Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 284875, 14 pages
doi:10.1155/2010/284875

Research Article

The Near-Ring of Lipschitz Functions on a Metric Space

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Received 6 August 2009; Revised 3 April 2010; Accepted 25 April 2010

Academic Editor: Francois Goichot

This paper treats near-rings of zero-preserving Lipschitz functions on metric spaces that are also abelian groups, using pointwise addition of functions as addition and composition of functions as multiplication. We identify a condition on the metric ensuring that the set of all such Lipschitz functions is a near-ring, and we investigate the complications that arise from the lack of left distributivity in the resulting right near-ring. We study the behavior of the set of invertible Lipschitz functions, and we initiate an investigation into the ideal structure of normed near-rings of Lipschitz functions. Examples are given to illustrate the results and to demonstrate the limits of the theory.

1. Introduction and Background

Banach spaces and Banach algebras of scalar-valued Lipschitz functions on a metric space have been studied in some depth by functional analysts for the past half of a century. The papers of Arens and Eells [1], de Leeuw [2], Sherbert [3, 4], and Johnson [5] contain some of the important early work on these topics. The book of Weaver [6] provides a systematic treatment of both the analytic and algebraic results concerning spaces of scalar-valued Lipschitz functions on a metric space. The Lipschitz functions considered therein are usually bounded and map a metric space (X, ρ) to a Banach space E (often $\mathbb{R}$ or $\mathbb{C}$), so that addition (and multiplication in case $E = \mathbb{R}$ or $E = \mathbb{C}$) of functions is defined. The Lipschitz number of a function f, denoted $\|f\|_L$, is used in combination with the infinity norm $\|\cdot\|_\infty$ to produce a norm $\|\cdot\| := \max(\|\cdot\|_L, \|\cdot\|_\infty)$. If one identifies a distinguished basepoint in X, then $\|\cdot\|_L$ is used as a norm on the set of basepoint preserving Lipschitz functions mapping (X, ρ) to E.

In this paper, we initiate an analogous study of zero-preserving Lipschitz functions on a metric space that is also an abelian group, using pointwise addition of functions as addition and function composition as multiplication. Our Lipschitz functions, therefore, map a metric space (X, ρ) to itself and may be regarded as a generalization of the bounded linear operators that are so important in analysis. Rather than taking X to be a Banach space, we only require $(X, +)$ to be an abelian group. By restricting the possible metrics ρ on X, we ensure that the set of zero-preserving Lipschitz functions on X is a near-ring under pointwise addition of functions and function composition. Near-rings and near-algebras, the nonlinear counterparts of rings and algebras, respectively, have a rich theory of their own. Basic near-ring definitions and results can be found in the books of Pilz [7], Clay [8], and Meldrum [9]; the dissertation of Brown [10] is the seminal work in near-algebras. Closely related to our present work is the dissertation of Irish [11], which considers near-algebras of Lipschitz functions on a Banach space.

In the next section, we give the required definitions and elementary results. Next, we study the behavior of the set of units and also of ideals in normed near-rings of Lipschitz functions under topological closure. We conclude the paper by investigating the ideal structure of near-rings of Lipschitz functions.

2. Definitions, Notation, and Elementary Results

We begin this section by recalling the definition of a Lipschitz function.

Definition 2.1 (see [6, 12]). A function f from a metric space (X_1, ρ_1) to a metric space (X_2, ρ_2) is *Lipschitz* if there exists a constant $K \geq 0$ such that for all $x, y \in X_1$, $\rho_2(f(x), f(y)) \leq K\rho_1(x, y)$. If $f : (X_1, \rho_1) \to (X_2, \rho_2)$ is Lipschitz, the *Lipschitz number* of f is defined as

$$\|f\|_L := \sup\left\{ \frac{\rho_2(f(x), f(y))}{\rho_1(x, y)} \mid x, y \in X_1,\ x \neq y \right\}. \tag{2.1}$$

Remark 2.2. If f is Lipschitz, then $\|f\|_L \geq 0$. Also, $\|f\|_L = 0$ if and only if f is constant. For Lipschitz functions $f : (X_1, \rho_1) \to (X_2, \rho_2)$ and $g : (X_2, \rho_2) \to (X_3, \rho_3)$, we have that $\|g \circ f\|_L \leq \|g\|_L \|f\|_L$.

Remark 2.3. It is well-known that a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous and therefore differentiable almost everywhere (a.e.). Also, the derivative of f is bounded a.e. in magnitude by the Lipschitz constant, and for $a \leq b$, the difference $f(b) - f(a)$ is equal to the integral of the derivative of f on the interval $[a, b]$. Conversely, if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous (and thus differentiable a.e.) and if $|f'(x)| \leq K$ a.e., then f is Lipschitz with Lipschitz constant at most K. We will only use these observations in the case where f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ that is everywhere, except possibly for finitely many points, differentiable; also, the derivative of f will be continuous at all points where it exists. In this case f is Lipschitz if and only if the set $\{|f'(x)| : f \text{ is differentiable at } x\}$ is bounded and

$$\|f\|_L = \sup\{|f'(x)| : f \text{ is differentiable at } x\}. \tag{2.2}$$

In this restricted case these facts about real-valued Lipschitz functions are elementary consequences of the definition of Lipschitz functions, the definition of derivatives of real-valued functions, and the mean value theorem. The interested reader can consult [12] or [6] for more on Lipschitz functions.

In what follows, all metric spaces (X, ρ) are also abelian groups under the operation $+$, with identity element 0. We exclude the trivial case $X = \{0\}$. If the metric ρ satisfies the condition in the next definition, the pointwise addition of two Lipschitz functions is again a Lipschitz function.

Definition 2.4. Let $K > 0$ be a real number. A metric ρ on a metric space X is *K-subadditive* on X if

$$\frac{\rho(a+b, c+d)}{\rho(a,c)+\rho(b,d)}, \tag{2.3}$$

for all $a, b, c, d \in X$ with $\rho(a,c)+\rho(b,d) \neq 0$, bounded, and

$$\sup_{\rho(a,c)+\rho(b,d)\neq 0} \frac{\rho(a+b, c+d)}{\rho(a,c)+\rho(b,d)} = K. \tag{2.4}$$

Remark 2.5. If ρ is a metric on X and we define the metric $\overline{\rho}$ on $X \times X$ via $\overline{\rho}((a,b),(c,d)) = \rho(a,c) + \rho(b,d)$, then K-subadditivity of ρ is equivalent to having the Lipschitz number of $+ : X \times X \to X$ equal to K.

Example 2.6. Let $(X, \|\cdot\|)$ be a normed vector space and let the metric ρ be defined on X by $\rho(x,y) = \|x-y\|$ for all $x, y \in X$. Then ρ is 1-subadditive.

Assume that ρ is a 1-subaddive metric on the abelian group X, and define $\|\cdot\| : X \to \mathbb{R}^+ \cup \{0\}$ by $\|x\| := \rho(x, 0)$ for all $x \in X$. We will show that $\|\cdot\|$ satisfies the properties given in the next definition.

Definition 2.7 (see, e.g., [13]). A function $\|\cdot\| : X \to \mathbb{R}^+ \cup \{0\}$ is a *norm* on the abelian group X if $\|\cdot\|$ satisfies the following criteria:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|x\| = \|-x\|$ for all $x \in X$;

(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Remark 2.8. Let $\|\cdot\|$ be a norm on an abelian group. Then the function $\rho : X \times X \to \mathbb{R}^+ \cup \{0\}$, defined by $\rho(x,y) := \|x-y\|$ for all $x, y \in X$, is a 1-subadditive metric on X.

Example 2.9. Let X be a multiplicative subgroup of the unit circle in the complex plane, and take "addition" in X to be complex multiplication (so that $1 \in X$ is the neutral element). If we denote by $\|z\|$ the Euclidean distance between the complex numbers z and 1, $\|\cdot\|$ is a norm on the abelian group X.

Next we give some of the elementary properties of K-subadditive metrics.

Proposition 2.10. *Assume that ρ is K-subadditive on the metric space X. Then*

(1) $\rho(x-y, 0) \leq K\rho(x,y)$ *and* $\rho(x,y) \leq K\rho(x-y,0)$;

(2) $\rho(x,y) \leq K^2\rho(-x,-y)$ *and* $\rho(-x,-y) \leq K^2\rho(x,y)$;

(3) $K \geq 1$;

(4) *if* $K = 1$, *then* $\rho(x, y) = \rho(x - y, 0)$ *and* $\rho(x, y) = \rho(-x, -y)$;

(5) *if* $K = 1$, *then* $\|\cdot\| : X \to \mathbb{R}^+ \cup \{0\}$, *defined by* $\|x\| := \rho(x, 0)$ *for all* $x \in X$, *is a norm on* X *and* $\|x - y\| = \rho(x, y)$ *for all* $x, y \in X$.

Proof. We prove (1) and (2). The other parts follow immediately from the following:

(1)

$$\begin{aligned}\rho(x-y,0) &= \rho(x-y,y-y)\\ &\leq K(\rho(x,y)+\rho(-y,-y))\\ &= K\rho(x,y),\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\rho(x,y) &= \rho((x-y)+y,0+y)\\ &\leq K(\rho(x-y,0)+\rho(y,y))\\ &= K\rho(x-y,0).\end{aligned} \tag{2.6}$$

(2)

$$\begin{aligned}\rho(x,y) &= \rho(y,x)\\ &= \rho(x-x+y,x-y+y)\\ &\leq K[\rho(x-x,x-y)+\rho(y,y)]\\ &\leq K^2[\rho(x,x)+\rho(-x,-y)]\\ &= K^2\rho(-x,-y).\end{aligned} \tag{2.7}$$

Thus $\rho(x, y) \leq K^2\rho(-x, -y)$ and therefore also $\rho(-x, -y) \leq K^2\rho(x, y)$. □

Remark 2.11. Note that from part (5) of Proposition 2.10 and Remark 2.8 we have that any 1-subadditive metric on an abelian group X is induced by a norm on X and, conversely, any norm on X induces a 1-subadditive metric.

The following is an example of a metric space with a metric that is not K-subadditive for any K.

Example 2.12. Consider the metric on $\mathbb{R}$ given by $\rho(a, b) := |2^a - 2^b|$ for $a, b \in \mathbb{R}$. Then if ρ is K-subadditive, we would have that $2^{n-1} = \rho(n, n - 1) \leq K$ for all $n \in \mathbb{N}$, which is clearly not possible.

Notation 2.13. For a metric space (X, ρ), we denote by $\mathcal{L}_{X,\rho}$, or simply $\mathcal{L}_X$ when there is no ambiguity about the metric, the set of zero-preserving Lipschitz functions on X.

Using some of the elementary properties of K-subadditive metrics, we obtain the following properties for $\|\cdot\|_L$.

Proposition 2.14. *Let ρ be a K-subadditive metric on X. Then for all $f, g \in \mathcal{L}_X$ one has*

(1) $\|f\|_L \geq 0$ *and* $\|f\|_L = 0$ *if and only if* $f = 0_X$*;*

(2) $\|-f\|_L \leq K^2\|f\|_L$*;*

(3) $\|f \circ g\|_L \leq \|f\|_L\|g\|_L$*;*

(4) $\|f + g\|_L \leq K(\|f\|_L + \|g\|_L)$*.*

Proof. The result follows from Definitions 2.1 and 2.4 and Proposition 2.10. □

We now recall the definition of a near-ring.

Definition 2.15. A triple $(N, +, *)$ is called a (right) *near-ring* if

(1) $(N, +)$ is a (not necessarily abelian) group,

(2) $(N, *)$ is a semigroup,

(3) for all $a, b, c \in N$, $(a + b) * c = a * c + b * c$.

A near-ring $(N, +, *)$ is called *zero-symmetric* if, for all $n \in N$, $n * 0_N = 0_N$, where 0_N is the neutral element of $(N, +)$.

If $(G, +)$ is any group, then $\mathcal{M}(G)$, the set of all self-maps of G, is a near-ring under pointwise addition and function composition. The set of all zero-preserving self-maps of G, $\mathcal{M}_0(G)$, is a zero-symmetric sub-near-ring of $\mathcal{M}(G)$. Further examples of near-rings, along with many of the basic results of the theory of near-rings, may be found in the books of Clay [8], Meldrum [9], and Pilz [7].

Following the definition of a normed ring as given in [14], we make the following analogous definition.

Definition 2.16. A *normed near-ring* $(N, \|\cdot\|)$ is a near-ring N with a function $\|\cdot\| : N \to \mathbb{R}^+ \cup \{0\}$, such that

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|x\| = \|-x\|$ for all $x \in N$;

(3) $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in N$;

(4) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in N$.

Proposition 2.17. *Assume that ρ is a K-subadditive metric on X. Then*

(1) $(\mathcal{L}_{X,\rho}, +, \circ)$ *("$+$" is pointwise addition and "$\circ$" is function composition) is a zero-symmetric near-ring with identity;*

(2) $(\mathcal{L}_{X,\rho}, \|\cdot\|_L)$ *is a normed near-ring if* $K = 1$*.*

Proof. (1) We show that if $f, g \in \mathcal{L}_X$, then $f \circ g, f + g, -f \in \mathcal{L}_X$. From Remark 2.2, $f \circ g \in \mathcal{L}_X$, and from Proposition 2.10, we have for $x, y \in X$,

$$\rho(-f(x), -f(y)) \leq K^2 \rho(f(x), f(y)), \tag{2.8}$$

and thus $f \in \mathcal{L}_X$ implies that $-f \in \mathcal{L}_X$. Also,

$$\rho(f(x) + g(x), f(y) + g(y)) \le K\rho(f(x), f(y)) + K\rho(g(x), g(y)), \tag{2.9}$$

since ρ is K-subadditive. Thus $f, g \in \mathcal{L}_X$ implies that $f + g \in \mathcal{L}_X$. Finally, note that since $\mathcal{L}_X$ contains only zero-preserving functions, it is a zero-symmetric near-ring.

(2) This result follows from (1) and Proposition 2.14. □

In some of our results, we assume that our metric space is a normed vector space over a field with an absolute value or norm. Recall that an absolute value on a field F is a function $|\cdot| : F \to \mathbb{R}^+ \cup \{0\}$, such that

(1) $|x| = 0$ if and only if $x = 0$;

(2) $|xy| = |x||y|$ for all $x, y \in F$;

(3) $|x + y| \le |x| + |y|$ for all $x, y \in F$.

If X is a normed vector space over a normed field, $\mathcal{L}_X$ is not only a normed near-ring but in fact a normed near-algebra. First we recall the definitions of a near-algebra and a normed near-algebra. We only consider near-algebras with identity.

Definition 2.18 (see [7]). A vector space A over a field F together with another binary operation "$\cdot$" is a (right) *near-algebra* over F if $(A, +, \cdot)$ is a (right) near-ring and for all $a, b \in A$ and all $k \in F$, $(ka) \cdot b = k(a \cdot b)$.

As with near-rings, near-algebras need not be zero-symmetric in general, as seen in the following example [15].

Example 2.19. Let X be a vector space over a field F. Then $\mathcal{M}(X)$, the set of all self-maps of X, is a near-algebra over F which is not zero-symmetric. Also, if R is any subalgebra of the F-algebra $\mathrm{End}_F(X)$, then the set of all affine transformations arising from elements of R and X, that is, $\mathcal{AF}_R(X) := \{A_v \mid A \in \mathrm{R}, v \in X\}$, where $A_v(x) := Ax + v$, is a sub-near-algebra of $\mathcal{M}(\mathrm{X})$ which is also not zero-symmetric.

Definition 2.20 (see [11]). A *normed near-algebra* $(A, \|\cdot\|)$ over a field $(F, |\cdot|)$ with an absolute value $|\cdot|$ is a near-algebra such that $(A, \|\cdot\|)$ is also a normed vector space over $(F, |\cdot|)$, with the norm $\|\cdot\|$ satisfying $\|f \cdot g\| \le \|f\|\|g\|$ for all $f, g \in A$.

Proposition 2.21. *Let* $(\mathrm{X}, \|\cdot\|)$ *be a normed vector space over a normed field* $(F, |\cdot|)$*. Then* $\mathcal{L}_X$ *is a normed near-algebra over* F*.*

Proof. This result follows from Definition 2.1 and Proposition 2.17. □

In the remainder of the paper, we use the induced topology obtained from $\|\cdot\|_L$ when making topological statements about $\mathcal{L}_X$.

3. Units in $\mathcal{L}_X$

We note for the reader that there is some overlap between this section and [11, Chapter 3]. However, our proofs are less involved and our results are more general since in [11] only the case where X is a Banach space over $\mathbb{R}$ or $\mathbb{C}$ is considered. In this section we show that if X is a complete and connected normed abelian group, then the set of units is an open subset of $\mathcal{L}_X$. Throughout this discussion, the metric on X will be the 1-subadditive metric induced by the norm on X, as described in Remark 2.11. We start with a technical lemma that is required for the proof of Lemma 3.2. We denote the identity function in $\mathcal{L}_X$ by 1_X.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed abelian group. Also, let $f \in \mathcal{L}_X$, with $\|f - 1_X\|_L < 1$, and $x, y \in X$ with $x \neq y$. Then*

$$0 < 1 - \|f - 1_X\|_L \leq \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq 1 + \|f - 1_X\|_L. \tag{3.1}$$

Proof. The statement follows from the fact that we have for all $x, y \in X$ with $x \neq y$ that

$$\begin{aligned} \|f - 1_X\|_L &\geq \frac{\|(f(x) - x) - (f(y) - y)\|}{\|x - y\|} \\ &\geq \frac{\left|\|f(x) - f(y)\| - \|x - y\|\right|}{\|x - y\|} \\ &= \left|\frac{\|f(x) - f(y)\|}{\|x - y\|} - 1\right|. \end{aligned} \tag{3.2}$$

□

The next lemma shows that if $f \in \mathcal{L}_X$ is surjective and "close enough" to 1_X, then f is a unit in $\mathcal{L}_X$.

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a normed abelian group. Also, let $f \in \mathcal{L}_X$ be such that $\|f - 1_X\|_L < 1$. Then f is injective and $\|f^{-1}\|_L \leq 1/(1 - \|f - 1_X\|_L)$, where $f^{-1} : f(X) \to X$.*

Proof. From Lemma 3.1 it follows that if $x \neq y$, but $f(x) = f(y)$, then $0 < 1 - \|f - 1_X\|_L \leq \|f(x) - f(y)\|/\|x - y\| = 0$, which is not possible. Thus f must be injective.

Note that

$$\sup\left\{\frac{\|f^{-1}(x) - f^{-1}(y)\|}{\|x - y\|} : x, y \in f(X),\ x \neq y\right\} \leq \sup\left\{\frac{\|x - y\|}{\|f(x) - f(y)\|} : x, y \in X,\ x \neq y\right\} \tag{3.3}$$

$$\leq \frac{1}{1 - \|f - 1_X\|_L}, \tag{3.4}$$

where (3.3) follows by replacing x and y by $f^{-1}(x)$ and $f^{-1}(y)$, respectively, in $\sup\{\|x - y\|/\|f(x) - f(y)\| : x, y \in X, x \neq y\}$, and (3.4) follows from Lemma 3.1. Thus f^{-1} is Lipschitz with $\|f^{-1}\|_L \leq 1/(1 - \|f - 1_X\|_L)$. □

The next two lemmas will be used in the proof of the main result of this section.

Notation 3.3. For $a \in X$ and $r \in \mathbb{R}^+$, we denote by $B_r(a)$ the set $\{x \in X : \|x - a\| < r\}$. Also, for $A \subseteq X$, we denote by $\overline{A}$ the topological closure of A.

Lemma 3.4. *Let $(X, \|\cdot\|)$ be a normed abelian group. Assume $f \in \mathcal{L}_X$ with $\|f - 1_X\|_L < 1$. Then there is an $\alpha \in \mathbb{R}^+$ such that $B_{\alpha r}(f(a)) \subseteq \overline{f(B_r(a))}$ for all $r \in \mathbb{R}^+$ and $a \in X$.*

Proof. If $\|f - 1_X\|_L = 0$, we have that $f = 1_X$, and the result follows trivially. Thus we assume that $0 < \|f - 1_X\|_L < 1$. Choose any α with $0 < \alpha < 1 - \|f - 1_X\|_L$. Suppose that $B_{\alpha r}(f(a))$ is not a subset of $\overline{f(B_r(a))}$, and let $c \in B_{\alpha r}(f(a)) \setminus \overline{f(B_r(a))}$. Also, let $d(c, f(B_r(a))) := \inf\{\|c - x\| : x \in f(B_r(a))\} > 0$. Choose $b \in B_r(a)$ with $\|c - f(b)\| < d(c, f(B_r(a)))/\|f - 1_X\|_L$. Let $d := b + c - f(b)$. We show next that

(1) $d \in B_r(a)$,

and

(2) $\|f(d) - c\| < d(c, f(B_r(a)))$.

Once we have (1) and (2), we have a contradiction with the definition of $d(c, f(B_r(a)))$, and we thus have that $B_{\alpha r}(f(a)) \subseteq \overline{f(B_r(a))}$. The fact that $d \in B_r(a)$ follows from the following:

$$\begin{aligned} \|d - a\| &= \left\|(f - 1_X)(a) - (f - 1_X)(b) + (c - f(a))\right\| \\ &\le \left\|f - 1_X\right\|_L \|a - b\| + \left\|f(a) - c\right\| \\ &< \left\|f - 1_X\right\|_L r + \alpha r \\ &< r. \end{aligned} \tag{3.5}$$

The fact that $\|f(d) - c\| < d(c, f(B_r(a)))$ follows from the following:

$$\begin{aligned} \left\|f(d) - c\right\| &= \left\|(f - 1_X)(b + c - f(b)) - (f - 1_X)(b)\right\| \\ &\le \left\|f - 1_X\right\|_L \left\|b + c - f(b) - b\right\| \\ &= \left\|f - 1_X\right\|_L \left\|c - f(b)\right\| \\ &< d(c, f(B_r(a))). \end{aligned} \tag{3.6}$$

□

Lemma 3.5. *Let $(X, \|\cdot\|)$ be a normed abelian group. Assume that X is complete, C is a closed subset of X, $f \in \mathcal{L}_X$, and $\|f - 1_X\|_L < 1$. Then $f(C)$ is closed in X.*

Proof. Suppose that $y \in X$ and $\|f(x_n) - y\| \to 0$ as $n \to \infty$, where $x_n \in C$ for all $n \in \mathbb{N}$. From Lemma 3.2 f is injective and $f^{-1} : f(X) \to X$ is Lipschitz. Note that $\langle x_n : n \in \mathbb{N}\rangle$ is Cauchy since $\langle f(x_n) : n \in \mathbb{N} \rangle$ is Cauchy and since we have the following:

$$\begin{aligned} \|x_n - x_m\| &= \left\| f^{-1} f(x_n) - f^{-1} f(x_m) \right\| \\ &\leq \left\| f^{-1} \right\|_L \| f(x_n) - f(x_m) \|. \end{aligned} \tag{3.7}$$

Since X is complete and C is closed, $\langle x_n : n \in N \rangle$ converges to, say, $x \in C$. Thus since f is continuous, $f(x) = y$, and we therefore have that $f(C)$ is closed in X. □

We introduce the main theorem of this section.

Theorem 3.6. *Let X be a complete and connected normed abelian group. Then the set of units is open in $\mathcal{L}_X$.*

Proof. In order to obtain the result, we show that if $g \in \mathcal{L}_X$ is a unit, then all $f \in \mathcal{L}_X$, with $\|f - g\|_L < 1/\|g^{-1}\|_L$, are also units. First note that $\|f - g\|_L < 1/\|g^{-1}\|_L$ implies that $\|f \circ g^{-1} - 1_X\|_L = \|(f - g) \circ g^{-1}\|_L \leq \|f - g\|_L \|g^{-1}\|_L < 1$. Since f is a unit in $\mathcal{L}_X$ if and only if $f \circ g^{-1}$ is a unit in $\mathcal{L}_X$, it is enough to show that $\|f - 1_X\|_L < 1$ implies that f is a unit in $\mathcal{L}_X$. So assume that $\|f - 1_X\|_L < 1$. From Lemma 3.2, f is injective. Lemma 3.5 implies that $f(X)$ is closed, thus $\overline{f(X)} = f(X)$. Also, Lemma 3.4 implies that the set $\overline{f(X)}$, which is equal to $f(X)$, is open. Now since X is connected, we have that $f(X)$ both open and closed implies that $f(X) = X$ and thus that f is surjective. To complete the argument, we recall that Lemma 3.2 implies that f^{-1} is Lipschitz. □

We conclude this section by giving an example to show that the completeness of X is an essential hypothesis in the preceding theorem.

Example 3.7. Define $f_n : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$f_n(x) = \begin{cases} x & x < 1 - \dfrac{1}{n}, \\ \dfrac{x^2}{2} + \left(1 - \dfrac{1}{n}\right) - \dfrac{(1 - 1/n)^2}{2} & x \in \left[1 - \dfrac{1}{n}, 1\right], \\ x - \dfrac{1}{2} + \left(1 - \dfrac{1}{n}\right) - \dfrac{(1 - 1/n)^2}{2} & x > 1. \end{cases} \tag{3.8}$$

Let $\overline{f}_n : \mathbb{R} \to \mathbb{R}$ be the continuous function such that the restriction of $\overline{f}_n$ to $\mathbb{Q}$ is equal to f_n. It follows from Remark 2.3, by calculating piecewise derivatives, that $f_n : \mathbb{Q} \to \mathbb{Q}$ is Lipschitz for $n > 1$. Let k_n be a rational number such that $1 - 1/n < k_n\sqrt{2} < 1$. Then $\overline{f}_n(k_n\sqrt{2})$ is rational but not in the range of f_n. Therefore f_n is not surjective and thus not a unit in $\mathcal{L}_{\mathbb{Q}}$ for any n. Next we show that f_n converges to $1_{\mathbb{Q}}$, the identity in $\mathcal{L}_{\mathbb{Q}}$, and thus the set of units in $\mathcal{L}_{\mathbb{Q}}$ is not open. Let $1_{\mathbb{R}}$ be the identity function on $\mathbb{R}$. We show that f_n converges to $1_{\mathbb{Q}}$, by showing that $\|\overline{f}_n - 1_{\mathbb{R}}\|_L$ converges to 0. From Remark 2.3 we have that it is enough to show that the absolute value of the derivative of $h_n = \overline{f}_n - 1_{\mathbb{R}}$ is bounded (where it is defined) by a constant M_n, where $M_n \to 0$ as $n \to \infty$. This is clearly the case on $(-\infty, 1 - 1/n)$ and on $(1, \infty)$. But also on $(1 - 1/n, 1)$ the derivative of h_n is $x - 1$; so on $(1 - 1/n, 1)$ the absolute value of the derivative of h_n is bounded by $1/n$. Thus we conclude that the required constants M_n exist.

4. Continuity of Multiplication and Closure of Ideals

In the first example in this section, we show that if $f, g, g_n \in \mathcal{L}_X$ for all $n \in \mathbb{N}$ with $g_n \to g$ in $\mathcal{L}_X$ as $n \to \infty$, then it is not necessarily the case that $f \circ g_n$ converges to $f \circ g$ in $\mathcal{L}_X$. Since

$$\|g_n \circ f - g \circ f\|_N = \|(g_n - g) \circ f\|_N \leq \|g_n - g\|_N \|f\|_N \tag{4.1}$$

in any normed (right) near-ring $(N, \|\cdot\|_N)$, we have that if $f, g, g_n \in N$ for all $n \in \mathbb{N}$ with $g_n \to g$ in N as $n \to \infty$, then $g_n \circ f$ converges to $g \circ f$. Thus right multiplication is a continuous function in a normed near-ring, but left multiplication is not. An example, similar to the next example, but more involved, is given in [11].

Example 4.1. In this example we show that it is not necessarily the case that $\|f \circ g - f \circ h\|_L \leq \|f\|_L \|g - h\|_L$ for $f, g, h \in \mathcal{L}_X$, and also it is not the case that if g_n converges to g as n approaches infinity, then $f \circ g_n$ converges to $f \circ g$.

Let $X = \mathbb{R}$ be endowed with the Euclidean metric d, so that $\mathcal{L}_X$ is a normed near-algebra, and define $f, g, h : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned} f(x) &:= d(x, [-1, 1]); \\ g(x) &:= x; \\ h(x) &:= kx, \quad \text{with } k > 1 \text{ fixed.} \end{aligned} \tag{4.2}$$

From Remark 2.3, $\|f \circ g - f \circ h\|_L = k$, whereas $\|f\|_L = 1$ and $\|g - h\|_L = k - 1$.

For each $n \in \mathbb{N}$, let $g_n(x) := (1 + 1/n)x$. Then by replacing h with g_n and k by $(1 + 1/n)$, we obtain that $\|f \circ g - f \circ g_n\|_L = 1 + 1/n$, whereas $\|f\|_L = 1$ and $\|g - g_n\|_L = 1/n$. Thus g_n converges to g, but it is not the case that $f \circ g_n$ converges to $f \circ g$.

Notation 4.2. Let I be a nonempty indexing set, and for $i \in I$, let A_i and B_i be nonempty subsets of X. Define $\mathcal{L}_X(A_i \to B_i : i \in I)$ by $\{f \in \mathcal{L}_X \mid f(A_i) \subseteq B_i \text{ for } i \in I\}$. If I is finite, we will use the notation $\mathcal{L}_X(A_1 \to B_1, \ldots, A_n \to B_n)$.

Remark 4.3. Note that we use the notation $\mathcal{L}_X(S \to \{0\})$, instead of the familiar notation $\mathrm{Ann}_{\mathcal{L}_X}(S)$.

The next proposition shows, for example, that the left ideal, obtained by considering the set of functions in $\mathcal{L}_X$ that annihilates a certain subset of X, is closed.

Proposition 4.4. *For $i \in I$, let A_i and B_i be nonempty subsets of the normed abelian group $(X, \|\cdot\|)$, with the B_i's closed. Then the set $\mathcal{L}_X(A_i \to B_i : i \in I)$ is a closed subset of $\mathcal{L}_X$.*

Proof. Let $\langle f_n : n \in \mathbb{N}\rangle$ be a sequence in $\mathcal{L}_X(A_i \to B_i : i \in I)$, and let $f \in \mathcal{L}_X$ be such that f_n converges f. To show that $\mathcal{L}_X(A_i \to B_i : i \in I)$ is closed, we need to show that $f \in \mathcal{L}_X(A_i \to B_i : i \in I)$. Let $a_i \in A_i$. Then $f_n(a_i) \in B_i$. Since $\|f(a_i) - f_n(a_i)\| \leq \|f - f_n\|_L \|a_i\|$ and $\|f - f_n\|_L$ converges to 0, we conclude that $f_n(a_i)$ converges to $f(a_i)$. Since each B_i is closed, we conclude that $f(a_i) \in B_i$, and thus $f \in \mathcal{L}_X(A_i \to B_i : i \in I)$. □

Theorem 4.5. *The closure of a right ideal of a normed near-ring N is again a right ideal of N.*

Proof. Denote the norm on N by $\|\cdot\|_N$. Let $I \subseteq N$ be a right ideal and f, g in $\overline{I}$ the closure of I. Assume that $\langle f_n \rangle$ and $\langle g_n \rangle$ are sequences in I converging to f and g, respectively. Then $\|(f-g)-(f_n-g_n)\|_N \leq \|f-f_n\|_N + \|g-g_n\|_N$, and the right side of this inequality converges to 0. Thus $(f-g) \in \overline{I}$. Next let $h \in N$. Then $\|f_n \circ h - f \circ h\|_N = \|(f_n - f)\circ h\|_N \leq \|f_n - f\|_N \|h\|_N$, and again the right side of the inequality converges to 0 as $n \to \infty$. Thus since $f_n \circ h \in I$ for all $n \in \mathbb{N}$, we conclude that $f \circ h \in \overline{I}$. It follows that $\overline{I}$ is a right ideal of N. □

Remark 4.6. Recall from the previous section that the set of units is open in $\mathcal{L}_X$ if X is a complete, connected, normed abelian group. In such a case, if S is a proper subset of $\mathcal{L}_X$ that is closed under either left or right function composition by an arbitrary function in $\mathcal{L}_X$, then the closure of S will also be a proper subset of $\mathcal{L}_X$.

5. Ideals in $\mathcal{L}_X$

This section contains some partial results on the ideal structure of $\mathcal{L}_X$. In the first example we show that ideals in $\mathcal{L}_X$ are in abundance.

Example 5.1. Let $\mathcal{F}$ be a set of functions from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$. Denote by $\mathcal{IF}_X$ the set

$$\left\{ f \in \mathcal{L}_X : \text{ there exists } F \in \mathcal{F} \text{ such that } \|f(x)\| \leq F(\|x\|) \text{ for } x \in X \right\}. \tag{5.1}$$

Assume that we have the following conditions on the functions in $\mathcal{F}$:

(i) if $F, G \in \mathcal{F}$, then there is an $H \in \mathcal{F}$ with $F(t) + G(t) \leq H(t)$ for all $t \in \mathbb{R}_0^+$;

(ii) if $r \in \mathbb{R}_0^+$ and $F \in \mathcal{F}$, then $rF \in \mathcal{F}$;

(iii) if $r \in \mathbb{R}_0^+$ and $F \in \mathcal{F}$, then there is a $G \in \mathcal{F}$ such that $F(rt) \leq G(t)$ for all $t \in \mathbb{R}_0^+$;

(iv) the functions in $\mathcal{F}$ are nondecreasing.

We assume that X is a normed abelian group and show next (in part) that $\mathcal{IF}_X$ is an ideal in $\mathcal{L}_X$.

We show that if $g \in \mathcal{IF}_X$ and $f, h \in \mathcal{L}_X$, then $f\circ(g+h) - f\circ h \in \mathcal{IF}_X$. The other cases are handled similarly. Since $g \in \mathcal{IF}_X$, there exists some $G \in \mathcal{IF}_X$ such that for all $x \in X$, $\|g(x)\| \leq G(\|x\|)$. We need to show that there is an $H \in \mathcal{IF}_X$, such that $\|(f\circ(g+h) - f\circ h)(x)\| \leq H(\|x\|)$. For any $x \in X$ we have

$$\begin{aligned} \left\| f \circ \left(g(x) + h(x)\right) - f \circ h(x) \right\| &\leq \|f\|_L \left\| g(x) + h(x) - h(x) \right\| \\ &= \|f\|_L \|g(x)\| \\ &\leq \|f\|_L G(\|x\|). \end{aligned} \tag{5.2}$$

Thus $\|(f \circ (g+h) - f \circ h)(x)\| \leq H(\|x\|)$ for $H = \|f\|_L G$.

In the next example we consider the set of bounded functions in $\mathcal{L}_X$.

Example 5.2. Let X be a normed abelian group. In this example we consider $\mathcal{BL}_X$, the set of bounded Lipschitz functions in $\mathcal{L}_X$. First we show that $\mathcal{BL}_X$ is a two-sided ideal. If $\mathcal{F}$ consists of all bounded nondecreasing functions from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$, then $\mathcal{IF}_X = \mathcal{BL}_X$, and it thus follows from the previous example that $\mathcal{BL}_X$ is a two-sided ideal in $\mathcal{L}_X$. Next we consider the case when $X = \mathbb{R}$. We show that $\mathcal{BL}_\mathbb{R}$ is not closed. Define for each $n \in \mathbb{N}$ the Lipschitz function $f_n : \mathbb{R} \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} 0 & x \leq 0, \\ x & x \in [0,1], \\ \sqrt{x} & x \in \left[1, n^2\right], \\ n & x > n^2. \end{cases} \tag{5.3}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & x \leq 0, \\ x & x \in [0,1], \\ \sqrt{x} & x > 1. \end{cases} \tag{5.4}$$

Then from Remark 2.3 it follows that $\|f - f_n\|_L \to 0$ as $n \to \infty$. Thus we have a sequence of bounded functions that converges to an unbounded function, which implies that $\mathcal{BL}_\mathbb{R}$ is not closed in $\mathcal{L}_\mathbb{R}$.

In the next few results we show that the Betsch-Wielandt density theorem for near-rings can be applied to $\mathcal{L}_X$.

Lemma 5.3. *Assume that X is a vector space over a field containing $\mathbb{R}$. Let $S \subseteq X$ with $0 \in S$. Fix x_0 in X, and define $d(S,x)$, for all $x \in X$, by $d(S,x) = \inf_{s \in S}\|s - x\|$. Then the function $f_{S,x_0} : X \to X$, defined by $f_{S,x_0}(x) := d(S,x)x_0$ for all $x \in X$, is Lipschitz. Also $f_{S,x_0} = f_{\overline{S},x_0}$, where $\overline{S}$ is the closure of S.*

Proof. We show that f_{S,x_0} is a Lipschitz function and leave the proof of the equality $f_{S,x_0} = f_{\overline{S},x_0}$ to the reader. First note that it is easy to verify that $|d(S,x) - d(S,y)| \leq \|x - y\|$. Thus

$$\begin{aligned} \left\| f_{S,x_0}(x) - f_{S,x_0}(y) \right\| &= \left\| d(S,x)x_0 - d(S,y)x_0 \right\| \\ &= \left| d(S,x) - d(S,y) \right| \|x_0\| \\ &\leq \left\| x - y \right\| \|x_0\|, \end{aligned} \tag{5.5}$$

and f_{S,x_0} is therefore a Lipschitz function. □

We will use the next result to conclude that if $x, y \in X$ with $x \neq 0$, then there is an $f \in \mathcal{L}_X$ such that $f(x) = y$.

Corollary 5.4. *Assume that* X *is a vector space over a field containing* $\mathbb{R}$*. Let* $x, y, z \in X$ *with* x *nonzero and* $x \neq z$*. Then there is an* $f \in \mathcal{L}_X$ *such that* $f(x) = y$ *and* $f(z) = 0$*.*

Proof. Note that $cf_{\{0,z\},y}(x) = y$ and $cf_{\{0,z\},y}(z) = 0$ for an appropriate $c \in F$, where $f_{\{0,z\},y}$ is as in Lemma 5.3. □

Corollary 5.5. *Assume that* X *is a vector space over a field containing* $\mathbb{R}$*. Then* $\mathcal{L}_X$ *is not a ring.*

Proof. Let $x \in X \setminus \{0\}$ and let $S = \{0, x\}$. Let $f_{S,x}$ be as in Lemma 5.3, and denote by 1_X the identity function on X. Now note that we have that $f_{S,x} \circ (1_X + 1_X)(x) \neq 0 = f_{S,x} \circ 1_X(x) + f_{S,x} \circ 1_X(x)$, since $(x + x) \notin S$. □

Corollary 5.6. *Assume that* X *is a vector space over a field containing* $\mathbb{R}$*. Then* X *is a type-2 primitive* $\mathcal{L}_X$*-module.*

Proof. From Corollary 5.4, $\mathcal{L}_X x = X$ for all $x \neq 0$. Also, $fX = 0$ for $f \in \mathcal{L}_X$ implies that $f = 0$. Thus X is a 2-primitive $\mathcal{L}_X$-module. □

Corollary 5.7. *Assume that* X *is a vector space over a field containing* $\mathbb{R}$*. Let* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *be elements in* X *with distinct and nonzero* a_i*'s. Then there exists an* $f \in \mathcal{L}_X$ *such that* $f(a_i) = b_i$ *for* $i = 1, \ldots, n$*.*

Proof. Since X is a type-2 primitive $\mathcal{L}_X$-module, the Betsch-Wielandt density theorem for near-rings (see, e.g., [7]) can be applied when $\mathcal{L}_X$ is not a ring. By the density theorem, if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are in X and the a_i's are distinct and all nonzero, then there exists $f \in \mathcal{L}_X$ such that $f(a_i) = b_i$. □

We conclude by exhibiting some of the maximal left ideals in $\mathcal{L}_X$. With the exception of the statement that $\mathcal{L}(\{x_0\} \to \{0\})$ is closed, the argument is solely based on the fact that X is a 2-primitive $\mathcal{L}_X$-module.

Theorem 5.8. *Assume that* X *is a vector space over a field containing* $\mathbb{R}$*. Let* $x_0 \in X$ *with* $x_0 \neq 0$*. Then* $\mathcal{L}(\{x_0\} \to \{0\})$*, often denoted by* $\mathrm{Ann}_{\mathcal{L}}(\{x_0\})$*, is a maximal closed left ideal that is not an ideal.*

Proof. From Proposition 4.4 we have that $\mathcal{L}_X(\{x_0\} \to \{0\})$ is closed. It is easy to verify that $\mathcal{L}_X(\{x_0\} \to \{0\})$ is a left ideal. Next we show that it is maximal. Assume that $f \in \mathcal{L}_X \setminus \mathcal{L}_X(\{x_0\} \to \{0\})$. Then from Corollary 5.4 we can find a $g \in \mathcal{L}_X$ such that $g(f(x_0)) = x_0$. But then $id_X = g \circ f + (id_X - g \circ f)$ and $(id_X - g \circ f) \in \mathcal{L}_X(\{x_0\} \to \{0\})$, implying that the left ideal generated by f and $\mathcal{L}_X(\{x_0\} \to \{0\})$ is all of $\mathcal{L}_X$. It follows that $\mathcal{L}_X(\{x_0\} \to \{0\})$ is a maximal left ideal.

Finally we show that $\mathcal{L}_X(\{x_0\} \to \{0\})$ is not an ideal. Let $0, x$ be two distinct elements in X, with $x \neq x_0$. From Corollary 5.4 we have functions $f, g \in \mathcal{L}_X$ with $f(x) = x$, $f(x_0) = 0$, and $g(x_0) = x$. Then it follows that $f \in \mathcal{L}_X(\{x_0\} \to \{0\})$, but $f \circ g \notin \mathcal{L}_X(\{x_0\} \to \{0\})$, since we have $f(g(x_0)) = x$. □

Acknowledgment

The authors would like to thank the anonymous referee for helping them to make a substantial improvement to the quality of this paper.

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