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Research Article

On Certain Classes of p-Valent Functions by Using Complex-Order and Differential Subordination

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The aim of the present paper is to study the p-valent analytic functions in the unit disk and satisfy the differential subordinations $z(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}/(p-j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)} \prec (a+(aB+(A-B)\beta)z)/a(1+Bz)$, where $I_p(r,\lambda)$ is an operator defined by Sălăgean and β is a complex number. Further we define a new related integral operator and also study the Fekete-Szego problem by proving some interesting properties.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_p$ denote the class of all analytic functions in the form of

$$f(z) = ez^p - \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} + {}_2F_1(a,b;c;z), \quad |z| < 1, \tag{1.1}$$

where $F_1(a,b;c;z)$ is Gaussian hypergeometric function defined by

$$\begin{gathered} {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n, \\ (a,n) = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1,n-1), \quad c > b > 0,\ c > a+b, \\ t_{n-p+1} = \frac{(a,n-p+1)(b,n-p+1)}{(c,n-p+1)(n-p+1)!}, \quad e > 0. \end{gathered} \tag{1.2}$$

Note that it is easy to see that these functions are analytic in the unit disk Δ; for more details on hypergeometric functions ${}_2F_1(a,b;c.z)$, see [1, 2].

Definition 1.1. A function $f \in \mathcal{A}_p$ is said to be in the class $S_p^*(\alpha)$, p-valently starlike functions of order α, if it satisfies $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$, $(0 \le \alpha < p, z \in \Delta)$. We write $S_p^*(0) = S_p^*$, the class of p-valently starlike functions in Δ.

Similarly, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_p(\alpha)$, p-valently convex of order α, if it satisfies $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > \alpha$, $(0 \le \alpha < p,\ z \in \Delta)$.

Let $h(z)$ be analytic and $h(0) = p$. A function $f \in \mathcal{A}_p$ is in the class $S_p^*(h)$ if

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in \Delta. \tag{1.3}$$

The class $S_p^*(h)$ and a corresponding convex class $\mathcal{C}_p(h)$ were defined by Ma and Minda in [3]. Similar results which are related to the convex class can also be obtained easily from the corresponding functions in $S_p^*(h)$. For example,

(i) if $p = 1$ and

$$h(z) = \frac{1+z}{1-z}, \tag{1.4}$$

then the classes reduce to the usual classes of starlike and convex functions;

(ii) if $h(z) = (1 + (1-2\alpha)z)/(1-z)$ where $0 \le \alpha < 1$, then the classes are reduced to the usual classes of starlike and convex functions of order α;

(iii) if $h(z) = p((1 + Az)/(1 + Bz))$, where $-1 \le B < A \le 1$, then the classes are reduced to the class of Janowski starlike functions $S_p^*[A,B]$ which is defined by

$$S_p^*[A,B] = \left\{ f \in \mathcal{A}_p : \frac{zf'}{f} \prec p\frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in \Delta \right\}; \tag{1.5}$$

(iv) if $h(z) = ((1+z)/(1-z))^{\alpha}$ where $p = 1$ and $0 < \alpha \le 1$, then the classes reduce to the classes of strongly starlike and convex functions of order α that consists of univalent functions $f \in \mathcal{A}$ satisfing

$$\left| \arg\left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \le 1,\ z \in \Delta \tag{1.6}$$

or equivalently we have

$$SS^*(\alpha) = \left\{ f \in \mathcal{A}_p : \frac{zf'}{f} \prec \left(\frac{1+z}{1-z} \right)^{\alpha},\ 0 < \alpha \le 1,\ z \in \Delta \right\}. \tag{1.7}$$

In the literature, there are several works and many researchers have been studying the related problems. For example, Obradović and Owa [4], Silverman [5], Obradowič and Tuneski [6], and Tuneski [7] have studied the properties of classes of functions which are defined in terms of the ratio of $1 + zf''(z)/f'(z)$ and $zf'(z)/f(z)$.

Definition 1.2. A function $f \in \mathcal{A}_p$ is said to be p-valent Bazilevic of type η and order α if there exists a function $g \in S_p^*$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f^{1-\eta}(z)g^{\eta}(z)}\right\} > \alpha \quad (z \in \Delta) \tag{1.8}$$

for some η $(\eta \geq 0)$ and α $(0 \leq \alpha < p)$. We denote by $\mathcal{B}_p(\eta, \alpha)$, the subclass of $\mathcal{A}_p$ consisting of all such functions. In particular, a function in $\mathcal{B}_p(1, \alpha) = \mathcal{B}_p(\alpha)$ is said to be p-valently close-to-convex of order α in Δ.

Definition 1.3. Let f and g be analytic functions in Δ, then we say f is subordinate to g and denoted by $f \prec g$ if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \Delta$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. Also, we say that g is superordinate to f; see [8].

Definition 1.4. Motivated by the multiplier transformation on $\mathcal{A}$, we define the operator $\mathcal{I}_p(r, \lambda)$; by the following infinite series when $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ then

$$\mathcal{I}_p(r, \lambda) f(z) = z^p + \sum_{n=1+p}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right)^r a_n z^n \quad (\lambda \geq 0). \tag{1.9}$$

Sălăgean derivative operator is closely related to the operator $\mathcal{I}_p(r, \lambda)$; see [9]. In [10], Uralegaddi and Somanatha also studied the case $\mathcal{I}_1(r, 1) = \mathcal{I}_r$. The operator $\mathcal{I}_1(r, \lambda) = \mathcal{I}_r^{\lambda}$ was studied recently by Cho and Srivastava [11] and Cho and Kim [12].

Definition 1.5. Differential operator, for each $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, we have

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} + \sum_{n=1+p}^{\infty} \frac{n!}{(n-j)!} a_n z^{n-j}, \tag{1.10}$$

where $n, p \in N$, $p > j$, and $j \in N_0 = \{0\} \cup N$. In particular, if $j = 0$ we have $f^{(0)}(z) = f(z)$.

Definition 1.6. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{A}_p(\lambda, r, j; h)$ if it satisfies the following subordination:

$$\frac{z\left(\mathcal{I}_p(r, \lambda) f(z)\right)^{(j+1)}}{(p-j)\left(\mathcal{I}_p(r, \lambda) f(z)\right)^{(j)}} \prec h(z), \tag{1.11}$$

and in this study we consider

$$h(z) = 1 + \frac{A-B}{a}\frac{\beta z}{1+Bz}, \quad z \in \Delta, \tag{1.12}$$

where $-1 \le B < A \le 1, a > 0$ and $\beta(\neq 0)$ is a complex number; so we denote $\overline{\mathcal{A}}_p(\lambda, r, j; h) = \overline{\mathcal{A}}_p(\lambda, r, j, \beta, a, A, B)$. Then we say that $f(z)$ is superordinate to $h(z)$ if $f(z)$ satisfies the following:

$$h(z) \prec \frac{z(\mathcal{O}_p(r,\lambda)f(z))^{(j+1)}}{(p-j)(\mathcal{O}_p(r,\lambda)f(z))^{(j)}}, \tag{1.13}$$

where $h(z)$ is analytic in Δ and $h(0) = 1$.

Further we note that if

$$\frac{z(\mathcal{O}_p(r,\lambda)f(z))^{(j+1)}}{(\mathcal{O}_p(r,\lambda)f(z))^{(j)}} \prec \frac{(p-j)[a+(aB+(A-B)\beta)z]}{a(1+Bz)} = h(z). \tag{1.14}$$

By choosing $j = r = 0,\ p = 1$, so $h(0) = 1$, then $f(z) \in S^*(h)$. For $a = A = \beta = 1, B = -1$, and $p \ge 1$, we have $f(z) \in S_p^*(1)$. But if $a = \beta = 1$ and $j = r = 0$ and $-1 \le B < A \le 1$, then $f(z) \in S^*[A, B]$, a class of Janowski starlike functions. If we put $p = a = \beta = A = 1$, $B = -1$, then $f(z) \in SS^*(1)$ classes of strongly starlike. By Definition 1.2, if $g(z) \in S^*$, univalent starlike, and $j = r = 0$ and $p = a = A = \beta = 1$, $B = -1$ and ifw $\mathrm{Re}\{zf'(z)/f(z)g^2(z)\} > 1$, then $f(z) \in \mathcal{B}(2,1)$ is a class Bazilevic functions of type $\eta = 2$ and order $\alpha = 1$.

2. Main Results

Theorem 2.1. *Let the function $f(z)$ be of the form* (1.1). *If some $A, B, (-1 < B < A \le 1)$, and $\beta(\neq 0)$ are complex numbers and*

$$\begin{aligned}\sum_{m=p+1}^{\infty} \gamma_\lambda^r(m,p)\left[a(1+B)(\delta(m,j+1) - \delta(m,j)(p-j)) - \delta(m,j)(A-B)(p-j)|\beta|\right]k_m \\ < |\beta| e (A-B)\delta(p, j+1),\end{aligned} \tag{2.1}$$

then $f(z) \in \overline{\mathcal{A}}_p(\lambda, r, j, \beta, a, A, B)$, where $\gamma_\lambda^r(m,p) = ((m+\lambda)/(p+\lambda))^r, \delta(m,j) = m!/(m-j)!$ and for $r, j \in \mathbb{N}_0,\ \lambda > 0,\ p \in \mathbb{N},\ j < p$. The result is sharp.

Proof. Since the function $f(z)$ in the theorem can be expressed in the form

$$f(z) = ez^p + \sum_{n=2p}^{\infty} k_{n-p+1} z^{n-p+1} \quad \text{or} \quad f(z) = ez^p + \sum_{m=p+1}^{\infty} k_m z^m, \tag{2.2}$$

where $m = n - p + 1$ and $k_m = (a,m)(b,m)/(c,m)m!$, and also we have for all $r, j \in \mathcal{N}_0$,

$$
\begin{aligned}
(\mathcal{I}_p(r,\lambda)f(z))^{(j)} &= \frac{ep!}{(p-j)!}z^{p-j} + \sum_{m=p+1}^{\infty}\left(\frac{m+\lambda}{p+\lambda}\right)^r \frac{m!}{(m-j)!}k_m z^{m-j} \\
&= e\delta(p,j)z^{p-j} + \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)\delta(m,j)k_m z^{m-j},
\end{aligned}
\tag{2.3}
$$

now, assume that the condition (2.1) holds true. We show that $f \in \overline{\mathcal{A}}_p(\lambda, r, j, \beta, a, A, B)$. Equivalently, we prove that

$$
\left| \frac{az(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)} - a(p-j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)}}{(p-j)R(\mathcal{I}_p(r,\lambda)f(z))^{(j)} - Baz(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}} \right| < 1, \tag{2.4}
$$

where $R = aB + (A-B)\beta$. But we have

$$
\begin{aligned}
&\left| \frac{az(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)} - a(p-j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)}}{(p-j)R(\mathcal{I}_p(r,\lambda)f(z))^{(j)} - Baz(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}} \right| \\
&= \left| \frac{\left[a\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(\delta(m,j+1) - \delta(m,j)(p-j))k_m z^{m-j}\right]}{\left[\beta(A-B)\delta(p,j+1)ez^{p-j} - \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(aBC - \delta(m,j)(A-B)\beta(p-j))k_m z^{m-j}\right]} \right| \\
&< \left\{ \frac{\left[a\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(\delta(m,j+1) - \delta(m,j)(p-j))k_m\right]}{\left[|\beta|e(A-B)\delta(p,j+1) - \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(aBC - \delta(m,j)(A-B)|\beta|(p-j))k_m\right]} \right\} < 1.
\end{aligned}
\tag{2.5}
$$

where C denotes $(\delta(m,j+1) - \delta(m,j)(p-j))$.

The last inequality is true by (2.1) and this completes the proof. The result is sharp for the functions $f_m(z)$ defined in Δ by

$$
\begin{aligned}
&f_m(z) \\
&= ez^p + \frac{|\beta|e(A-B)\delta(p,j+1)}{\gamma_\lambda^r(m,p)\left[a(1+B)(\delta(m,j+1) - \delta(m,j)(p-j)) - \delta(m,j)(A-B)(p-j)|\beta|\right]}z^m
\end{aligned}
\tag{2.6}
$$

for $m \geq p+1$. □

Remark 2.2. We observe that if $B \neq 0$, the converse of the above theorem needs not be true. For instance, consider the function $f(z)$ defined by

$$\frac{z(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}{((p-j)\mathcal{I}_p(r,\lambda)f(z))^{(j)}} \prec \frac{a-\mathrm{Sgn}(B)(pB+(A-B)\beta)z}{a(1-\mathrm{Sgn}(B)Bz)}, \tag{2.7}$$

where $\mathrm{Sgn}(B) = 1, 0, -1$ thus accordingly $B > 0$, $B = 0$ and $B < 0$. It is easily seen that $f(z) \in \overline{\mathcal{A}}_p(\lambda, r, j, \beta, a, A, B)$ and

$$k_m = -\frac{(1+B)\ \beta e(A-B)\delta(p,j+1)\mathrm{Sgn}(B)^{m-p-1}B^{m-p-2}}{[a(1+B)(\delta(m,j+1)+\delta(m,j)(p-j))+\delta(m,j)(A-B)(p-j)\beta]\gamma_\lambda^r(m,p)} \quad (m \geq p+1) \tag{2.8}$$

so that

$$\begin{aligned}\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)&\left[\frac{a(1+B)(\delta(m,j+1)-\delta(m,j)(p-j))-\beta(A-B)\delta(m,j)(p-j)}{\beta e(A-B)\delta(p,j+1)}\right]|k_m| \\ &= (1+B)\sum_{m=p+1}^{\infty}(B)^{m-p-2} = \frac{1+B}{1-B} > 1,\end{aligned} \tag{2.9}$$

where A, B are satisfying the conditions $-1 \leq B < A \leq 1$, $0 < B < 1$. This establishes our claim.

Theorem 2.3. *If the function $f(z) \in \overline{\mathcal{A}}_p(\lambda, r, j, \beta, a, A, B)$, then*

$$|k_m| \leq \frac{|\beta|(A-B)\delta(p,j+1)e}{a\gamma_\lambda^r(m,p)(\delta(m,j+1)-\delta(m,j)(p-j))}, \quad m \geq p-1, \tag{2.10}$$

where $-1 \leq B < A \leq 1$ and $0 < a < A-B \leq 1$, and the estimate is sharp.

Proof. We have

$$\frac{z\left(\mathcal{I}_p(r,\lambda)f(z)\right)^{(j+1)}}{(p-j)\left(\mathcal{I}_p(r,\lambda)f(z)\right)^{(j)}}=\frac{a+(aB+(A-B)\beta)w(z)}{a(1+Bw(z))}, \tag{2.11}$$

where $w(z)=\sum_{i=p+1}^{\infty}w_{i-p}z^{i-p}$ is defined as in the Definition 1.3. Now we can write

$$\begin{aligned}&a\sum_{i=p+1}^{\infty}\gamma_{\lambda}^{r}(i,p)\left(\delta(i,j+1)-\delta(i,j)(p-j)\right)k_i z^{i-j}\\&=\left\{\beta(A-B)\delta(p,j+1)ez^{p-j}-\sum_{i=p+1}^{\infty}\gamma_{\lambda}^{r}(i,p)\left(\delta(i,j+1)Ba-\delta(i,j)(p-j)R\right)k_i z^{i-j}\right\}\sum_{i=p+1}^{\infty}w_{i-p}z^{i-p},\end{aligned} \tag{2.12}$$

where $R=aB+(A-B)\beta$. Now if we equalize the coefficients of the same power of z in both sides, then we have

$$\begin{aligned}&a\sum_{i=p-1}^{m}\gamma_{\lambda}^{r}(i,p)\left(\delta(i,j+1)-\delta(i,j)(p-j)\right)k_i z^{i-j}+\sum_{i=m+1}^{\infty}c_i z^{i-j}\\&\quad=\left\{\beta(A-B)\delta(p,j+1)ez^{p-j}-\sum_{i=p+1}^{m-1}\gamma_{\lambda}^{r}(i,p)\left(\delta(i,j+1)Ba-\delta(i,j)(p-j)R\right)k_i z^{i-j}\right\}w(z),\end{aligned} \tag{2.13}$$

where c_i's are suitable constants. By multiplying each side of the above equation by its conjugate and letting $|z|=1$, $r\to 1^-$, we get

$$\begin{aligned}&a^2\sum_{i=p-1}^{m}\gamma_{\lambda}^{2r}(i,p)\left(\delta(i,j+1)-\delta(i,j)(p-j)\right)^2|k_i|^2\\&\quad\le\left[|\beta|(A-B)\delta(p,j+1)e\right]^2+\sum_{i=p+1}^{m-1}\gamma_{\lambda}^{2r}(i,p)\left(\delta(i,j+1)Ba-\delta(i,j)(p-j)R\right)^2|k_i|^2\end{aligned} \tag{2.14}$$

so that

$$\begin{aligned}&a^2\gamma_{\lambda}^{2r}(m,p)\left(\delta(m,j+1)-\delta(m,j)(p-j)\right)^2|k_m|^2\\&\quad\le\left[|\beta|(A-B)\delta(p,j+1)e\right]^2-\left(1-a^2\right)\sum_{i=p+1}^{m-1}\gamma_{\lambda}^{2r}(i,p)\left(\delta(i,j+1)Ba-\delta(i,j)(p-j)R\right)^2|k_i|^2.\end{aligned} \tag{2.15}$$

Since $-1 < B < A \leq 1$ and $0 < a < A - B < 1$, we have

$$|k_m| \leq \frac{|\beta|(A-B)\delta(p,j+1)e}{a\gamma_\lambda^r(m,p)(\delta(m,j+1)-\delta(m,j)(p-j))}, \quad m \geq p-1 \tag{2.16}$$

and this completes the proof. Note that the estimate in (2.10) is sharp for the functions $f_m(z)$ defined in Δ; when $j = r = 0$ in (1.3), then

$$f_m(z) = \exp\left[\int_0^z \frac{p(\psi(t)R+a)}{t(1+B\psi(t))}dt\right], \quad m \geq 1+p, \tag{2.17}$$

where $|\psi(t)| < 1$, $z \in \Delta$ and $R = aB + (A-B)\beta$. We can choose $\psi(t) = t^m$. □

Theorem 2.4 ([Fekete-Szego Problem]). *Let the function $f(z)$, given by* (2.2)*, be in the class $\overline{\mathcal{A}}_p(\lambda, j, \beta, a, A, B)$ and μ any complex number. Then*

$$\begin{aligned}
&\left|k_{p+2} - \mu k_{p+1}^2\right| \\
&\leq \frac{(A-B)\delta(p,j+1)e|\beta|}{2a\gamma_\lambda^r(p+2,p)} \\
&\quad\times \max\left\{1, \left|\frac{2\gamma_\lambda^r(p+2,p)\delta(p+2,j)(a\gamma_\lambda^r(p+1,p)\delta(p+1,j)+\mu(A-B)\delta(p,j+1)e\beta)}{a(\gamma_\lambda^r(p+1,p))^2}\right|\right\}.
\end{aligned} \tag{2.18}$$

Proof. On using the coefficients of z^{p+1} and z^{p+2}, we get

$$\begin{aligned}
k_{p+1} &= \frac{(A-B)\beta e\delta(p,j+1)}{a\gamma_\lambda^r(p+1,p)\delta(p+1,j)}w_1, \\
k_{p+2} &= \frac{(A-B)e\beta\delta(p,j+1)}{a\gamma_\lambda^r(p+2,p)2\delta(p+2,j)}w_2 - \frac{(A-B)e\beta\delta(p,j+1)}{a\gamma_\lambda^r(p+1,p)\delta(p+1,j)}w_1^2.
\end{aligned} \tag{2.19}$$

By using [13] that

$$\left|w_2 - \rho w_1^2\right| \leq \max\{1, |\rho|\} \tag{2.20}$$

for every complex number ρ, then we can write

$$\begin{aligned}
&\left|k_{p+2}-\mu k_{p+1}^2\right| \\
&\quad = \left|\frac{1}{a}(A-B)\delta(p,j+1)e\beta\left(\frac{w_2}{2\gamma_\lambda^r(p+2,p)\delta(p+2,j)}-\frac{w_1^2}{\gamma_\lambda^r(p+1,p)\delta(p+1,j)}\right)\right. \\
&\qquad \left. -\ \mu\left(\frac{(A-B)\delta(p,j+1)e\beta}{a\gamma_\lambda^r(p+1,p)\delta(p+1,j)}\right)^2 w_1^2\right| \\
&\quad = \left|\frac{(A-B)\delta(p,j+1)e\beta}{2a\gamma_\lambda^r(p+2,p)\delta(p+2,j)}w_2\right. \\
&\qquad \left. -\frac{(A-B)\delta(p,j+1)e\beta a\gamma_\lambda^r(p+1,p)\delta(p+1,j)-\mu((A-B)\delta(p,j+1)e\beta)^2}{\left(a\gamma_\lambda^r(p+1,p)\delta(p+1,j)\right)^2}w_1^2\right| \\
&\quad = \frac{(A-B)\delta(p,j+1)e|\beta|}{2a\gamma_\lambda^r(p+2,p)}\left|w_2-hw_1^2\right|,
\end{aligned} \tag{2.21}$$

where

$$h=\frac{2\gamma_\lambda^r(p+2,p)\delta(p+2,j)\left(a\gamma_\lambda^r(p+1,p)\delta(p+1,j)-\mu(A-B)e\beta\delta(p,j+1)\right)}{a\left(\gamma_\lambda^r(p+1,p)\delta(p+1,j)\right)^2}. \tag{2.22}$$

□

3. Integral Operator

Now, we introduce a new integral operator which is denoted by $\mathcal{G}_{\eta,p}(z)$ on functions belonging to $\overline{\mathcal{A}}_p$ as follows:

$$\mathcal{G}_{\eta,p}(z)=e(1-\eta)z^p+\eta p\int_\epsilon^z\frac{f(t)}{t}dt \quad \left(0<\eta<1,\ \epsilon\longrightarrow 0^+\right), \tag{3.1}$$

and we verify the effect of this operator on (1.11); with a simple calculation, we have

$$\begin{aligned}
\mathcal{G}_{\eta,p}(z) &= e(1-\eta)z^p+\eta p\left[\int_\epsilon^z\left(et^{p-1}+\sum_{m=p+1}^\infty k_m t^{m-1}\right)dt\right] \quad \left(0<\eta<1,\ \epsilon\longrightarrow 0^+\right) \\
&= e(1-\eta)z^p+\eta e z^p+\sum_{m=p+1}^\infty\frac{\eta p}{m}k_m z^m \\
&= ez^p+\sum_{m=p+1}^\infty d_m z^m,
\end{aligned} \tag{3.2}$$

where $d_m = (\eta p/m)k_m$. If we put $r = j = 0$ in (1.11), then we obtain

$$\frac{zf'(z)}{pf(z)} \prec \frac{a + \left(aB + (A-B)\beta\right)z}{a(1+Bz)} \tag{3.3}$$

that is denoted by $\overline{\mathcal{A}}_p(\lambda, 0, 0, \beta, a, A, B) = \overline{\mathcal{A}}_p(\lambda, \beta, a, A, B)$.

Now if we let $\overline{\mathcal{A}}_{p,\eta}(\lambda, \beta, a, A, B)$ be a class of functions $\mathcal{G}_{\eta,p}(z)$ analytic in Δ and defined by (3.1) where $f(z) \in \overline{\mathcal{A}}_p(\lambda, \beta, a, A, B)$. Then on using (3.1) and definition of subordination, we have the following theorem.

Theorem 3.1. *$\mathcal{G}_{\eta,p}(z) \in \overline{\mathcal{A}}_{p,\eta}(\lambda, \beta, a, A, B)$ if and only if*

$$\frac{zcg'_{\eta,p}(z) + z^2\mathcal{G}''_{\eta,p}(z) - ep^2(1-\eta)z^p}{pz\mathcal{G}'_{\eta,p} - ep^2(1-\eta)z^p} \prec \frac{a + \left(aB + (A-B)\beta\right)z}{a(1+Bz)}. \tag{3.4}$$

Proof. The conditions (3.3) and (3.1) give

$$\begin{aligned}
\mathcal{G}'_{\eta,p}(z) &= ep(1-\eta)z^{p-1} + \eta p\frac{f(z)}{z} \quad \text{or} \quad f(z) = \frac{z\mathcal{G}'_{\eta,p}(z)}{\eta p} - \frac{e(1-\eta)z^p}{\eta}, \\
f'(z) &= \frac{1}{\eta, p}\left(\mathcal{G}'_{\eta,p}(z) + z\mathcal{G}''_{\eta,p}(z)\right) - \frac{ep}{\eta}(1-\eta)z^{p-1} \\
&= \frac{1}{\eta p}\left(H_{\eta,p}(z)\right)' - \frac{ep}{\eta}(1-\eta)z^{p-1},
\end{aligned} \tag{3.5}$$

where $H_{\eta,p}(z) = z\mathcal{G}'_{\eta,p}(z)$. By putting $f'(z)$ and $f(z)$ in (3.3), we obtain

$$\begin{aligned}
\frac{zf'(z)}{pf(z)} &= \frac{zH'_{\eta,p}(z) - ep^2(1-\eta)z^p}{H_{\eta,p}(z) - ep^2(1-\eta)z^p} \\
&= \frac{z\mathcal{G}'_{\eta,p}(z) + z^2\mathcal{G}''_{\eta,p}(z) - ep^2(1-\eta)z^p}{pz\mathcal{G}'_{\eta,p}(z) - ep^2(1-\eta)z^p} \prec \frac{a + \left(aB + (A-B)\beta\right)z}{a(1+Bz)}.
\end{aligned} \tag{3.6}$$

With a simple calculation on $F_\xi(z)$, we have

$$\begin{aligned}
F_\xi(z) &= \frac{p+\xi}{z^\xi}\left[\int_0^z s^{\xi-1}\left(es^p + \sum_{m=p+1}^{\infty} k_m s^m\right)ds\right] \\
&= ez^p + \sum_{m=p+1}^{\infty} \frac{p+\xi}{m+\xi}k_m z^m \\
&= ez^p + \sum_{p+1}^{\infty} b_m z^m \quad \text{where } b_m = \frac{p+\xi}{m+\xi}k_m.
\end{aligned} \tag{3.7}$$

Let $\overline{\mathcal{A}}_{p,\xi}(\lambda,\beta,a,A,B)$ be the class of functions $F_{\xi}(z)$ analytic in Δ defined by $f\in\overline{\mathcal{A}}_{p}(\lambda,\beta,a,A,B)$. We can write next theorem on using (3.3) and definition of subordination. □

Theorem 3.2. *The $F_{\xi}(z)\in\overline{\mathcal{A}}_{p,\xi}(\lambda,\beta,a,A,B)$ if and only if*

$$\frac{z\left((\xi+1)F_{\xi}'(z)+zF_{\xi}''(z)\right)}{p\left(\xi F_{\xi}(z)+zF_{\xi}'(z)\right)}\prec\frac{a+(aB+(A-B)\beta)z}{a(1+Bz)}. \tag{3.8}$$

Proof. Since

$$F_{\xi}(z)=\frac{p+\xi}{z^{\xi}}\int_{0}^{z}s^{\xi-1}f(s)ds, \tag{3.9}$$

then we have

$$\begin{aligned} f(z)&=\frac{1}{p+\xi}\left(\xi F_{\xi}(z)+zF_{\xi}'(z)\right),\\ f'(z)&=\frac{1}{p+\xi}\left((\xi+1)F_{\xi}'(z)+zF_{\xi}''(z)\right). \end{aligned} \tag{3.10}$$

Now by making substitution $f'(z)$ and $f(z)$ in (3.3), we obtain

$$\begin{aligned} \frac{zf'(z)}{pf(z)}&=\frac{(z/(p+\xi))\left((\xi+1)F_{\xi}'(z)+zF_{\xi}''(z)\right)}{(p/(p+\xi))\left(\xi F_{\xi}(z)+zF_{\xi}'(z)\right)}\\ &=\frac{z\left((\xi+1)F_{\xi}'(z)+zF_{\xi}''(z)\right)}{p\left(\xi F_{\xi}(z)+zF_{\xi}'(z)\right)}\prec\frac{a+(aB+(A-B)\beta)z}{a(1+Bz)}. \end{aligned} \tag{3.11}$$

□

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