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Research Article

On Maximal Ideals of Compact Connected Topological Semigroups

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Several results concerning ideals of a compact topological semigroup S with $S^2 = S$ can be found in the literature. In this paper, we further investigate in a compact connected topological semigroup S how the conditions $S^2 = S$ and $S^2 \neq S$ affect the structure of ideals of S, especially the maximal ideals.

1. Introduction

First, we list some standard definitions which can be found in [1–3].

Definition 1.1. A *topological semigroup* is a topological space S together with a continuous function $m : S \times S \to S$ such that S is Hausdorff and m is associative.

A *subsemigroup* of a semigroup S is a nonvoid set $A \subset S$ such that $A^2 \subset A$, and A is called a *subgroup* of S if it is a group with respect to m.

An element e of a topological semigroup S is called an *idempotent* if $e^2 = e$. Similarly, an element e of S is called a *left identity* (*right identity*) if $ea = a$ ($ae = a$) for all $a \in S$. An element of S is called an *identity* of S if it is both a left and a right identity of S.

The set of all idempotents of S will be denoted by E throughout this paper. For each $e \in E$, let $H(e)$ be the union of all subgroups of S containing e. It is shown in [3] that $H(e)$ is the maximal subgroup of S containing e.

Definition 1.2. A nonempty subset A of a semigroup S is called a *left ideal* (*right ideal*) of S if $SA \subset A$ ($AS \subset A$) and an *ideal* if it is both a left and a right ideal. A left ideal (right ideal, ideal) is said to be *proper* if it is not S itself.

An (*left, right*) *ideal* M of a semigroup S is called *minimal* if it does not properly contain any (*left, right*) ideal of S. It follows that there can be at most one minimal ideal of S. If S has a minimal ideal K, then K is called the *kernel* of S.

A *maximal* (left, right) *ideal* of a semigroup S is a proper (left, right) ideal of S that is not properly contained in any other (left, right) ideal.

Definition 1.3. Let A be a subset of a topological semigroup S, then $J_0(A)$ is defined as follows:

$$J_0(A) = \begin{cases} \emptyset & \text{if } A \text{ contains no ideal of } S, \\ \cup & \{I : I \text{is an ideal of } S \text{ and } I \subset A\}. \end{cases} \tag{1.1}$$

Theorem 1.4. *Let S be a compact connected topological semigroup without zero, and let K be the kernel of S. Then, either $K \cap E$ is infinite or K is a topological subgroup of S.*

Proof. Since S is a compact topological semigroup, $K = \cup\{H(e) : e \in K \cap E\}$, and $H(e) = eSe$ by [3, Theorem1.2.6]. Suppose that $K \cap E$ is finite and K is not a topological subgroup of S. Let $e_K \in K \cap E$. Then, $K \setminus H(e_K) \neq \emptyset$. Otherwise, $K = H(e_K)$ is both the kernel and a maximal subgroup of S by [3, Theorem 1.3.14], and hence K is topological subgroup of S with the relative topology, which contradicts our assumption.

Furthermore, since $K \cap E$ is finite and $K \setminus H(e_K) = \cup\{H(e) : e \in K \cap E, e \neq e_K\}$, it follows that $K \setminus H(e_K)$ and $H(e_K)$ form a separation of K. Hence, K is disconnected, which contradicts [1, Theorem 1.28]. Therefore, we can deduce that either $K \cap E$ is infinite or K is a maximal subgroup of S. □

2. Maximal Ideals of Compact Connected Topological Semigroups

The following theorem is a summary of the results found in [1]. It lists necessary and sufficient conditions for $S^2 = S$ in a compact topological semigroup S. In this section, we characterize maximal ideals in a compact connected topological semigroup S with $S^2 = S$ and $S^2 \neq S$.

Theorem 2.1. *Let S be a compact connected topological semigroup. The following are equivalent:*

(a) $S^2 = S$,

(b) $E \cap (S \setminus I) \neq \emptyset$ *for each proper ideal I of S,*

(c) $S = SES$.

The following theorem and corollary are results from [3], which are useful for our discussion.

Theorem 2.2. *Let S be a compact topological semigroup. Then, any proper (left, right) ideal of S is contained in a maximal (left, right) ideal of S, and each maximal (left, right) ideal is open.*

Corollary 2.3. *If S is a compact connected topological semigroup and J a maximal ideal of S, then J is dense in S.*

Theorem 2.4. *Suppose that S is a compact topological semigroup and $S^2 \neq S$.*

(a) *For each $a \in S \setminus S^2$, $S \setminus \{a\}$ is a maximal ideal of S.*

(b) *If S has more than one connected maximal ideal, then, S is connected.*

Proof. (a) Let $a \in S \setminus S^2$. For every $x \in S \setminus \{a\}$ and $y \in S$, $\{xy, yx\} \subset S^2 \subset S \setminus \{a\}$ implies that $S \setminus \{a\}$ is a proper ideal of S. (b) Let M_1 and M_2 be two distinct connected maximal ideals of S. Suppose that S is disconnected. Then, $M_1 \cup M_2 = S = P \cup Q$ such that $\overline{P} \cap Q = \emptyset = P \cap \overline{Q}$. Since M_1 and M_2 are connected, $M_1 \subset P$ and $M_2 \subset P$. It follows that $M_1 \cap M_2 = \emptyset$, and hence $M_1 \subset S \setminus M_2 = \{a_2\}$ and $M_2 \subset S \setminus M_1 = \{a_1\}$. On the other hand, since M_1 and M_2 are ideals, $a_1a_2 = a_2a_1 = a_2$ and $a_1a_2 = a_2a_1 = a_1$, and hence $M_1 = \{a_2\} = \{a_1\} = M_2$ contradicting M_1 and M_2 being distinct. Therefore, S is connected, and hence K is connected.

The following example shows that the condition S having more than one connected maximal ideal is a necessary condition for Theorem 2.4(b). □

Example 2.5. Let $S = [0, 1/4] \cup \{1/2\}$ with the usual topology and the usual multiplication. Then, $S^2 = [0, 1/8] \cup \{1/4\} \neq S$, $K = \{0\}$ is connected, $M = [0, 1/4]$ is the only connected maximal ideal of S, and S is disconnected.

The next theorem is Theorem 2.4.3 of [3], and hence the proof is omitted.

Theorem 2.6. *If S is a connected topological semigroup and I an ideal of S, then one and only one component of I is an ideal of S.*

One will call the ideal in Theorem 2.6 the component ideal of I.

Theorem 2.7. *Let S be a compact connected topological semigroup and$C= \bigcup\{M_C : M_C$is the ideal component of a maximal proper ideal $M\}$. Then either $C = S$ or C is the maximal proper connected ideal of S. Furthermore, if $C \neq S$, then C is the component ideal of a maximal ideal of S.*

Proof. For each maximal ideal M of S, let M_C be its component ideal. Since K is the kernel and $K \subset M_C$ for each M_C, $C{=}\bigcup\{M_C : M_C$ is the ideal component of a maximal proper ideal $M\}$ is a connected ideal.

Suppose that there is a connected ideal I such that $C \subset I \subsetneq S$, then I is contained in a maximal ideal M of S. Since $K \subset I \cap M_C$, $I \cup M_C$ is a connected ideal of S and is contained in M, and hence $I \cup M_C \subset M_C \subset C$, a contradiction. Thus, if $C \neq S$, then C is the maximal connected proper ideal of S. Furthermore, there exists a maximal ideal M of S such that $C \subset M$. Let M_C be the component ideal of M. Then, $M_C = C$. □

Lemma 2.8. *Let S be a compact connected topological semigroup, M a maximal ideal of S, and M_C the component ideal of M. If $S^2 \neq S$, then M_C is not closed in S.*

Proof. If $M_C = M$, then the result follows from Theorem 2.4(b).

If $M_C \subsetneq M$, then $M = M_C \cup K_M$ where K_M is the union of all components of M except M_C. If M_C were closed in S, then K_M is open in S because $K_M = M \cap (S \setminus M_C)$ and M are both open. Therefore, for $a \in S \setminus M$, $S = \overline{M} = M_C \cup (K_M \cup \{a\})$, and hence S is disconnected, which is a contradiction. □

The next theorem provides a necessary and sufficient condition for a compact connected topological semigroup S satisfying $S^2 \neq S$ by means of the component ideals of its maximal ideals.

Theorem 2.9. *Let S be a compact connected topological semigroup. Then, $S^2 \neq S$ if and only if there exists a maximal ideal M of S with $M = S \setminus \{b\}$, $b \in S \setminus S^2$ such that $S^2 \subset M_C$ where M_C is a component ideal of M.*

Proof. Suppose that $S^2 \neq S$. It follows from Theorem 2.1(a) that there exists a maximal ideal M of S such that $E \cap (S \setminus M) = \emptyset$. By [3, Theorem 1.3.8], S/M is either the zero semigroup of order two or else completely 0-simple.

Suppose that S/M is the zero semigroup of order two. Then, $S \setminus M = \{b\}$ for some $b \in S$. If $b \in S^2$, then $b = xy$ with $x, y \in S \setminus M$. It is because if $\{x, y\} \cap M \neq \emptyset$, then $b \in M$ contradicting $S \setminus M = \{b\}$. It follows that $x = y = b$, and hence $b \in E$. This contradicts $E \cap (S \setminus M) = \emptyset$. Therefore, $b \in S \setminus S^2$ and $S^2 \subset M_C \subset M \setminus \{b\}$. Note that the semigroup S/M is not completely 0-simple because if S/M were completely 0-simple, then S/M contains a nonzero primitive idempotent, which contradicts $E \cap (S \setminus M) = \emptyset$.

The converse is obviously true.The next example shows that the component ideal M_C of a maximal ideal M can be M itself. □

Example 2.10. Let $S = [0, 1/2]$ with the usual multiplication and the usual topology. Then, S is a compact connected topological semigroup, and $S^2 \neq S$. Let $M = [0, 1/2)$ and $M^\# = S \setminus \{5/16\}$. Then, M and $M^\#$ are maximal ideals of S, and $M_C = [0, 1/2) = M$ and $M_C^\# = [0, 5/16) \subsetneq M^\#$.

The next theorem is Theorem 1.40 of [1], and hence the proof is omitted.

Theorem 2.11. *Let S be a compact connected topological semigroup. Then, $S^2 = S$ if and only if each dense (left, right) ideal (containing K) is connected.*

When $S^2 = S$, it is possible that $aS = S$ for some $a \in S$. Existence of the set $P = \{\alpha \in S : \alpha S = S\}$ and its relationship to maximal ideals have been discussed in [3]. The following theorem provides a few additional properties of the set P of a compact topological semigroup S.

Theorem 2.12. *Suppose that S is a compact topological semigroup such that $aS = S$ for some $a \in S$. Let $P = \{\alpha \in S : \alpha S = S\}$. Then, the following is considered.*

(a) *P is a right group.*

(b) *If $P \neq S$, Then $S \setminus P$ is dense in S or S is disconnected.*

(c) *$J_0(S \setminus \{a\})$ is dense in S for each $a \in P$ if S is connected and $P \neq S$.*

Proof. (a) According to [3, Theorem 1.4.6], $P = \bigcup_{e \in E \cap P} H(e)$, and P is a subtopological semigroup of S. Then, $eS = S$ for all $e \in E \cap P$, and hence e is a left identity of S. For each $a \in P$, $a \in H(e)$ for some $e \in E \cap P$, and hence there exists $a^{-1} \in H(e)$ such that $aa^{-1} = e$. For any $x \in P$, $x = (aa^{-1})x = a(a^{-1}x) \in aP$. It follows that $P = aP$ for every $a \in P$, and hence P is right simple since S is compact and P is closed. The result follows from Theorem 1 of [4]. □

(b) Since P is a nonempty closed subtopological semigroup of S and the kernel K exists, $S \setminus P$ is nonempty. In fact, by [3, Theorem 1.4.7], $S \setminus P$ is the only maximal ideal of S because $S \neq P \neq \emptyset$. If $\overline{S \setminus P} \neq S$, then $S \setminus P$ is both open and closed by the maximality, and hence S is disconnected.

(c)The result follows immediately from part (b) and the fact that $S \setminus P \subset J_0(S \setminus \{a\})$ for every $a \in P$.

The following example shows that the condition $S \neq P$ is necessary for Theorem 2.12(b) and (c).

Example 2.13. Let $S = [0, 1]$ with the usual topology and the multiplication $xy = y$ for $x, y \in S$. Then, $S = P = K$.

Definition 2.14. A topological semigroup S has the *left maximal property* (*right maximal property*) if there exists a maximal left (right) ideal L^* (R^*) containing every proper left (right) ideal of S.

In [3], Paalmande Miranda presented several results showing how a compact connected topological semigroup S with the left or right maximal property is related to the condition $S = aS$, where $a \in S$. In the same spirit of these results and Theorem 2.11, the following theorem characterizes a compact connected topological semigroup satisfying the maximal property and the condition $S = Sa \cup aS \cup SaS$ by means of its maximal ideals.

Theorem 2.15. *Let S be a compact connected topological semigroup. Then, the following are equivalent.*

(a) *There is an idempotent e such that $e \in S \setminus M$ for every maximal ideal M of S.*

(b) *The semigroup S has the maximal property and $S = Sa \cup aS \cup SaS$ for some $a \in S$.*

Proof. (a) $\Rightarrow$ (b) Since $K \subset S \setminus \{e\}$ and $I \subset J_0(S \setminus \{e\})$ for every proper ideal I of S, S has the maximal property with the maximal ideal $J_0(S \setminus \{e\})$.

Let $a \in S \setminus J_0(S \setminus \{e\})$. Then, $J_0(S \setminus \{e\})$ is properly contained by the ideal $Sa \cup aS \cup SaS \cup \{a\}$. Hence, $Sa \cup aS \cup SaS \cup \{a\} = S$. Since S is connected and Sa, aS, SaS, and $\{a\}$ are closed, $a \in Sa \cup aS \cup SaS$, and hence, $S = Sa \cup aS \cup SaS$.

(b) $\Rightarrow$ (a) Suppose that S has the maximal property with the maximal ideal M^* and S does not satisfy the condition in part (a). Then, $E \subset M^*$, and hence it follows from Theorem 2.9 that $S^2 \subset M^*$. On the other hand, $S = Sa \cup aS \cup SaS \subset S^2 \subset M^*$, which contradicts M^* being the maximal ideal of S.

The following corollary to Theorem 1.4.12 of [3] implies that the maximal ideal M in Theorem 2.9 is not unique. □

Corollary 2.16. *A necessary and sufficient condition that a compact connected topological semigroup S has the maximal ideal property is that S has at least one idempotent e with $S = SeS$ and S is not simple.*

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