G(\alpha, k, \gamma)$ is univalent for $k + 2(\gamma - \Re) \leq 4$, since

$$G(\alpha, k, \gamma) \subset K \left(\gamma + \frac{k}{2} - 1 - \Re \right). \quad (2.23)$$

The functions in $K(\gamma + k/2 - 1 - \Re)$ need not even be finitely valent in E for $k + 2(\gamma - \Re) > 4$.

Remark 2.4. From Theorem 2.2 and [11, Lemma 1.3] by Pommerenke, it follows that $G(\alpha, k, \gamma)$ is a linearly invariant family of order $(\gamma + k/2 - \Re)$. Therefore, the image of E under functions in $G(\alpha, k, \gamma)$ contains the schlicht disk $|z| < 1/(k + 2(\gamma - \Re))$.

Theorem 2.5. Let $f \in G_\beta(\alpha, k, \gamma)$, $\gamma \in (0, 1)$, $|\beta| < \pi/2$, be of the form (1.1). Then $|a_2| \leq k/2 + ((1 + \gamma)/2) |\cos(\beta/\gamma)|$. This estimate is best possible, extremal function being $f_0(z)$ defined by (2.4).

Proof. Let $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ in (1.5).

Now, it is known that, for functions p of positive real part with $\gamma \in (0, 1)$, p^γ is subordinate to $((1 + z)/(1 - z))^\gamma$. Also $|b_2| \leq k/2$, see [1, 12]. Therefore, from (1.5), we have $2a_2 = 2b_2 + (e^{-i\beta} \cos(\beta/\gamma))(1 + \gamma)$, and this gives us the required result. $\square$

Remark 2.6. Let $f \in G(\alpha, k, \gamma)$, for $2 \leq k \leq 4 - 2(\gamma - \Re)$, and be given by (1.1). Then f is univalent in E by Remark 2.3 and $w_0 f(z)/(w_0 - f(z))$ is univalent in E for $w_0 \neq 0$, $w_0 \neq f(z)$. Now

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0} \right) z^2 + \cdots, \quad (2.24)$$

and therefore $|a_2 + 1/w_0| \leq 2$ and so $|1/w_0| \geq 2/(4+k+(1+\gamma)\cos(\beta/\gamma))$, on using Theorem 2.5. Hence it follows that the image of E under $f \in G(\alpha, k, \gamma)$ with $2 \leq k + 2(\gamma - \Re) \leq 4$ contains the schlicht disc $|z| < 2/(4+k+(1+\gamma)\cos(\beta/\gamma))$.

From Remark 2.3, and the results proved for the class $K(\sigma)$, $\sigma \geq 0$ in [4], we at once have the following.

Theorem 2.7. Let $f \in G(\alpha, k, \gamma)$ and be given by (1.1). Let F_σ be defined by

$$F_\sigma(z) = \frac{1}{2(\sigma+1)} \left[\left(\frac{1+z}{1-z} \right)^{\sigma+1} - 1 \right] = z + \sum_{n=2}^{\infty} A_n(\sigma) z^n, \quad (2.25)$$

where $\sigma = (\gamma + k/2 - 1 - \Re)$, and $\Re$ is given by (2.13). Clearly $F_\sigma \in G(\alpha, k, r)$.

(i) Denote by $L(r, f)$ the length of the image of the circle $|z| = r$ under f and by $A(r, f)$ the area of $f(|z| \leq r)$. Then, for $0 \leq r < 1$

- (a) $L(r, f) \leq L(r, F_\sigma)$,
- (b) $A(r, f) \leq A(r, F_\sigma)$.

(ii) For $z = re^{i\theta}$, $0 \leq r < 1$, $|f(z)| = (1/2(\sigma+1))[(1+z)/(1-z))^{\sigma+1} - 1]$.

The function F_σ , defined by (2.25), shows that this upper bound is sharp.

Theorem 2.8. Let $f \in G(\alpha, k, \gamma)$. Then, for $0 < r < 1$, $\alpha, r \in (0, 1)$, $k \geq 2$,

$$L(r, f) \leq c(\alpha, k, r) \left(\frac{1}{1-r} \right)^{k/2+\gamma}, \quad (2.26)$$

where $c(\alpha, k, r)$ is a constant depending upon α , k , and γ only.

Proof. With $z = re^{i\theta}$,

$$\begin{aligned} L(r, f) &= \int_0^{2\pi} |zf'(z)| d\theta \\ &= \int_0^{2\pi} r \left| \frac{e^{-i\beta} \phi'(z) (p(z) \cos(\beta/\gamma) - i \sin(\beta/\gamma))^{\gamma}}{1 - \alpha^2 z^2} \right| d\theta, \quad \phi \in V_k, p \in P, z \in E. \end{aligned} \quad (2.27)$$

For $\phi \in V_k$, it is known [10] that there exist $s_1, s_2 \in S^*$ such that

$$z\phi'(z) = \frac{(s_1(z))^{k/4+1/2}}{(s_2(z))^{k/4-1/2}}. \quad (2.28)$$

Also, for $p \in P$ we have for $z = re^{i\theta}$,

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2} \quad (2.29)$$

(see [13]). Now, from (2.27), (2.28), and (2.29), we have

$$\begin{aligned} L(r, f) &\leq \frac{c_1(\alpha, k, \gamma)}{r^{(k/4-1/2)}} \left(\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{(k/4+1/2)(2/(2-\gamma))} d\theta \right)^{(2-\gamma)/2} \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{\gamma/2} \\ &\leq c(\alpha, k, \gamma) \left(\frac{1}{1-r} \right)^{k/2+\gamma}, \end{aligned} \quad (2.30)$$

where we have used distortion theorems, subordination for the starlike functions, and Holder's inequality, and c and c_1 are constants. $\square$

Theorem 2.9. Let $f \in G(\alpha, k, \gamma)$ and be given by (1.1). Then, for $\alpha, \gamma \in [0, 1]$, $k \geq 2$, one has $a_n = o(1)n^{k/2+\gamma-1}$, ($n \rightarrow \infty$) where $o(1)$ is a constant depending only on k, α , and γ .

Proof. With $z = re^{i\theta}$, Cauchy's theorem gives

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} z f'(z) e^{-in\theta} d\theta. \quad (2.31)$$

Thus

$$n|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |z f'(z)| d\theta = (1/2\pi r^n) L(r, f). \quad (2.32)$$

Using Theorem 2.8 and putting $r = 1 - 1/n$, we prove this result. $\square$

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References

- [1] V. Paate, "Über die konforme abbildung von gebieten deren rander von beschränkter drehung sind," *Annales Academiae Scientiarum Fennicae A*, vol. 33, 77 pages, 1933.
- [2] K. Inayat Noor, "On a generalization of close-to-convexity," *International Journal of Mathematics and Mathematical Sciences*, vol. 6, no. 2, pp. 327–334, 1983.
- [3] Ch. Pommerenke, "On close-to-convex analytic functions," *Transactions of the American Mathematical Society*, vol. 114, pp. 176–186, 1965.
- [4] A. W. Goodman, "On close-to-convex functions of higher order," *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica*, vol. 25, pp. 17–30, 1972.
- [5] A. W. Goodman and E. B. Saff, "On the definition of a close-to-convex function," *International Journal of Mathematics and Mathematical Sciences*, vol. 2, no. 1, pp. 125–132, 1978.
- [6] A. Lecko, "Some classes of close-to-convex functions," *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka i Fizyka*, vol. 60, no. 9, pp. 62–70, 1989.
- [7] W. Hengartner and G. Schober, "On Schlicht mappings to domains convex in one direction," *Commentarii Mathematici Helvetici*, vol. 45, pp. 303–314, 1970.
- [8] A. Lecko, "Some quasi-close-to-convex functions," *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka i Fizyka*, vol. 101, no. 14, pp. 15–31, 1992.

- [9] W. Kaplan, "Close-to-convex schlicht functions," *The Michigan Mathematical Journal*, vol. 1, pp. 169–185, 1952.
- [10] D. A. Brannan, "On functions of bounded boundary rotation," *Proceedings of the Edinburgh Mathematical Society*, vol. 2, pp. 339–347, 1969.
- [11] Ch. Pommerenke, "Linear-invariante familien analytischer funktionen I," *Mathematische Annalen*, vol. 155, no. 2, pp. 108–154, 1964.
- [12] A. W. Goodman, *Univalent Functions*, vol. 1-2, Polygonal Publishing House, Washington, DC, USA, 1983.
- [13] Ch. Pommerenke, "On starlike and close-to-convex functions," *Proceedings of the London Mathematical Society*, vol. 13, pp. 290–304, 1963.