Research Article

Convolution Operators and Bochner-Riesz Means on Herz-Type Hardy Spaces in the Dunkl Setting

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We study the Dunkl convolution operators on Herz-type Hardy spaces $\mathcal{H}_{\alpha,2}^p$ and we establish a version of multiplier theorem for the maximal Bochner-Riesz operators on the Herz-type Hardy spaces $\mathcal{H}_{\alpha,\infty}^p$.

1. Introduction

The classical theory of Hardy spaces on $\mathbb{R}^n$ has received an important impetus from the work of Fefferman and Stein, Lu and Yang [1, 2]. Their work resulted in many applications involving sharp estimates for convolution and multiplier operators.

By using the technique of Herz-type Hardy spaces for the Dunkl operator Λ_α , we are attempting in this paper to study the Dunkl convolution operators, and we establish a version of multiplier theorem for the maximal Bochner-Riesz operators on these spaces.

The Dunkl operator Λ_α , $\alpha > -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha f(x) := \frac{d}{dx} f(x) + \frac{2\alpha + 1}{x} \left[\frac{f(x) - f(-x)}{2} \right], \quad (1.1)$$

is the operator devised by Dunkl [3] in connection with a generalization of the classical theory of spherical harmonics. The Dunkl analysis with respect to $\alpha \geq -1/2$ concerns the Dunkl operator Λ_α , the Dunkl transform $\mathcal{F}_\alpha$, the Dunkl convolution $*_\alpha$, and a certain measure μ_α on $\mathbb{R}$.

In this paper we define a Herz-type Hardy spaces $\mathcal{H}_{\alpha,q}^p$, $0 < p \leq 1 < q \leq \infty$, in the Dunkl setting. Next, we consider the Dunkl convolution operators $T_k f := k *_\alpha f$, where k is

a locally integrable function on $\mathbb{R}$. We use the atomic decomposition of the Herz-type Hardy spaces $\mathcal{H}_{\alpha,q}^p$ to study the $\mathcal{H}_{\alpha,2}^p - \mathcal{H}_{\alpha,2}^q$ -bounded and the $\mathcal{H}_{\alpha,2}^1 - L^1$ -bounded of the operators T_k . Finally, we establish a version of multiplier theorem for the maximal Bochner-Riesz operators σ_α^η , $t > 0$, $\eta > \alpha + 1/2$:

$$\sigma_\alpha^\eta(f) := \sup_{t>0} \left| \Phi_{\alpha,t}^\eta * f \right|, \quad (1.2)$$

where

$$\Phi_{\alpha,t}^\eta(x) := \frac{\Gamma(\eta+1)}{2^{\alpha+1}} t^{2\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! \Gamma(n+\alpha+\eta+2)} (tx)^{2n}, \quad (1.3)$$

on the Herz-type Hardy spaces $\mathcal{H}_{\alpha,\infty}^p$. In this version we prove the $\mathcal{H}_{\alpha,\infty}^p - L^p$ -bounded of the operators σ_α^η , for $\alpha + 1/2 < \eta < \alpha + 3/2$.

The content of this work is the following. In Section 2, we recall some results about harmonic analysis and we define a Herz-type Hardy spaces $\mathcal{H}_{\alpha,q}^p$, $0 < p \leq 1 < q \leq \infty$ for the Dunkl operator Λ_α . In Section 3, we study the $\mathcal{H}_{\alpha,2}^p - \mathcal{H}_{\alpha,2}^q$ -bounded and the $\mathcal{H}_{\alpha,2}^1 - L^1$ -bounded of the convolution operators T_k . In Section 4, we prove the $\mathcal{H}_{\alpha,\infty}^p - L^p$ -bounded of the maximal Bochner-Riesz operators σ_α^η .

Throughout the paper we use the classic notation. Thus $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are the Schwartz space on $\mathbb{R}$ and the space of tempered distributions on $\mathbb{R}$, respectively. Finally, C will denote a positive constant not necessary the same in each occurrence.

2. The Dunkl Harmonic Analysis on $\mathbb{R}$

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial problem

$$\Lambda_\alpha f(x) = \lambda f(x), \quad f(0) = 1, \quad (2.1)$$

has a unique analytic solution $E_\alpha(\lambda x)$ called Dunkl kernel [4–6] given by

$$E_\alpha(\lambda x) = \mathcal{I}_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(\lambda x), \quad x \in \mathbb{R}, \quad (2.2)$$

where

$$\mathcal{I}_\alpha(\lambda x) := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(\lambda x)^{2n}}{2^{2n} n! \Gamma(n+\alpha+1)}, \quad (2.3)$$

is the modified spherical Bessel function of order α .

We notice that, the Dunkl kernel $E_\alpha(\lambda x)$ can be expanded in a power series [7] in the form

$$E_\alpha(\lambda x) = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{b_n(\alpha)}, \quad (2.4)$$

where

$$b_{2n}(\alpha) = \frac{2^{2n}n!}{\Gamma(\alpha+1)}\Gamma(n+\alpha+1), \quad b_{2n+1}(\alpha) = 2(\alpha+1)b_{2n}(\alpha+1). \quad (2.5)$$

Note 1. Let μ_α be the measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) := \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}|x|^{2\alpha+1}dx. \quad (2.6)$$

We denote by $L^p(\mathbb{R}, \mu_\alpha)$, $p \in]0, \infty]$, the space of measurable functions f on $\mathbb{R}$, such that

$$\begin{aligned} \|f\|_{L_\alpha^p} &:= \left[\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right]^{1/p} < \infty, \quad p \in]0, \infty[, \\ \|f\|_{L_\alpha^\infty} &:= \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty. \end{aligned} \quad (2.7)$$

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on $\mathbb{R}$, which was introduced by Dunkl in [8], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu in [5].

The Dunkl transform of a function $f \in L^1(\mathbb{R}, \mu_\alpha)$, is given by

$$\mathcal{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x) f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}. \quad (2.8)$$

For $T \in \mathcal{S}'(\mathbb{R})$, we define the Dunkl transform $\mathcal{F}_\alpha(T)$ of T , by

$$\langle \mathcal{F}_\alpha(T), \varphi \rangle := \langle T, \mathcal{F}_\alpha(\varphi) \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}). \quad (2.9)$$

Note 2. For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x, y, z) := \{1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\} \Delta_\alpha(|x|, |y|, |z|), \quad (2.10)$$

where

$$\begin{aligned} \sigma_{x,y,z} &:= \begin{cases} \frac{x^2 + y^2 - z^2}{2xy}, & \text{if } x, y \in \mathbb{R} \setminus \{0\} \\ 0, & \text{otherwise} \end{cases} \\ \Delta_\alpha(|x|, |y|, |z|) &:= \begin{cases} d_\alpha \frac{\left[\left((|x| + |y|)^2 - z^2 \right) \left(z^2 - (|x| - |y|)^2 \right) \right]^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases} \quad (2.11) \\ d_\alpha &= \frac{2^{1-\alpha} (\Gamma(\alpha+1))^2}{\sqrt{\pi} \Gamma(\alpha+1/2)}, \quad A_{x,y} = [|x| - |y|, |x| + |y|]. \end{aligned}$$

We denote by $\nu_{x,y}$ the following signed measure:

$$d\nu_{x,y}(z) := \begin{cases} W_\alpha(x, y, z) d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R} \setminus \{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \quad (2.12)$$

The Dunkl translation operators τ_x , $x \in \mathbb{R}$ (see [6]) are defined for $f \in \mathcal{C}(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$), by

$$\tau_x f(y) := \int_{||x|-|y||}^{|x|+|y|} f(z) d\nu_{x,y}(z) + \int_{-(|x|+|y|)}^{-||x|-|y||} f(z) d\nu_{x,y}(z). \quad (2.13)$$

Let f and g be two functions in $\mathcal{S}(\mathbb{R})$. We define the Dunkl convolution product $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R}. \quad (2.14)$$

For $T \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, we define the Dunkl convolution product $T *_\alpha f$ by

$$T *_\alpha f(x) := \langle T(y), \tau_x f(-y) \rangle, \quad x \in \mathbb{R}. \quad (2.15)$$

We begin by recalling the definition of the Herz-type Hardy space in the Dunkl setting. Firstly we introduce a class of fundamental functions that we will call atoms.

Let $0 < p \leq 1 < q \leq \infty$. A measurable function a on $\mathbb{R}$ is called a (p, q) atom, if a satisfies the following conditions:

- (i) there exists $r > 0$ such that $\text{supp}(a) \subset [-r, r]$;
- (ii) $\|a\|_{L^q_a} \leq r^{-2(\alpha+1)(1/p-1/q)}$, where r is given in (i);
- (iii) $\int_{\mathbb{R}} a(x)x^j d\mu_\alpha(x) = 0$, for all $j = 0, 1, \dots, 2s+1$,

where $s = [(\alpha+1)(1/p-1)]$, (the integer part of $(\alpha+1)(1/p-1)$).

Let $0 < p \leq 1 < q \leq \infty$. Our Herz-type Hardy space $\mathcal{H}^p_{\alpha,q}$ is constituted by all those $f \in \mathcal{S}'(\mathbb{R})$ that can be represented by

$$f = \sum_{j=0}^{\infty} \lambda_j a_j, \quad (2.16)$$

where $\lambda_j \in \mathbb{C}$ and a_j is a (p, q) atom, for all $j \in \mathbb{N}$, such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$ and the series in (2.16) converges in $\mathcal{S}'(\mathbb{R})$.

We define on $\mathcal{H}^p_{\alpha,q}$ the norm $\|\cdot\|_{\mathcal{H}^p_{\alpha,q}}$ by

$$\|f\|_{\mathcal{H}^p_{\alpha,q}} := \inf \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p}, \quad (2.17)$$

where the infimum is taken over all those sequences $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ such that f is given by (2.16) for certain (p, q) atoms a_j , $j \in \mathbb{N}$.

As the same in [7], we prove the following theorem.

Theorem 2.1. *Let $0 < p \leq 1 < q \leq \infty$ and $f \in \mathcal{H}^p_{\alpha,q}$. Then*

$$|\mathcal{F}_\alpha(f)(y)| \leq C|y|^{2(\alpha+1)(1/p-1)} \|f\|_{\mathcal{H}^p_{\alpha,q}}, \quad y \in \mathbb{R}. \quad (2.18)$$

3. Dunkl Convolution Operators on $\mathcal{H}^p_{\alpha,2}$

In the following, we study on $\mathcal{H}^p_{\alpha,2}$, $0 < p \leq 1$ the Dunkl convolution operators defined by $T_k f := k *_\alpha f$, where k is a locally integrable function on $\mathbb{R}$.

Theorem 3.1. *Let $0 < p \leq q \leq 1$. Assume that for every $n \in \mathbb{N}$ we are given $\xi_n > 0$ and a function g_n such that*

- (i) $g_n(x) = 0$, $|x| \geq 2^{-n}$,
- (ii) $\|g_n\|_{L^1_\alpha} \leq \xi_n 2^{2(\alpha+1)(1/q-1/p)n}$,
- (iii) $\|x^{2s+2} \mathcal{F}_\alpha(g_n)\|_{L^2_\alpha} \leq \xi_n 2^{2(\alpha+1)(1/q-1/2)n}$, $s = [(\alpha+1)(1/p-1)]$.

Suppose also that $\sum_{n=0}^{\infty} (\xi_n)^q < \infty$ and define $k = \sum_{n=0}^{\infty} g_n$. Then T_k defines a bounded linear mapping from $\mathcal{H}^p_{\alpha,2}$ into $\mathcal{H}^q_{\alpha,2}$.

To prove this theorem the following lemma is needed.

Lemma 3.2. For $y, z \in \mathbb{R}$ and $j \in \mathbb{N}$, there are constants $A_{j,\alpha}$, such that

$$\tau_y(x^j)(z) = A_{j,\alpha} \sum_{i=0}^j \frac{j!}{b_i(\alpha)b_{j-i}(\alpha)} y^i z^{j-i}, \quad (3.1)$$

where $b_j(\alpha)$ are the constants given by (2.4).

Proof. Let $y, z \in \mathbb{R}$. By dominated convergence theorem, we can write

$$\tau_y(x^j)(z) = \lim_{t \rightarrow 0} \int_{\mathbb{R}} x^j E_{\alpha}(tx) d\nu_{y,z}(x), \quad (3.2)$$

and by derivation under the integral sign, we get

$$\tau_y(x^j)(z) = \lim_{t \rightarrow 0} \Lambda_{\alpha,t}^j \int_{\mathbb{R}} E_{\alpha}(tx) d\nu_{y,z}(x). \quad (3.3)$$

Then, from Theorem 2.4, [6] we obtain

$$\tau_y(x^j)(z) = \lim_{t \rightarrow 0} \Lambda_{\alpha,t}^j (E_{\alpha}(ty) E_{\alpha}(tz)). \quad (3.4)$$

Let $F(t) = E_{\alpha}(ty) E_{\alpha}(tz)$ for $|t| < \xi$, where $\xi > 0$. According to (2.11), [9],

$$\Lambda_{\alpha,t}^j F(t) = F^{(j)}(t) + \sum_{i=1}^{j-1} \left\{ 2^i P_{j-1}(t_1^0, \dots, t_i^0) F^{(j)}(t \cdot t_1^0 \dots t_i^0) + 2^j Q_{j-1}(t_1^1, \dots, t_{j-1}^1) F^{(j)}(t \cdot t_1^1 \dots t_{j-1}^1) \right\}, \quad (3.5)$$

where $P_{j-1}(t_1^0, \dots, t_i^0)$, $i = 1, 2, \dots, j-1$, and $Q_{j-1}(t_1^1, \dots, t_{j-1}^1)$ are polynomials of degree at most $j-1$ with respect to each variable. Moreover,

$$F^{(j)}(t) = \sum_{i=0}^j \binom{j}{i} y^i z^{j-i} E_{\alpha}^{(i)}(ty) E_{\alpha}^{(j-i)}(tz). \quad (3.6)$$

Then, from (2.4) we deduce

$$\lim_{t \rightarrow 0} F^{(j)}(t) = \sum_{i=0}^j \binom{j}{i} \frac{i!(j-i)!}{b_i(\alpha)b_{j-i}(\alpha)} y^i z^{j-i} = \sum_{i=0}^j \frac{j!}{b_i(\alpha)b_{j-i}(\alpha)} y^i z^{j-i}. \quad (3.7)$$

Therefore,

$$\tau_y(x^j)(z) = \lim_{t \rightarrow 0} \Lambda_{\alpha,t}^j F(t) = A_{j,\alpha} \sum_{i=0}^j \frac{j!}{b_i(\alpha)b_{j-i}(\alpha)} y^i z^{j-i}, \quad (3.8)$$

where

$$A_{j,\alpha} = 1 + \sum_{i=1}^{j-1} \left\{ 2^i P_{j-1}(t_1^0, \dots, t_i^0) + 2^j Q_{j-1}(t_1^1, \dots, t_{j-1}^1) \right\}. \quad (3.9)$$

This finishes the proof of the lemma. $\square$

Proof of Theorem 3.1. Firstly, notice that $\|g_n\|_{L_\alpha^1} \leq \xi_n$, $n \in \mathbb{N}$. Hence, the series defining k converges in $L^1(\mathbb{R}, \mu_\alpha)$ and $k \in L^1(\mathbb{R}, \mu_\alpha)$.

Let a be a $(p, 2)$ atom. Suppose that $a(x) = 0$, $|x| > r$ and that $\|a\|_{L_\alpha^2} \leq r^{-2(\alpha+1)(1/p-1/2)}$, where $r > 0$. We can write

$$T_k a = \sum_{n=0}^{\infty} g_n *_{\alpha} a. \quad (3.10)$$

Step 1. Let $n \in \mathbb{N}$. From (2.13), for $||y| - |z|| \geq 2^{-n}$, we have

$$\tau_y g_n(z) = \int_{||y|-|z||}^{|y|+|z|} g_n(x) d\nu_{y,z}(x) + \int_{-(|y|+|z|)}^{-||y|-|z||} g_n(x) d\nu_{y,z}(x) = 0. \quad (3.11)$$

Hence, for $|y| > r + 2^{-n}$, we deduce

$$g_n *_{\alpha} a(y) = \int_{-r}^r a(-z) \tau_y g_n(z) d\mu_\alpha(z) = 0. \quad (3.12)$$

Step 2. Firstly, let us consider that $r \geq 2^{-n}$. From (Proposition 3(i), [10]) and condition (ii) of the theorem, we have

$$\|g_n *_{\alpha} a\|_{L_\alpha^2} \leq 4 \|g_n\|_{L_\alpha^1} \|a\|_{L_\alpha^2} \leq 4 \xi_n (r + 2^{-n})^{-2(\alpha+1)(1/q-1/2)}. \quad (3.13)$$

Assume now that $r < 2^{-n}$. Since $\int_{\mathbb{R}} a(x) x^j d\mu_\alpha(x) = 0$, $j = 0, 1, \dots, 2s+1$, with $s = [(\alpha+1)(1/p-1)]$, we have

$$g_n *_{\alpha} a(x) = \int_{\mathbb{R}} a(-y) \left[\tau_y g_n(x) - \sum_{j=0}^{2s+1} \frac{y^j}{b_j(\alpha)} \Lambda_\alpha^j g_n(x) \right] d\mu_\alpha(y), \quad x \in \mathbb{R}, \quad (3.14)$$

where $b_j(\alpha)$ are the constants given by (2.4).

Using the properties of the Dunkl transform established by de Jeu [5] (see also [7, 10]), we deduce

$$\begin{aligned}
\|g_n *_{\alpha} a\|_{L_{\alpha}^2} &\leq \int_{\mathbb{R}} |a(-y)| \left\| \tau_y g_n - \sum_{j=0}^{2s+1} \frac{y^j}{b_j(\alpha)} \Lambda_{\alpha}^j g_n \right\|_{L_{\alpha}^2} d\mu_{\alpha}(y) \\
&= \int_{\mathbb{R}} |a(-y)| \left\| \mathcal{F}_{\alpha}(\tau_y g_n) - \sum_{j=0}^{2s+1} \frac{y^j}{b_j(\alpha)} \mathcal{F}_{\alpha}(\Lambda_{\alpha}^j g_n) \right\|_{L_{\alpha}^2} d\mu_{\alpha}(y) \\
&= \int_{\mathbb{R}} |a(-y)| \left\| \left[E_{\alpha}(ixy) - \sum_{j=0}^{2s+1} \frac{(ixy)^j}{b_j(\alpha)} \right] \mathcal{F}_{\alpha}(g_n) \right\|_{L_{\alpha}^2} d\mu_{\alpha}(y).
\end{aligned} \tag{3.15}$$

According to page 302, [7]

$$\left| E_{\alpha}(ixy) - \sum_{j=0}^{2s+1} \frac{(ixy)^j}{b_j(\alpha)} \right| \leq \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} |xy|^{2s+2}. \tag{3.16}$$

Using condition (iii) of the theorem and Hölder's inequality, we get

$$\begin{aligned}
\|g_n *_{\alpha} a\|_{L_{\alpha}^2} &\leq \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} \|x^{2s+2} \mathcal{F}_{\alpha}(g_n)\|_{L_{\alpha}^2} \int_{-r}^r |a(y)| y^{2s+2} d\mu_{\alpha}(y) \\
&\leq \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} \|x^{2s+2} \mathcal{F}_{\alpha}(g_n)\|_{L_{\alpha}^2} \|a\|_{L_{\alpha}^2} \left[2 \int_0^r y^{4s+4} d\mu_{\alpha}(y) \right]^{1/2} \\
&\leq \gamma_n 2^{2(\alpha+1)(1/q-1/2)n} r^{2s-2(\alpha+1)(1/p-1)+2},
\end{aligned} \tag{3.17}$$

where

$$\gamma_n := \frac{\xi_n}{2^{2\alpha}\Gamma(\alpha+1)\sqrt{(2s+\alpha+3)2^{\alpha+1}\Gamma(\alpha+1)}} = c\xi_n. \tag{3.18}$$

Using the fact that $s - (\alpha+1)(1/p-1) + 1 > 0$ and $r < 2^{-n}$, we obtain

$$\|g_n *_{\alpha} a\|_{L_{\alpha}^2} \leq \gamma_n 2^{2(\alpha+1)(1/q-1/2)n} \leq \gamma_n (r + 2^{-n})^{-2(\alpha+1)(1/q-1/2)}. \tag{3.19}$$

Step 3. We now prove for all $j = 0, \dots, 2s+1$; $s = [(\alpha+1)(1/p-1)]$, that

$$\int_{\mathbb{R}} x^j (g_n *_{\alpha} a)(x) d\mu_{\alpha}(x) = 0. \tag{3.20}$$

Fubini's theorem and [6] (see also page 20, [10]) lead to

$$\begin{aligned}
 & \int_{\mathbb{R}} x^j (g_n *_{\alpha} a)(x) d\mu_{\alpha}(x) \\
 &= \int_{\mathbb{R}} x^j \int_{\mathbb{R}} a(y) \left[\int_{\mathbb{R}} g_n(z) d\nu_{x,-y}(z) \right] d\mu_{\alpha}(y) d\mu_{\alpha}(x) \\
 &= \int_{\mathbb{R}} a(y) \int_{\mathbb{R}} g_n(z) \left[\int_{\mathbb{R}} x^j d\nu_{y,z}(x) \right] d\mu_{\alpha}(z) d\mu_{\alpha}(y) \\
 &= \int_{\mathbb{R}} a(y) \int_{\mathbb{R}} g_n(z) \tau_y(x^j)(z) d\mu_{\alpha}(z) d\mu_{\alpha}(y).
 \end{aligned} \tag{3.21}$$

Hence, by Lemma 3.2 and by taking into account that $\int_{\mathbb{R}} a(x) x^j d\mu_{\alpha}(x) = 0$, $j = 0, 1, \dots, 2s + 1$, with $s = [(\alpha + 1)(1/p - 1)]$, we get

$$\int_{\mathbb{R}} x^j (g_n *_{\alpha} a)(x) d\mu_{\alpha}(x) = A_{j,\alpha} \sum_{i=0}^j \frac{j!}{b_i(\alpha) b_{j-i}(\alpha)} \left[\int_{\mathbb{R}} y^i a(y) d\mu_{\alpha}(y) \right] \left[\int_{\mathbb{R}} g_n(z) z^{j-i} d\mu_{\alpha}(z) \right] = 0. \tag{3.22}$$

According to the previous three steps, we conclude that $(1/\gamma_n)(g_n *_{\alpha} a)$ is a $(q, 2)$ atom. Then, $T_k a \in \mathcal{H}_{\alpha,2}^q$, and

$$\|T_k a\|_{\mathcal{H}_{\alpha,2}^q} \leq c \left(\sum_{n=0}^{\infty} (\xi_n)^q \right)^{1/q}. \tag{3.23}$$

Let now f be in $\mathcal{H}_{\alpha,2}^p$. Assume that $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where $\lambda_j \in \mathbb{C}$ and a_j is a $(p, 2)$ atom, for every $j \in \mathbb{N}$, and such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. The series defining f converges in $L^1(\mathbb{R}, \mu_{\alpha})$. In fact, it is sufficient to note that $\|a\|_{L_{\alpha}^1} \leq (1/\sqrt{2^{\alpha}\Gamma(\alpha+1)}) r^{2(\alpha+1)(1-1/p)}$. Hence $f \in L^1(\mathbb{R}, \mu_{\alpha})$. Moreover, $k \in L^1(\mathbb{R}, \mu_{\alpha})$. Then by (Proposition 3(i), [10]), the operator T_k is bounded from $L^1(\mathbb{R}, \mu_{\alpha})$ into itself, and from this, we deduce that $T_k f = \sum_{j=0}^{\infty} \lambda_j T_k a_j$. Using the fact that $\sum_{j=0}^{\infty} |\lambda_j| \leq (\sum_{j=0}^{\infty} |\lambda_j|^p)^{1/p}$, we obtain

$$\|T_k f\|_{\mathcal{H}_{\alpha,2}^q} \leq c \left(\sum_{n=0}^{\infty} (\xi_n)^q \right)^{1/q} \|f\|_{\mathcal{H}_{\alpha,2}^p}. \tag{3.24}$$

This completes the proof of the Theorem 3.1. $\square$

We now study the Dunkl convolution operators T_k on the Herz-type Hardy spaces $\mathcal{H}_{\alpha,2}^1$.

Theorem 3.3. Let k be a locally integrable function on $\mathbb{R}$. Assume that the following three conditions are satisfied:

- (i) T_k defines a bounded linear operator from $L^2(\mathbb{R}, \mu_\alpha)$ into itself.
- (ii) T_k defines a bounded linear operator from $L^1(\mathbb{R}, \mu_\alpha)$ into $\mathcal{S}'(\mathbb{R})$.
- (iii) There exist A and $c > 1$ such that

$$\int_{|z|>cR} |\tau_x k(z) - k(z)| d\mu_\alpha(z) \leq A, \quad |x| \in (0, R), \quad R > 0. \quad (3.25)$$

Then T_k defines a bounded linear mapping from $\mathcal{H}_{\alpha,2}^1$ into $L^1(\mathbb{R}, \mu_\alpha)$.

Proof. Let a be a $(1, 2)$ atom. We choose $r > 0$ such that $\text{supp}(a) \subset [-r, r]$ and $\|a\|_{L_\alpha^2} \leq r^{-(\alpha+1)}$. We can write

$$\int_{\mathbb{R}} |T_k a(x)| d\mu_\alpha(x) = \left[\int_{|x|<cr} + \int_{|x|\geq cr} \right] |T_k a(x)| d\mu_\alpha(x) := I_1 + I_2. \quad (3.26)$$

Here $c > 1$ is the one given in (iii).

From condition (i) of the theorem and Hölder's inequality, we deduce that

$$I_1 \leq \left[\int_{\mathbb{R}} |T_k a(x)|^2 d\mu_\alpha(x) \right]^{1/2} \left[2 \int_0^{cr} d\mu_\alpha(x) \right]^{1/2} \leq C \|a\|_{L_\alpha^2} r^{\alpha+1} \leq C. \quad (3.27)$$

Also, by taking into account that $\int_{\mathbb{R}} a(y) d\mu_\alpha(y) = 0$, the condition (iii) of the theorem allows us to write

$$\begin{aligned} I_2 &= \int_{|x|\geq cr} \left| \int_{\mathbb{R}} \tau_x k(y) a(-y) d\mu_\alpha(y) \right| d\mu_\alpha(x) \\ &= \int_{|x|\geq cr} \left| \int_{\mathbb{R}} \{ \tau_x k(y) - k(x) \} a(-y) d\mu_\alpha(y) \right| d\mu_\alpha(x) \\ &\leq \int_{-r}^r |a(-y)| \int_{|x|\geq cr} |\tau_y k(x) - k(x)| d\mu_\alpha(x) d\mu_\alpha(y) \\ &\leq C \int_{-r}^r |a(y)| d\mu_\alpha(y) \leq C \|a\|_{L_\alpha^2} \left[2 \int_0^r d\mu_\alpha(y) \right]^{1/2} \leq C. \end{aligned} \quad (3.28)$$

Hence, it concludes that

$$\|T_k a\|_{L_\alpha^1} \leq C. \quad (3.29)$$

Note that the positive constant C is not depending on the $(1, 2)$ atom a .

Let now f be in $\mathcal{H}_{\alpha,2}^1$. Then $f \in \mathcal{S}'(\mathbb{R})$ and $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where $\lambda_j \in \mathbb{C}$ and a_j is a $(1, 2)$ atom, for every $j \in \mathbb{N}$ and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$.

The series defining f converges in $L^1(\mathbb{R}, \mu_\alpha)$. In fact, it is sufficient to note that $\|a\|_{L^1_\alpha} \leq 1/\sqrt{2^a\Gamma(\alpha+1)}$, for every $(1,2)$ atom a . Hence $f \in L^1(\mathbb{R}, \mu_\alpha)$. Then the condition (ii) of the theorem implies that

$$T_k f = \sum_{j=0}^{\infty} \lambda_j T_k a_j. \quad (3.30)$$

By (3.29) the series in (3.30) converges in $L^1(\mathbb{R}, \mu_\alpha)$ and $\|T_k f\|_{L^1_\alpha} \leq C \sum_{j=0}^{\infty} |\lambda_j|$. Hence $\|T_k f\|_{L^1_\alpha} \leq C \|f\|_{\mathcal{H}^1_{\alpha,2}}$. $\square$

4. Maximal Bochner-Riesz Operators on $\mathcal{H}^p_{\alpha,\infty}$

The Bochner-Riesz mean $\sigma_{\alpha,t}^\eta$, for $t > 0$ and $\eta > \alpha + 1/2$ associated to the Dunkl transform $\mathcal{F}_\alpha$ is defined by

$$\sigma_{\alpha,t}^\eta(f)(x) := \int_{-t}^t \left(1 - \frac{y^2}{t^2}\right)^\eta E_\alpha(ixy) \mathcal{F}_\alpha(f)(y) d\mu_\alpha(y), \quad f \in \mathcal{S}(\mathbb{R}). \quad (4.1)$$

The maximal operator σ_α^η , $\eta > \alpha + 1/2$ associated to the Bochner-Riesz means $\sigma_{\alpha,t}^\eta$, $t > 0$, is defined by

$$\sigma_\alpha^\eta(f) := \sup_{t>0} \left| \sigma_{\alpha,t}^\eta(f) \right|. \quad (4.2)$$

Lemma 4.1. For $t > 0$ and $\eta > \alpha + 1/2$, one has

$$(i) \quad \sigma_{\alpha,t}^\eta(f) = \Phi_{\alpha,t}^\eta *_\alpha f, \text{ where}$$

$$\Phi_{\alpha,t}^\eta(x) := \frac{\Gamma(\eta+1)}{2^{\alpha+1}\Gamma(\alpha+\eta+2)} t^{2\alpha+2} \mathfrak{J}_{\alpha+\eta+1}(itx). \quad (4.3)$$

Here $\mathfrak{J}_\alpha$ is the modified spherical Bessel function given by (2.3).

$$(ii) \quad \text{The operator } \sigma_\alpha^\eta \text{ is bounded from } L^p(\mathbb{R}, \mu_\alpha), 1 \leq p \leq \infty \text{ into itself.}$$

Proof. Let $t > 0$ and $\eta > \alpha + 1/2$.

$$(i) \quad \text{By taking into account that the functions } |z|^{\alpha+1/2} \mathfrak{J}_\alpha(iz) \text{ and } \mathfrak{J}_\alpha(iz) \text{ are bounded on } \mathbb{R} \text{ it is not hard to see that}$$

$$\left\| \Phi_{\alpha,t}^\eta \right\|_{L^1_\alpha} = \left\| \Phi_{\alpha,1}^\eta \right\|_{L^1_\alpha} = \frac{\Gamma(\eta+1)}{2^{\alpha+1}\Gamma(\alpha+\eta+2)} \int_{\mathbb{R}} |\mathfrak{J}_{\alpha+\eta+1}(ix)| d\mu_\alpha(x) < \infty. \quad (4.4)$$

On the other hand, from [11], we have

$$\int_0^1 (1-y^2)^\eta \mathfrak{I}_\alpha(itxy) y^{2\alpha+1} dy = \frac{\Gamma(\alpha+1)\Gamma(\eta+1)}{2\Gamma(\alpha+\eta+2)} \mathfrak{I}_{\alpha+\eta+1}(itx). \quad (4.5)$$

Thus,

$$\begin{aligned} \Phi_{\alpha,t}^\eta(x) &= \frac{t^{2\alpha+2}}{2^\alpha \Gamma(\alpha+1)} \int_0^1 (1-y^2)^\eta \mathfrak{I}_\alpha(itxy) y^{2\alpha+1} dy \\ &= \int_{-t}^t \left(1 - \frac{y^2}{t^2}\right)^\eta E_\alpha(ixy) d\mu_\alpha(y). \end{aligned} \quad (4.6)$$

Applying Inversion Theorem[5], we obtain

$$\mathcal{F}_\alpha(\Phi_{\alpha,t}^\eta)(y) = \left(1 - \frac{y^2}{t^2}\right)^\eta \chi_{(-t,t)}(y), \quad (4.7)$$

where $\chi_{(-t,t)}$ is the characteristic function of the set $(-t, t)$. Thus,

$$\sigma_{\alpha,t}^\eta(f)(x) = \int_{\mathbb{R}} E_\alpha(ixy) \mathcal{F}_\alpha(\Phi_{\alpha,t}^\eta)(y) \mathcal{F}_\alpha(f)(y) d\mu_\alpha(y), \quad (4.8)$$

and from (Proposition 3 (ii), [10]), we deduce that

$$\sigma_{\alpha,t}^\eta(f)(x) = \Phi_{\alpha,t}^\eta *_\alpha f(x). \quad (4.9)$$

(ii) Using (i) and (Proposition 3 (i), [10]), we obtain

$$\|\sigma_\alpha^\eta(f)\|_{L_\alpha^p} \leq 4 \|\Phi_{\alpha,1}^\eta\|_{L_\alpha^1} \|f\|_{L_\alpha^p}. \quad (4.10)$$

This clearly yields the result. $\square$

Theorem 4.2. Let $0 < p \leq 1 < q \leq \infty$ and $f \in \mathcal{H}_{\alpha,q}^p$. For $t > 0$ and $\eta > \alpha + 1/2$, the operator $\sigma_{\alpha,t}^\eta$ extended to a bounded operator from $\mathcal{H}_{\alpha,q}^p$ into $\mathcal{S}'(\mathbb{R})$.

Proof. According to Theorem 2.1, if $f \in \mathcal{H}_{\alpha,q}^p$, then $\sigma_{\alpha,t}^\eta(f)$ is in $\mathcal{S}'(\mathbb{R})$ and it is defined by

$$\langle \sigma_{\alpha,t}^\eta(f), \varphi \rangle = \int_{-t}^t \left(1 - \frac{y^2}{t^2}\right)^\eta \mathcal{F}_\alpha(f)(y) \mathcal{F}_\alpha(\varphi)(y) d\mu_\alpha(y), \quad \varphi \in \mathcal{S}(\mathbb{R}). \quad (4.11)$$

Moreover,

$$\left| \left\langle \sigma_{\alpha,t}^{\eta}(f), \varphi \right\rangle \right| \leq C \|f\|_{\mathcal{H}_{\alpha,q}^p} \int_{\mathbb{R}} |y|^{2(\alpha+1)(1/p-1)} |\mathcal{F}_{\alpha}(\varphi)(y)| d\mu_{\alpha}(y), \quad \varphi \in \mathcal{S}(\mathbb{R}). \quad (4.12)$$

Hence $\sigma_{\alpha,t}^{\eta}$ is a bounded operator from $\mathcal{H}_{\alpha,q}^p$ into $\mathcal{S}'(\mathbb{R})$. $\square$

We now study the behavior of the maximal Bochner-Riesz operator σ_{α}^{η} on $\mathcal{H}_{\alpha,\infty}^p$.

Theorem 4.3. *Let $2(\alpha+1)/(\alpha+\eta+3/2) < p \leq 1$. Then the maximal Bochner-Riesz operator σ_{α}^{η} , $\alpha+1/2 < \eta < \alpha+3/2$ is bounded from $\mathcal{H}_{\alpha,\infty}^p$ into $L^p(\mathbb{R}, \mu_{\alpha})$.*

To prove this theorem we need the following lemma.

Lemma 4.4. (i) *For $x, y \in \mathbb{R}$ and $\eta > \alpha+1/2$,*

$$\left| \tau_x \Phi_{\alpha,t}^{\eta}(y) \right| \leq C t^{\alpha-\eta+1/2} ||x| - |y||^{-(\alpha+\eta+3/2)}. \quad (4.13)$$

(ii) *For $0 < |y| < |x|$ and $\eta > \alpha+1/2$,*

$$\left| \tau_x \Phi_{\alpha,t}^{\eta}(y) - \Phi_{\alpha,t}^{\eta}(x) \right| \leq C |y| t^{\alpha-\eta+3/2} ||x| - |y||^{-(\alpha+\eta+3/2)}. \quad (4.14)$$

Proof. (i) Since the function $|z|^{\alpha+1/2} \mathcal{J}_{\alpha}(iz)$ is bounded on $\mathbb{R}$, it follows that

$$\left| \Phi_{\alpha,t}^{\eta}(z) \right| \leq C t^{\alpha-\eta+1/2} |z|^{-(\alpha+\eta+3/2)}; \quad t > 0, \quad z \in \mathbb{R}. \quad (4.15)$$

According to [6] and the fact that $\Phi_{\alpha,t}^{\eta}$ is even, we obtain

$$\tau_x \Phi_{\alpha,t}^{\eta}(y) = \int_0^{\pi} \Phi_{\alpha,t}^{\eta}((x, y)_{\theta}) d\rho_{x,y}(\theta), \quad (4.16)$$

where

$$d\rho_{x,y}(\theta) := \frac{\Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+1/2)} \{1 - \operatorname{sgn}(xy) \cos \theta\} \sin^{2\alpha} \theta d\theta, \quad (4.17)$$

$$(x, y)_{\theta} := \sqrt{x^2 + y^2 - 2|xy| \cos \theta}.$$

By (4.15) and using the fact that $(x, y)_{\theta} \geq ||x| - |y||$, we deduce that

$$\begin{aligned} \left| \tau_x \Phi_{\alpha,t}^{\eta}(y) \right| &\leq C t^{\alpha-\eta+1/2} \int_0^{\pi} (x, y)_{\theta}^{-(\alpha+\eta+3/2)} d\rho_{x,y}(\theta) \\ &\leq C t^{\alpha-\eta+1/2} ||x| - |y||^{-(\alpha+\eta+3/2)}. \end{aligned} \quad (4.18)$$

(ii) From [11], we have

$$\frac{d}{dz}\Phi_{\alpha,t}^{\eta}(z) = -\frac{\Gamma(\eta+1)}{2^{\alpha+2}\Gamma(\alpha+\eta+3)}t^{2\alpha+4}z\mathfrak{I}_{\alpha+\eta+2}(itz). \quad (4.19)$$

Then, similarly to the proof in (i), we have

$$\left|\frac{d}{dz}\Phi_{\alpha,t}^{\eta}(z)\right| \leq Ct^{\alpha-\eta+3/2}|z|^{-(\alpha+\eta+3/2)}; \quad t > 0, \quad z \in \mathbb{R}, \quad (4.20)$$

$$\tau_x\Phi_{\alpha,t}^{\eta}(y) - \Phi_{\alpha,t}^{\eta}(x) = \int_0^{\pi} \left[\Phi_{\alpha,t}^{\eta}((x,y)_{\theta}) - \Phi_{\alpha,t}^{\eta}((x,0)_{\theta}) \right] d\rho_{x,y}(\theta), \quad (4.21)$$

which can be written as

$$\begin{aligned} \tau_x\Phi_{\alpha,t}^{\eta}(y) - \Phi_{\alpha,t}^{\eta}(x) &= \int_0^{\pi} \int_0^1 \frac{d}{ds} \left[\Phi_{\alpha,t}^{\eta}((x,sy)_{\theta}) \right] ds d\rho_{x,y}(\theta) \\ &\leq |y| \int_0^{\pi} \int_0^1 \frac{d}{ds} \Phi_{\alpha,t}^{\eta}((x,sy)_{\theta}) d\rho_{x,y}(\theta) ds. \end{aligned} \quad (4.22)$$

Then from (4.20), it follows

$$\begin{aligned} \left| \tau_x\Phi_{\alpha,t}^{\eta}(y) - \Phi_{\alpha,t}^{\eta}(x) \right| &\leq C|y|t^{\alpha-\eta+3/2} \int_0^1 \int_0^{\pi} (x,sy)_{\theta}^{-(\alpha+\eta+3/2)} d\rho_{x,y}(\theta) ds \\ &\leq C|y|t^{\alpha-\eta+3/2} \int_0^1 ||x| - s|y||^{-(\alpha+\eta+3/2)} ds. \end{aligned} \quad (4.23)$$

Hence, if $0 < |y| < |x|$, we obtain

$$\left| \tau_x\Phi_{\alpha,t}^{\eta}(y) - \Phi_{\alpha,t}^{\eta}(x) \right| \leq C|y|t^{\alpha-\eta+3/2}||x| - |y||^{-(\alpha+\eta+3/2)}, \quad (4.24)$$

which completes the proof of the lemma. $\square$

Proof of Theorem 4.3. Let us first show that $C > 0$, exists such that

$$\left\| \sigma_{\alpha}^{\eta}(a) \right\|_{L_{\alpha}^p} \leq C, \quad (4.25)$$

for every (p, ∞) atom a .

Let a be an (p, ∞) atom. Suppose that $a(x) = 0$, $|x| > r$ and $\|a\|_{L^\infty_\alpha} \leq r^{-2(\alpha+1)/p}$. We choose $\ell \in \mathbb{Z}$ such that $2^{\ell-1} < r \leq 2^\ell$, we write

$$\int_{|x| \geq 4r} \left| \sigma_\alpha^\eta(a)(x) \right|^p d\mu_\alpha(x) \leq I_1 + I_2, \quad (4.26)$$

where

$$I_1 := \sum_{i=1}^{\infty} \int_{(i+1)2^\ell \leq |x| \leq (i+2)2^\ell} \sup_{t \geq \delta_i} \left| \sigma_{\alpha,t}^\eta(a)(x) \right|^p d\mu_\alpha(x), \quad (4.27)$$

$$I_2 := \sum_{i=1}^{\infty} \int_{(i+1)2^\ell \leq |x| \leq (i+2)2^\ell} \sup_{t < \delta_i} \left| \sigma_{\alpha,t}^\eta(a)(x) \right|^p d\mu_\alpha(x), \quad (4.28)$$

being $\delta_i = 2^{-\ell}/i^b$, where b will be specified later.

According to Lemma 4.4 (i), for $|x| \in [(i+1)2^\ell, (i+2)2^\ell]$, $i = 1, 2, \dots$, we get

$$\begin{aligned} \left| \sigma_{\alpha,t}^\eta(a)(x) \right| &\leq \int_{-r}^r |a(-y)| \left| \tau_x \Phi_{\alpha,t}^\eta(y) \right| d\mu_\alpha(y) \\ &\leq C t^{\alpha-\eta+1/2} r^{-2(\alpha+1)/p} \int_0^{2^\ell} ||x| - |y||^{-(\alpha+\eta+3/2)} d\mu_\alpha(y) \\ &\leq C \frac{(2^\ell t)^{\alpha-\eta+1/2}}{i^{\alpha+\eta+3/2} r^{2(\alpha+1)/p}}. \end{aligned} \quad (4.29)$$

Then, using the fact that $2^{\ell-1} < r \leq 2^\ell$, we obtain

$$I_1 \leq C \sum_{i=1}^{\infty} \left(\frac{\delta_i^{\alpha-\eta+1/2} 2^{\ell\{\alpha-\eta+1/2-2(\alpha+1)/p\}}}{i^{\alpha+\eta+3/2}} \right)^p i^{2\alpha+1} 2^{2\ell(\alpha+1)}, \quad (4.30)$$

and hence, it concludes that

$$I_1 \leq C \sum_{i=1}^{\infty} i^{2\alpha+1-\{\alpha+\eta+3/2+(\alpha-\eta+1/2)b\}p}. \quad (4.31)$$

Note that the last series is convergent provide that $b < (p(\alpha+\eta+3/2)-2(\alpha+1)/p(\eta-\alpha-1/2))$.

On the other hand, since $\int_{\mathbb{R}} a(y) d\mu_{\alpha}(y) = 0$, from Lemma 4.4 (ii) we have

$$\begin{aligned} \left| \sigma_{\alpha,t}^{\eta}(a)(x) \right| &\leq \int_{-r}^r |a(-y)| \left| \tau_x \Phi_{\alpha,t}^{\eta}(y) - \Phi_{\alpha,t}^{\eta}(x) \right| d\mu_{\alpha}(y) \\ &\leq C t^{\alpha-\eta+3/2} r^{-2(\alpha+1)/p} \int_0^{2^{\ell}} ||x| - |y||^{-(\alpha+\eta+3/2)} |y| d\mu_{\alpha}(y) \\ &\leq C \frac{(2^{\ell} t)^{\alpha-\eta+3/2}}{i^{\alpha+\eta+3/2} r^{2(\alpha+1)/p}}. \end{aligned} \quad (4.32)$$

Then, for $|x| \geq 4r$, we obtain

$$I_2 \leq C \sum_{i=1}^{\infty} \left(\frac{\delta_i^{\alpha-\eta+3/2} 2^{\ell\{\alpha-\eta+3/2-2(\alpha+1)/p\}}}{i^{\alpha+\eta+3/2}} \right)^p i^{2\alpha+1} 2^{2\ell(\alpha+1)}, \quad (4.33)$$

and in fact, we deduce that

$$I_2 \leq C \sum_{i=1}^{\infty} i^{2\alpha+1-\{\alpha+\eta+3/2+(\alpha-\eta+3/2)b\}p}. \quad (4.34)$$

The last series converges provided that $b > \frac{p(\alpha+\eta+3/2)-2(\alpha+1)}{p(\eta-\alpha-3/2)}$.

Note that we can find b such that the series in (4.31) and (4.34) converge if and only if $p > (2(\alpha+1)/\alpha+\eta+3/2)$. By combining (4.31) and (4.34) we show that

$$\int_{|x| \geq 4r} \left| \sigma_{\alpha}^{\eta}(a)(x) \right|^p d\mu_{\alpha}(x) \leq C, \quad (4.35)$$

for a certain $C > 0$ that is not depending on a .

From Lemma 4.1 (ii) and (4.35) we deduce that

$$\begin{aligned} \left\| \sigma_{\alpha}^{\eta}(a) \right\|_{L_{\alpha}^p}^p &\leq \left[\int_{|x| < 4r} + \int_{|x| \geq 4r} \right] \left| \sigma_{\alpha}^{\eta}(a)(x) \right|^p d\mu_{\alpha}(x) \\ &\leq C \left[\|a\|_{L_k^{\infty}}^p \int_{|x| < 4r} d\mu_{\alpha}(x) + 1 \right] \leq C, \end{aligned} \quad (4.36)$$

that is, (4.25) holds. Let now f be in $\mathcal{A}_{\alpha,\infty}^p$. Assume that $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where the series converges in $\mathcal{S}'(\mathbb{R})$, and for every $j \in \mathbb{N}$, a_j is a (p, ∞) atom and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. From Theorem 4.2, for $0 < p \leq 1$, we can write

$$\sigma_{\alpha,t}^{\eta}(f)(x) = \sum_{j=0}^{\infty} \lambda_j \sigma_{\alpha,t}^{\eta}(a_j)(x); \quad t > 0, \quad x \in \mathbb{R}. \quad (4.37)$$

Hence, from (4.25) it follows $\|\sigma_\alpha^\eta(f)\|_{L_\alpha^p}^p \leq C \sum_{j=0}^\infty |\lambda_j|^p$. Thus we conclude that $\|\sigma_\alpha^\eta(f)\|_{L_\alpha^p} \leq C \|f\|_{\mathcal{H}_{\alpha,\infty}^p}$. $\square$

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References

- [1] C. Fefferman and E. M. Stein, " H^p spaces of several variables," *Acta Mathematica*, vol. 129, no. 3-4, pp. 137-193, 1972.
- [2] S. Lu and D. Yang, "Multiplier theorems for Herz type Hardy spaces," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3337-3346, 1998.
- [3] C. F. Dunkl, "Differential-difference operators associated to reflection groups," *Transactions of the American Mathematical Society*, vol. 311, no. 1, pp. 167-183, 1989.
- [4] C. F. Dunkl, "Integral kernels with reflection group invariance," *Canadian Journal of Mathematics*, vol. 43, no. 6, pp. 1213-1227, 1991.
- [5] M. F. E. de Jeu, "The Dunkl transform," *Inventiones Mathematicae*, vol. 113, no. 1, pp. 147-162, 1993.
- [6] M. Rösler, "Bessel-type signed hypergroups on $\mathbb{R}$," in *Probability Measures on Groups and Related Structures, XI (Oberwolfach, 1994)*, H. Heyer and A. Mukherjea, Eds., pp. 292-304, World Scientific, River Edge, NJ, USA, 1995.
- [7] A. Gasmi, M. Sifi, and F. Soltani, "Herz-type Hardy spaces for the Dunkl operator on the real line," *Fractional Calculus & Applied Analysis*, vol. 9, no. 3, pp. 287-306, 2006.
- [8] C. F. Dunkl, "Hankel transforms associated to finite reflection groups," in *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications*, vol. 138 of *Contemporary Mathematics*, pp. 123-138, American Mathematical Society, Providence, RI, USA, 1992.
- [9] H. Ben Mohamed and K. Trimèche, "Dunkl transform on $\mathbb{R}$ and convolution product on new spaces of distributions," *Integral Transforms and Special Functions*, vol. 14, no. 5, pp. 437-458, 2003.
- [10] F. Soltani, " L^p -Fourier multipliers for the Dunkl operator on the real line," *Journal of Functional Analysis*, vol. 209, no. 1, pp. 16-35, 2004.
- [11] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, Mass, USA, 1966.