Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 989603, 15 pages
doi:10.1155/2009/989603

Research Article

On Certain Sufficient Condition Involving Gaussian Hypergeometric Functions

H. Silverman,[1] Thomas Rosy,[2] and S. Kavitha[2]

[1] *Department of Mathematics, College of Charleston, Charleston, SC 29424, USA*
[2] *Department of Mathematics, Madras Christian College, East Tambaram, Chennai 600 059, India*

Correspondence should be addressed to H. Silverman, silvermanh@cofc.edu

Received 25 May 2009; Accepted 26 October 2009

Recommended by Teodor Bulboacă

The authors define a new subclass of $\mathcal{A}$ of functions involving complex order in the open unit disk $\mathbb{U}$. For this new class, we obtain certain inclusion properties involving the Gaussian hypergeometric functions.

1. Introduction and Motivation

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}. \tag{1.2}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ $(0 \le \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \le \alpha < 1). \tag{1.3}$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order α in $\mathbb{U}$ $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leq \alpha < 1). \tag{1.4}$$

The class of convex functions is denoted by the class $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha). \tag{1.5}$$

A function $f \in \mathcal{A}$ is said to be in the class UCV of uniformly convex functions in $\mathbb{U}$ if f is a normalized convex function in $\mathbb{U}$ and has the property that, for every circular arc δ contained in the unit disk $\mathbb{U}$, with center ζ also in U, the image curve $f(\delta)$ is a convex arc. The function class UCV was introduced by Goodman [1].

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.6}$$

Furthermore, we denote by $k - UCV$ and $k - ST$ two interesting subclasses of $\mathcal{S}$ consisting, respectively, of functions which are k-uniformly convex and k-starlike in $\mathbb{U}$. Thus, we have

$$\begin{aligned} k - UCV &:= \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \ (z \in \mathbb{U};\ 0 \leq k < \infty) \right\}, \\ k - ST &:= \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \ (z \in \mathbb{U};\ 0 \leq k < \infty) \right\}. \end{aligned} \tag{1.7}$$

The class $k - UCV$ was introduced by Kanas and Wiśniowska [2], where its geometric definition and connections with the conic domains were considered. The class $k - ST$ was investigated in [3]. In fact, it is related to the class $k - UCV$ by means of the well-known Alexander equivalence between the usual classes of convex and starlike functions; see also the work of Kanas and Srivastava [4] for further developments involving each of the classes $k - UCV$ and $k - ST$. In particular, when $k = 1$, we obtain

$$1 - UCV \equiv UCV, \qquad 1 - ST = SP, \tag{1.8}$$

where UCV and SP are the familiar classes of uniformly convex functions and parabolic starlike functions in $\mathbb{U}$, respectively (see for details, [1, 5]). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [6] presented a systematic and unified study of the classes UCV and SP.

A function $f \in \mathcal{A}$ is said to be in the class $P_\gamma^\tau(A,B) \subset \mathcal{A}$ if it satisfies the inequality

$$\left| \frac{f'(z) + \gamma z f''(z) - 1}{(A-B)\tau - B\left[f'(z) + \gamma z f''(z) - 1\right]} \right| < 1 \quad \left(z \in \mathbb{U};\ \tau \in \mathbb{C} \setminus \{0\},\ -1 \le B < A \le 1,\ 0 \le \gamma < 1\right). \tag{1.9}$$

The class $P_0^\tau(A,B)$ was introduced earlier by Dixit and Pal [7]. Two of the many interesting subclasses of the class $P_\gamma^\tau(A,B)$ are worthy of mention here. First of all, by setting

$$\gamma = 0, \quad \tau = e^{i\eta}\cos\eta\left(-\frac{\pi}{2} < \eta < \frac{\pi}{2}\right), \quad A = 1 - 2\beta \quad (0 \le \beta < 1), \quad B = -1, \tag{1.10}$$

the class $P_\gamma^\tau(A,B)$ reduces essentially to the class $R_\eta(\beta)$ introduced and studied by Ponnusamy and Rønning [8], where

$$R_\eta(\beta) = \left\{ f \in \mathcal{A} : \Re\left(e^{i\eta}(f'(z) - \beta)\right) > 0 \ \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \eta < \frac{\pi}{2},\ 0 \le \beta < 1\right) \right\}. \tag{1.11}$$

Secondly, if we put

$$\gamma = 0, \quad \tau = 1, \quad A = \beta, \quad B = -\beta \quad (0 < \beta \le 1), \tag{1.12}$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{U};\ 0 < \beta \le 1) \tag{1.13}$$

which was studied by (among others) Padmanabhan [9] and Caplinger and Causey [10].

Finally, many of the authors have also studied the class $P_\gamma^1(A,B)$. For details of these works one can refer to the works of Ding Gong [11], R. Singh and S. Singh [12], Owa and Wu [13], and also the references cited by them. Although, many mapping properties of the class $P_\gamma^1(A,B)$ have been studied by these authors, they did not study any mapping properties involving the hypergeometric functions.

The Gaussian hypergeometric function $F(a,b;c,z)$, $z \in \mathbb{U}$ is given by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.14}$$

is the solution of the homogeneous hypergeometric differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0 \tag{1.15}$$

and has rich applications in various fields such as conformal mappings, quasiconformal theory, and continued fractions.

Here, a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer n, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. In the case of $c = -k$, $k = 0, 1, 2, \ldots$, $F(a,b;c;z)$ is defined if $a = -j$ or $b = -j$, where $j \leq k$. In this situation, $F(a,b;c;z)$ becomes a polynomial of degree j in z. Results regarding $F(a,b;c;z)$ when $\Re\,(c-a-b)$ is positive, zero, or negative are abundant in the literature. In particular when $\Re\,(c-a-b) > 0$, the function is bounded. This and the zero balanced case $\Re\,(c-a-b) = 0$ are discussed in detail by many authors (see [14, 15]). The hypergeometric function $F(a,b;c;z)$ has been studied extensively by various authors and it plays an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters a, b, and c. We refer to [8, 16–19] and references therein for some important results.

In particular, the close-to-convexity (in turn the univalency), convexity, starlikeness, (for details on these technical terms we refer to [5]), and various other properties of these hypergeometric functions were examined based on the conditions on a, b, and c in [8]. For more interesting properties of hypergeometric functions, one can also refer to [20, 21].

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$ and $g(z)$ univalent. Then we say that $f(z)$ is subordinate to $g(z)$ written as $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For $f \in \mathcal{A}$, we recall that the operator $I_{a,b,c}(f)$ of Hohlov [22] which maps $\mathcal{A}$ into itself defined by

$$I_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z), \tag{1.16}$$

where $*$ denotes usual Hadamard product of power series. Therefore, for a function f defined by (1.1), we have

$$I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n. \tag{1.17}$$

Using the integral representation,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{dt}{(1-tz)^a}, \quad \Re(c) > R(b) > 0, \tag{1.18}$$

we can write

$$\left[I_{a,b,c}(f)\right](z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} dt * \frac{z}{(1-tz)^a}. \tag{1.19}$$

When $f(z)$ equals the convex function $z/(1-z)$, then the operator $I_{a,b,c}(f)$ in this case becomes $zF(a,b;c;z)$. For $a = 1$, $b = 1+\delta$, $c = 2+\delta$ with $\Re(\delta) > -1$ then the convolution operator $I_{a,b,c}(f)$ turns into Bernardi operator

$$B_f(z) = \left[I_{a,b,c}(f)\right](z) = \frac{1+\delta}{z^\delta} \int_0^1 t^{\delta-1} f(t)\,dt. \tag{1.20}$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.

Let $0 \leq k < \infty$, and let $f \in \mathcal{A}$ be of the form (1.1). If $f \in k - UCV$, then the following coefficient inequalities hold true (cf. [2]):

$$|a_n| \leq \frac{(P_1)_{n-1}}{(1)_n}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{1.21}$$

where $P_1 = P_1(k)$ is the coefficient of z in the function

$$p_k(z) = 1 + \sum_{n=1}^{\infty} P_n(k) z^n, \tag{1.22}$$

which is the extremal function for the class $P(p_k)$ related to the class $k - UCV$ by the range of the expression

$$1 + \frac{zf''(z)}{f'(z)} \quad (z \in \mathbb{U}), \tag{1.23}$$

where $P_1 = P_1(k)$ is given, as above, by (1.22).

Similarly, if f of the form (1.1) belong to the class $k - ST$, then (cf. [3])

$$|a_n| \leq \frac{(P_1)_{n-1}}{(1)_{n-1}}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{1.24}$$

where $P_1 = P_1(k)$ is given, as above by (1.22).

2. Properties of $P_\gamma^\tau(A, B)$

Theorem 2.1. *Let $f \in \mathcal{S}$ and be of the form* (1.1). *If $f \in P_\gamma^\tau(A, B)$, then*

$$|a_n| \leq \frac{(A - B)|\tau|}{n(1 + \gamma(n - 1))}. \tag{2.1}$$

The estimate is sharp.

Proof. Since $f \in P_\gamma^\tau(A, B)$, we have

$$1 + \frac{1}{\tau}\left[f'(z) + \gamma z f''(z) - 1\right] = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.2}$$

where $w(z)$ is analytic in $\mathbb{U}$ and satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. Hence, we have

$$\frac{1}{\tau}\left[f'(z) + \gamma z f''(z) - 1\right] = w(z)\left[(A - B) - \frac{B}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right)\right]. \tag{2.3}$$

Using $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $w(z) = \sum_{n=1}^{\infty} b_n z^n$, we have

$$\left\{(A-B) - \frac{B}{\tau}\sum_{n=2}^{\infty}(1+\gamma(n-1))na_n z^{n-1}\right\}\left[\sum_{n=1}^{\infty} b_n z^n\right] = \frac{1}{\tau}\left[\sum_{n=2}^{\infty}(1+\gamma(n-1))na_n z^{n-1}\right]. \tag{2.4}$$

By equating the coefficients, we observe that the coefficient a_n in the right-hand side depends only on $a_2, a_3, \ldots, a_{n-1}$ on the left-hand side of the above expression. This gives

$$\begin{aligned}&\left[(A-B) - \frac{B}{\tau}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]na_n z^{n-1}\right)\right]w(z)\\ &\quad = \frac{1}{\tau}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]na_n z^{n-1}\right) + \sum_{n=k+1}^{\infty} d_n z^{n-1}.\end{aligned} \tag{2.5}$$

By using $|w(z)| < 1$, we get

$$\begin{aligned}&\left|(A-B) - \frac{B}{\tau}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]na_n z^{n-1}\right)\right|\\ &\quad \geq \left|\frac{1}{\tau}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]na_n z^{n-1}\right) + \sum_{n=k+1}^{\infty} d_n z^{n-1}\right|.\end{aligned} \tag{2.6}$$

Squaring both sides of (2.6) and integrating around $|z| = r$, $0 < r < 1$, we obtain

$$\begin{aligned}&(A-B)^2 + \frac{B^2}{|\tau|^2}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]^2 n^2|a_n|^2 r^{2n-2}\right)\\ &\quad \geq \frac{1}{|\tau|^2}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]^2 n^2|a_n|^2 r^{2n-2}\right) + \sum_{n=k+1}^{\infty}|d_n|^2 r^{2n-2}.\end{aligned} \tag{2.7}$$

By letting $r \to 1$, we conclude that

$$(A-B)^2 + \frac{B^2}{|\tau|^2}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]^2 n^2|a_n|^2\right) \geq \frac{1}{|\tau|^2}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]^2 n^2|a_n|^2\right) \tag{2.8}$$

or

$$\left(\sum_{n=2}^{k}[1+\gamma(n-1)]^2 n^2|a_n|^2\right) \leq (A-B)^2|\tau|^2 + B^2\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]^2 n^2|a_n|^2\right). \tag{2.9}$$

By making use of the fact that $-1 \leq B < 1$, we get

$$[1+\gamma(n-1)]^2 n^2|a_n|^2 \leq (A-B)^2|\tau|^2. \tag{2.10}$$

This gives

$$|a_n| \leq \frac{(A-B)|\tau|}{n\big(1+\gamma(n-1)\big)}, \quad n = 2, 3, \ldots. \tag{2.11}$$

The result is sharp for the function

$$f(z) = \begin{cases} \dfrac{z}{\tau}\displaystyle\iint_0^1 u^{1/\gamma-1}\frac{1+A\tau(uvz)^{n-1}+B(1-\tau)(uvz)^{n-1}}{1+B(uvz)^{n-1}}\,du\,dv, & \gamma > 0, \\ z\displaystyle\int_0^1 \frac{1+A\tau(uz)^{n-1}+B(1-\tau)(uz)^{n-1}}{1+B(uz)^{n-1}}\,du, & \gamma = 0. \end{cases} \tag{2.12}$$

□

Theorem 2.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then a sufficient condition for $f \in P_\gamma^\tau(A,B)$ is*

$$\sum_{n=2}^{\infty} n(1+|B|)\big[1+\gamma(n-1)\big]|a_n| \leq (A-B)|\tau|. \tag{2.13}$$

The result is sharp for the function

$$f(z) = z + \frac{(A-B)|\tau|}{n(1+|B|)\big[1+\gamma(n-1)\big]}z^n, \quad n \geq 2. \tag{2.14}$$

Proof. In view of (2.13),

$$\begin{aligned} &\left|\frac{1}{\tau}\big(f'(z)+\gamma z f''(z)-1\big)\right| - \left|(A-B) - \frac{B}{\tau}\big(f'(z)+\gamma z f''(z)-1\big)\right| \\ &\quad = \left|\frac{1}{\tau}\left(1+\sum_{n=2}^{\infty} n a_n z^{n-1} + \gamma\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} - 1\right)\right| \\ &\qquad - \left|(A-B) - \frac{B}{\tau}\left(1+\sum_{n=2}^{\infty} n a_n z^{n-1} + \gamma\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} - 1\right)\right| \\ &\quad \leq \frac{1}{|\tau|}\sum_{n=2}^{\infty} n\big(1+\gamma(n-1)\big)|a_n||z|^{n-1} \\ &\qquad - \left\{(A-B) - \frac{|B|}{|\tau|}\left(\sum_{n=2}^{\infty} n\big(1+\gamma(n-1)\big)\right)|a_n||z|^{n-1}\right\} \end{aligned} \tag{2.15}$$

which is clearly less than or equal to zero for all $|z| = r$, $0 < r < 1$. Letting $r \to 1$, we get

$$\left|\frac{f'(z)+\gamma z f''(z)-1}{(A-B)\tau - B\big[f'(z)+\gamma z f''(z)-1\big]}\right| < 1. \tag{2.16}$$

Thus, $f \in P_\gamma^\tau(A,B)$. □

3. Results Involving Gaussian Hypergeometric Function

Theorem 3.1. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + 2$. Then a sufficient condition for the function $zF(a,b;c;z)$ to be in the class $P_\gamma^\tau(A,B)$ is that*

$$S \leq \frac{(A-B)|\tau|}{1+|B|} + 1, \tag{3.1}$$

where

$$\begin{aligned} S = {} & \frac{\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\ & \times \left[\gamma|ab|(1+|a|)(1+|b|) + (1+2\gamma)|ab|(c-|a|-|b|-2) + (c-|a|-|b|-2)(c-|a|-|b|-1)\right]. \end{aligned} \tag{3.2}$$

Proof. $zF(a,b;c;z)$ has the series representation given by

$$zF(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n. \tag{3.3}$$

In view of Theorem 2.2, it suffices to show that

$$S(a,b,c,\gamma) := \sum_{n=2}^{\infty} n(1+|B|)(1+\gamma(n-1)) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \leq (A-B)|\tau|. \tag{3.4}$$

From the fact that $|(a)_n| \leq (|a|)_n$, we observe that c is real and positive, under the hypothesis

$$S(a,b,c,\gamma) \leq \sum_{n=2}^{\infty} n[1+\gamma(n-1)] \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \tag{3.5}$$

By writing $n[1+\gamma(n-1)]$ as, $\gamma(n-1)(n-2) + (n-1)(1+2\gamma) + 1$, we get

$$\begin{aligned} S(a,b,c,\gamma) \leq {} & \gamma \sum_{n=2}^{\infty} (n-1)(n-2) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ & + (1+2\gamma) \sum_{n=2}^{\infty} (n-1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ = {} & \gamma \sum_{n=3}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} + (1+2\gamma) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \end{aligned} \tag{3.6}$$

Using the fact that

$$(a)_n = a(a+1)_{n-1}, \tag{3.7}$$

it is easy to see that

$$S(a,b,c,\gamma) \le \gamma \frac{|ab|(1+|a|)(1+|b|)}{c(1+c)} \sum_{n=3}^{\infty} \frac{(2+|a|)_{n-3}(2+|b|)_{n-3}}{(2+c)_{n-3}(1)_{n-3}} + (1+2\gamma)\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(1+|a|)_{n-2}(1+|b|)_{n-2}}{(1+c)_{n-2}(1)_{n-2}} + \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \tag{3.8}$$

From (1.14),

$$S(a,b,c,\gamma) \le \gamma \frac{|ab|(1+|a|)(1+|b|)}{c(1+c)} F(2+|a|,2+|b|;2+c;1) + (1+2\gamma)\frac{|ab|}{c}F(1+|a|,1+|b|;1+c;1) + F(|a|,|b|;c;1) - 1. \tag{3.9}$$

By using the Gauss summation theorem

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \tag{3.10}$$

we get

$$S(a,b,c,\gamma) \le \gamma \frac{|ab|(1+|a|)(1+|b|)}{c(1+c)} \frac{\Gamma(c-|a|-|b|-2)\Gamma(c+2)}{\Gamma(c-|a|)\Gamma(c-|b|)} + (1+2\gamma)\frac{|ab|}{c}\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)} + \frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1. \tag{3.11}$$

Equation (3.4) now follows by an application of (3.1) and (3.2). □

Theorem 3.2. *Let $a,b \in \mathbb{C}\setminus\{0\}$. Also, let c be a real number such that $c > |a|+|b|$. If $f \in P_\gamma^\tau(A,B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \le \frac{1}{1+|B|} + 1 \tag{3.12}$$

is satisfied, then $zF(a,b;c;z^k) * f(z) \in P_\gamma^\tau(A,B)$, where $k \in \mathbb{N}$.

Proof. Let f be of the form (1.1) belong to the class $P_\gamma^\tau(A,B)$. By virtue of Theorem 2.2, it suffices to show that

$$S_0 := \sum_{n=2}^{\infty}(k(n-1)+1)(1+|B|)(1+\gamma k(n-1))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_{k(n-1)+1}\right| \le (A-B)|\tau|. \tag{3.13}$$

Taking into account inequality (2.1) and the relation $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\begin{aligned} S_0 &\leq (1+|B|)\sum_{n=2}^{\infty}(k(n-1)+1)(1+\gamma k(n-1))\frac{(A-B)|\tau|}{(k(n-1)+1)(1+\gamma k(n-1))}\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \\ &\leq (1+|B|)(A-B)|\tau|\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= (1+|B|)(A-B)|\tau|(F(|a|,|b|;c;1)-1) \end{aligned} \tag{3.14}$$

which is bounded previously by $(A-B)|\tau|$, in view of inequality (3.12). □

Repeating the previous reasoning for $b = \overline{a}$, we can improve the assertion of Theorem 3.2 as follows.

Theorem 3.3. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > \max\{0, 2\Re(a)\}$. If $f \in P_\gamma^\tau(A,B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-2\Re(a))}{\Gamma(c-|a|)\Gamma(c-|\overline{a}|)} \leq \frac{1}{1+|B|}+1 \tag{3.15}$$

is satisfied, then $zF(a,\overline{a};c;z^k) * f(z) \in P_\gamma^\tau(A,B)$, where $k \in \mathbb{N}$.

In the special case when $b = 1$, Theorem 3.2 immediately yields the following new result.

Theorem 3.4. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a|+1$. If $f \in P_\gamma^\tau(A,B)$, and if the inequality*

$$\frac{c-1}{c-|a|-1} \leq \frac{1}{1+|B|}+1 \tag{3.16}$$

is satisfied, then $zF(a,1;c;z^k) * f(z) \in P_\gamma^\tau(A,B)$, where $k \in \mathbb{N}$.

Theorem 3.5. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a|+|b|+3$. If $f \in \mathcal{S}$, and if the inequality*

$$\begin{aligned} &\frac{|a||b|\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[4+\frac{(1+|a|)(1+|b|)}{c-|a|-|b|-2}\left(5+\frac{(2+|a|)(2+|b|)}{c-|a|-|b|-3}\right)\right] \\ &\quad+\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+3\frac{|a||b|}{c-|a|-|b|-1}+\frac{|a||b|(1+|a|)(1+|b|)}{c(1+c)}\right] \\ &\leq \frac{(A-B)|\tau|}{1+|B|}+1 \end{aligned} \tag{3.17}$$

is satisfied, then $I_{a,b,c}(f) \in P_\gamma^\tau(A,B)$.

Proof. Let $f \in \mathcal{S}$. Applying the well-known estimate for the coefficients of the functions $f \in \mathcal{S}$, due to de Branges [23], we need to show that

$$\sum_{n=2}^{\infty} n^2 \left[1+\gamma(n-1)\right] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \leq \frac{(A-B)|\tau|}{1+|B|}. \tag{3.18}$$

The left-hand side of (3.18) can be written as

$$\sum_{n=2}^{\infty} n^2 \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| + \gamma \sum_{n=2}^{\infty} n^2 (n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right|. \tag{3.19}$$

The second expression of (3.19), by virtue of the triangle inequality for the pochhammer symbol $|(a)_{n-1}| \leq (|a|)_{n-1}$, is less than or equal to

$$\gamma \sum_{n=2}^{\infty} n^2 (n-1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \gamma \sum_{n=1}^{\infty} (n+1)^2 \frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}} =: S_1. \tag{3.20}$$

Now, making use of the relation (3.7), we get

$$\begin{aligned}
S_1 &= \gamma \frac{|ab|}{c} \sum_{n=1}^{\infty} (n+1)^2 \frac{(|a|+1)_{n-1}(|b|+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} \\
&= \gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} (n+2)^2 \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\
&= \gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} n(n-1) \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\
&\quad + 5\gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} n \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\
&\quad + 4\gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n},
\end{aligned} \tag{3.21}$$

where we are writing $(n+2)^2 = n(n-1)+5n+4$. By repeating the use of (3.7) and the Gauss summation formula, we have

$$\begin{aligned}
S_1 &\leq \frac{|ab|(|a|+1)(|b|+1)(|a|+2)(|b|+2)\Gamma(c)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&\quad + \frac{5|ab|(|a|+1)(|b|+1)\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} + \frac{4|ab|\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&= \frac{|ab|\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[4 + \frac{(1+|a|)(1+|b|)}{c-|a|-|b|-2}\left(5 + \frac{(2+|a|)(2+|b|)}{c-|a|-|b|-3}\right)\right].
\end{aligned} \tag{3.22}$$

As a next step, we consider the first expression of equation. By making use of the triangle inequality for the pochhammer symbol as stated in evaluating S_1, we get

$$\begin{aligned}\sum_{n=2}^{\infty} n^2 \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| &= \sum_{n=0}^{\infty} (n+2)^2 \left| \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+-1}} \right| \\ &\leq \sum_{n=0}^{\infty} (n+2)^2 \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} := S_2.\end{aligned} \tag{3.23}$$

Now making use of relation (3.7), we obtain

$$\begin{aligned}S_2 = &\sum_{n=0}^{\infty} (n+1)^2 \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &+ 2\sum_{n=0}^{\infty} (n+1) \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty} \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}},\end{aligned} \tag{3.24}$$

where we write $(n+2)^2 = (n+1)^2 + 2(n+1) + 1$. By repeating the use of (3.7) and the Gauss summation formula, we have

$$S_2 \leq \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[\frac{|ab|(|a|+1)(|b|+1)}{c(1+c)} + \frac{3|ab|}{c-|a|-|b|} + 1 \right]. \tag{3.25}$$

The proof of Theorem 3.5 now follows by an application of the inequalities of the terms dealing with S_1, S_2 and inequality (3.17). □

Repeating the previous reasoning for $b = \overline{a}$, we can improve the assertion of Theorem 3.5 as follows.

Theorem 3.6. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > \max\{0, 2\Re(a) + 3)\}$. If $f \in \mathcal{S}$, and if the inequality*

$$\begin{aligned}&\frac{|a|^2\Gamma(c)\Gamma(c-2\Re(a)-1)}{\Gamma(c-|a|)\Gamma(c-|\overline{a}|)} \left[4 + \frac{(1+|a|)(1+|\overline{a}|)}{c-2\Re(a)-2} \left(5 + \frac{(2+|a|)(2+|\overline{a}|)}{c-2\Re(a)-3} \right) \right] \\ &\quad + \frac{\Gamma(c)\Gamma(c-2\Re(a))}{\Gamma(c-|a|)\Gamma(c-|\overline{a}|)} \left[1 + 3\frac{|a|^2}{c-2\Re(a)-1} + \frac{|a|^2(1+|a|)(1+|\overline{a}|)}{c(1+c)} \right] \\ &\leq \frac{(A-B)|\tau|}{1+|B|} + 1\end{aligned} \tag{3.26}$$

is satisfied, then $I_{a,\overline{a},c}(f) \in P_\gamma^\tau(A,B)$.

In the special case when $b = 1$, Theorem 3.2 immediately yields a result concerning the Carlson-Shaffer operator $\mathcal{L}(a,c)$.

Theorem 3.7. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + 4$. If $f \in \mathcal{S}$, and if the inequality*

$$\begin{aligned}
&\frac{|a|(c-1)}{(c-|a|-1)(c-|a|-2)}\left[4+\frac{2(1+|a|)}{c-|a|-3}\left(5+\frac{3(2+|a|)}{c-|a|-4}\right)\right] \\
&\quad+\frac{c}{(c-|a|-1)}\left[1+\frac{3|a|}{c-|a|-2}+\frac{2|a|(1+|a|)}{c(1+c)}\right] \\
&\quad\le \frac{(A-B)|\tau|}{1+|B|}+1
\end{aligned} \tag{3.27}$$

is satisfied, then $\mathcal{L}(a,c)(f) \in P_\gamma^\tau(A,B)$.

Theorem 3.8. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + P_1$, where $P_1 = P_1(k)$ is given with (1.22). If, for some k $(0 \le k < \infty)$, $f \in k - \mathcal{UCV}$, and the inequality*

$$ {}_3F_2(|a|,|b|,P_1;c,1;1)+\frac{P_1\gamma|ab|}{c}\,{}_3F_2(|a|+1,|b|+1,P_1+1;c+1,2;1)\le\frac{(A-B)|\tau|}{1+|B|}+1 \tag{3.28}$$

is satisfied, then $I_{a,b,c}(f) \in P_\gamma^\tau(A,B)$.

Proof. By means of (1.17) and (2.13), the following inequality must be satisfied:

$$\sum_{n=2}^{\infty} n\left[1+\gamma(n-1)\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le \frac{(A-B)|\tau|}{1+|B|}. \tag{3.29}$$

Applying the estimates for the coefficients given by (1.21), and making use of the relations (3.7) and $|(d)_n| \le (|d|)_n$, condition (3.29) will be satisfied if

$$\begin{aligned}
&(1+|B|)\left[{}_3F_2(|a|,|b|,P_1;c,1;1)-1\right] \\
&\quad+(1+|B|)\frac{P_1\gamma|ab|}{c}\,{}_3F_2(|a|+1,|b|+1,P_1+1;c+1,2;1)\le(A-B)|\tau|
\end{aligned} \tag{3.30}$$

provided $c > |a| + |b| + P_1$. The proof of the Theorem 3.8 is now completed by virtue of hypothesis (3.28). □

Theorem 3.9. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + P_1$, where $P_1 = P_1(k)$ is given with* (1.22). *If, for some k $(0 \le k < \infty)$, $f \in k - ST$, and the inequality*

$$\begin{aligned} {}_3F_2(|a|, |b|, P_1; c, 1; 1) &+ \frac{P_1(1+\gamma)|ab|}{c} {}_3F_2(|a|+1, |b|+1, P_1+1; c+1, 2; 1) \\ &+ \frac{P_1\gamma|ab|}{c} {}_3F_2(|a|+1, |b|+1, P_1+1; c+1, 1; 1) \le \frac{(A-B)|\tau|}{1+|B|} + 1 \end{aligned} \tag{3.31}$$

is satisfied, then $I_{a,b,c}(f) \in P_\gamma^\tau(A, B)$.

Proof. Proceeding as in the proof of Theorem 3.8, and applying the estimates for the coefficients given by (1.24) instead of (1.21), and making use of relations (3.7) and $|(d)_n| \le (|d|)_n$, the proof of the theorem by virtue of hypothesis (3.31) is complete. □

Acknowledgment

The authors sincerely thank the referees for their suggestions.

References

[1] A. W. Goodman, "On uniformly convex functions," *Annales Polonici Mathematici*, vol. 56, no. 1, pp. 87–92, 1991.
[2] S. Kanas and A. Wiśniowska, "Conic regions and k-uniform convexity," *Journal of Computational and Applied Mathematics*, vol. 105, no. 1-2, pp. 327–336, 1999.
[3] S. Kanas and A. Wiśniowska, "Conic domains and starlike functions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 45, no. 4, pp. 647–657, 2000.
[4] S. Kanas and H. M. Srivastava, "Linear operators associated with k-uniformly convex functions," *Integral Transforms and Special Functions*, vol. 9, no. 2, pp. 121–132, 2000.
[5] A. W. Goodman, *Univalent Functions, Vols. I and II*, Polygonal Publishing, Washington, NJ, USA, 1983.
[6] H. M. Srivastava and A. K. Mishra, "Applications of fractional calculus to parabolic starlike and uniformly convex functions," *Computers & Mathematics with Applications*, vol. 39, no. 3-4, pp. 57–69, 2000.
[7] K. K. Dixit and S. K. Pal, "On a class of univalent functions related to complex order," *Indian Journal of Pure and Applied Mathematics*, vol. 26, no. 9, pp. 889–896, 1995.
[8] S. Ponnusamy and F. Rønning, "Duality for Hadamard products applied to certain integral transforms," *Complex Variables Theory and Application*, vol. 32, no. 3, pp. 263–287, 1997.
[9] K. S. Padmanabhan, "On a certain class of functions whose derivatives have a positive real part in the unit disc," *Annales Polonici Mathematici*, vol. 23, pp. 73–81, 1970.
[10] T. R. Caplinger and W. M. Causey, "A class of univalent functions," *Proceedings of the American Mathematical Society*, vol. 39, no. 2, pp. 357–361, 1973.
[11] Y. Ding Gong, "Properties of a class of analytic functions," *Mathematica Japonica*, vol. 41, no. 2, pp. 371–381, 1995.
[12] R. Singh and S. Singh, "Convolution properties of a class of starlike functions," *Proceedings of the American Mathematical Society*, vol. 106, no. 1, pp. 145–152, 1989.
[13] S. Owa and Z. Wu, "A note on certain subclass of analytic functions," *Mathematica Japonica*, vol. 34, no. 3, pp. 413–416, 1989.
[14] R. Balasubramanian, S. Ponnusamy, and M. Vuorinen, "On hypergeometric functions and function spaces," *Journal of Computational and Applied Mathematics*, vol. 139, no. 2, pp. 299–322, 2002.
[15] S. Ponnusamy, "Hypergeometric transforms of functions with derivative in a half plane," *Journal of Computational and Applied Mathematics*, vol. 96, no. 1, pp. 35–49, 1998.
[16] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.

[17] A. Gangadharan, T. N. Shanmugam, and H. M. Srivastava, "Generalized hypergeometric functions associated with k-uniformly convex functions," *Computers & Mathematics with Applications*, vol. 44, no. 12, pp. 1515–1526, 2002.
[18] Y. C. Kim and F. Rønning, "Integral transforms of certain subclasses of analytic functions," *Journal of Mathematical Analysis and Applications*, vol. 258, no. 2, pp. 466–489, 2001.
[19] T. N. Shanmugam, "Hypergeometric functions in the geometric function theory," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 433–444, 2007.
[20] H. Silverman, "Convolutions of univalent functions with negative coefficients," *Annales Universitatis Mariae Curie-Skłodowska Section A*, vol. 29, pp. 99–107, 1975.
[21] H. Silverman, "Starlike and convexity properties for hypergeometric functions," *Journal of Mathematical Analysis and Applications*, vol. 172, no. 2, pp. 574–581, 1993.
[22] Ju. E. Hohlov, "Operators and operations on the class of univalent functions," *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika*, no. 10, pp. 83–89, 1978.
[23] L. de Branges, "A proof of the Bieberbach conjecture," *Acta Mathematica*, vol. 154, no. 1-2, pp. 137–152, 1985.