

Research Article

On Reverses of Some Inequalities in n -Inner Product Spaces

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We present some new reverses of Cauchy-Bunyakovsky-Schwarz inequality, and Triangle and Boas-Bellman Type inequalities in n -inner product spaces. The results obtained generalize the results of Dragomir(2003–2005) in n -inner product spaces. Also we provide some applications for determinantal integral inequalities.

1. Introduction

In 1964, Gähler [1] introduced the concept of 2-norm and 2-inner product spaces as generalization of norm and inner product spaces. A systematic presentation of the results related to the theory of 2-inner product spaces can be found in the book in [2, 3] and in list of references in it. Generalization of 2-inner product space for $n \geq 2$ was developed by Misiak [4] in 1989. Gunawan and Mashadi [5] in 2001 introduced the concept of n -normed linear spaces. Gunawan [6] obtained Cauchy-Bunyakovsky inequality in these spaces. Cho et al. [7] extended parallelogram law and proved Hlwaka's type inequality in n -inner product spaces. In this paper by, using same ideas of [8–13] we establish some results regarding new reverses of Cauchy-Bunyakovsky-Schwarz inequality, and Triangle and Boas-Bellman Type inequalities in n -inner product spaces and we also provide some applications for determinantal integral inequalities.

2. Preliminaries

Definition 2.1 (see [4]). Assume that n is a positive integer and X is a vector space over the field $K = R$ of real numbers or the field $K = C$ of complex numbers, such that $\dim X \geq n$ and $\langle \cdot, \cdot | \underbrace{\cdot, \dots, \cdot}_{n-1} \rangle$ is a K valued function defined on $\underbrace{X \times X \times \dots \times X}_{n+1}$ such that:

- (nI₁) $\langle x_1, x_1 \mid x_2, \dots, x_n \rangle \geq 0$, for any $x_1, x_2, \dots, x_n \in X$ and $\langle x_1, x_1 \mid x_2, \dots, x_n \rangle = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent vectors;
- (nI₂) $\langle a, b \mid x_1, \dots, x_{n-1} \rangle = \langle a, b \mid \pi(x_1), \dots, \pi(x_{n-1}) \rangle$, for any $a, b, x_1, x_2, \dots, x_{n-1} \in X$ and for any bijections $\pi : \{x_1, x_2, \dots, x_{n-1}\} \rightarrow \{x_1, x_2, \dots, x_{n-1}\}$;
- (nI₃) If $n > 1$, then $\langle x_1, x_1 \mid x_2, \dots, x_n \rangle = \langle x_2, x_2 \mid x_1, x_3, \dots, x_n \rangle$, for any $x_1, x_2, \dots, x_n \in X$;
- (nI₄) $\langle a, b \mid x_1, \dots, x_{n-1} \rangle = \overline{\langle b, a \mid x_1, \dots, x_{n-1} \rangle}$
- (nI₅) $\langle \alpha a, b \mid x_1, \dots, x_{n-1} \rangle = \alpha \langle a, b \mid x_1, \dots, x_{n-1} \rangle$, for any $a, b, x_1, \dots, x_{n-1} \in X$ and any scalar $\alpha \in R$;
- (nI₆) $\langle a + a_1, b \mid x_1, \dots, x_{n-1} \rangle = \langle a, b \mid x_1, \dots, x_{n-1} \rangle + \langle a_1, b \mid x_1, \dots, x_{n-1} \rangle$, for any $a, b, a_1, x_1, \dots, x_{n-1} \in X$.

Then $\langle \cdot, \cdot \mid \underbrace{\cdot, \dots, \cdot}_{n-1} \rangle$ is called the n -inner product and $(X, \langle \cdot, \cdot \mid \underbrace{\cdot, \dots, \cdot}_{n-1} \rangle)$ is called the n -prehilbert space. If $n = 1$, then Definition 2.1 reduces to the ordinary inner product. In any given n -inner product space $(X, \langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle)$, we can define a function $\|\cdot, \dots, \cdot\|$ on $X^n = \underbrace{X \times X \times \dots \times X}_{n\text{-times}}$ as

$$\|x_1, x_2, \dots, x_n\| = \sqrt{\langle x_1, x_1 \mid x_2, \dots, x_n \rangle} \quad (2.1)$$

for any $x_1, x_2, \dots, x_n \in X$. It is easy to see that this function satisfies the following conditions:

- (nN₁) $\|x_1, x_2, \dots, x_n\| \geq 0$ and $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (nN₂) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation,
- (nN₃) $\|x_1, x_2, \dots, ax_n\| = |a| \|x_1, x_2, \dots, x_n\|$, for any $a \in K$,
- (nN₄) $\|x_1, x_2, \dots, x_n - 1, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$.

A function $\|\cdot, \dots, \cdot\|$ defined on X^n and satisfying the above conditions is called an n -norm on X and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called n -normed linear space. Whenever an n -inner product space $(X, \langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle)$ is given, we consider it as an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ with the n -norm defined by (2.1).

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field K . The following inequality is known as Cauchy-Schwarz's inequality:

$$|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x, y \in H, \quad (2.2)$$

where $\|z\|^2 = \langle z, z \rangle$, $z \in H$. The equality occurs in (2.2) if and only if x and y are linearly dependent.

In [8], Dragomir obtained the following reverse of Cauchy-Schwarz's inequality:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4, \quad (2.3)$$

provided $x, y \in H$ and $a, A \in K$ are so that either

$$\operatorname{Re}\langle Ay - x, x - ay \rangle \geq 0, \quad (2.4)$$

or, equivalently

$$\left\| x - \frac{a + A}{2}y \right\| \leq \frac{1}{2}|A - a|\|y\|, \quad (2.5)$$

holds. The constant $1/4$ is best possible in (2.3) in the sense that it cannot be replaced by a smaller quantity.

If x, y, A, a satisfy either (2.4) or (2.5), then the following reverse of Cauchy-Schwarz's inequality also holds:

$$\|x\|\|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}[A\overline{\langle x, y \rangle} + \bar{a}\langle x, y \rangle]}{[\operatorname{Re}(\bar{a}A)]^{1/2}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\bar{a}A)]^{1/2}} |\langle x, y \rangle| \quad (2.6)$$

provided that, the complex numbers a and A satisfy the condition $\operatorname{Re}(\bar{a}A) > 0$. In both inequalities in (2.6), the constant $1/2$ is best possible.

The following reverse of the triangle inequality in inner product space was also obtained by Dragomir [9]

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over K and $x, y \in H$, $M \geq m > 0$ such that either $\operatorname{Re}\langle Ay - x, x - ay \rangle \geq 0$ or, equivalently $\|x - (a + A)/2, y\| \leq 1/2|A - a|\|y\|$, holds.

$$0 \leq \|x\| + \|y\| - \|x + y\| \leq \frac{\sqrt{M} - \sqrt{m}}{\sqrt{mM}} \sqrt{\operatorname{Re}\langle x, y \rangle} \quad (2.7)$$

holds.

Gunawan [6] generalized the Cauchy-Bunyakovsky-Schwarz inequality (shortly, the CBS inequality) for inner product space to n -inner product space and obtained the following:

$$|\langle x, y \mid z_2, \dots, z_n \rangle|^2 \leq \langle x, x \mid z_2, \dots, z_n \rangle \langle y, y \mid z_2, \dots, z_n \rangle. \quad (2.8)$$

Moreover, the equality holds if and only if x, y, z_2, \dots, z_n are linearly dependent. In terms of the n -norms, the (CBS)-inequality (2.8) can be written as

$$|\langle x, y \mid z_2, \dots, z_n \rangle|^2 \leq \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2. \quad (2.9)$$

The equality holds if and only if x, y, z_2, \dots, z_n are linearly dependent.

3. Main Results

The aim of the present paper is to generalize the above mentioned results of Dragomir [8, 9], that is, reverses of CBS inequality, Triangle inequality and Boas-bellman type inequalities in inner product space to n -inner product spaces.

3.1. Reverses of the CBS Inequality

Theorem 3.1. Let $A, a \in K (K = C, R)$ and $x, y, z_2, \dots, z_n \in X$, where $(X, \langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle)$ is n -inner product space over K . If

$$\operatorname{Re}(\langle Ay - x, x - ay \mid z_2, \dots, z_n \rangle) \geq 0. \quad (3.1)$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} y, z_2, \dots, z_n \right\| \leq \frac{1}{2} |A - a| \|y, z_2, \dots, z_n\|. \quad (3.2)$$

holds, then one has

$$0 \leq \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - |\langle x, y \mid z_2, \dots, z_n \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y, z_2, \dots, z_n\|^4. \quad (3.3)$$

The constant $1/4$ in (3.3) cannot be replaced by a smaller constant.

Proof. Using (nI₂)–(nI₆), we get

$$\begin{aligned} \langle z_2, z_2 \mid x \pm y, \dots, z_n \rangle &= \langle x \pm y, x \pm y \mid z_2, \dots, z_n \rangle \\ &= \langle x, x \mid z_2, \dots, z_n \rangle + \langle y, y \mid z_2, \dots, z_n \rangle \pm 2 \operatorname{Re} \langle x, y \mid z_2, \dots, z_n \rangle, \end{aligned} \quad (3.4)$$

$$\begin{aligned} &\operatorname{Re} \langle x, y \mid z_2, \dots, z_n \rangle \\ &= \frac{1}{4} [\langle z_2, z_2 \mid x + y, \dots, z_n \rangle - \langle z_2, z_2 \mid x - y, \dots, z_n \rangle]. \end{aligned} \quad (3.5)$$

Considering the vectors $x, u, U, z_2, \dots, z_n \in X$ and using (3.5) and (nN1–nN6), we have

$$\begin{aligned} &\operatorname{Re} \langle U - x, x - u \mid z_2, \dots, z_n \rangle \\ &= \frac{1}{4} [\langle z_2, z_2 \mid U - u, \dots, z_n \rangle - \langle z_2, z_2 \mid -2x + U + u, \dots, z_n \rangle] \\ &= \frac{1}{4} \left[\langle U - u, U - u \mid z_2, \dots, z_n \rangle - 4 \left\langle x - \frac{u + U}{2}, x - \frac{u + U}{2} \mid z_2, \dots, z_n \right\rangle \right] \\ &= \frac{1}{4} \|U - u, z_2, \dots, z_n\|^2 - \left\| x - \frac{u + U}{2}, z_2, \dots, z_n \right\|^2. \end{aligned} \quad (3.6)$$

Therefore, $\operatorname{Re} \langle U - x, x - u \mid z_2, \dots, z_n \rangle \geq 0$ if and only if

$$\left\| x - \frac{u + U}{2}, z_2, \dots, z_n \right\| \leq \frac{1}{2} \|U - u, z_2, \dots, z_n\|. \quad (3.7)$$

Applying this to the vectors $U = Ay$ and $u = ay$, we obtain that the (3.1) and (3.2) are equivalent. If we consider the real numbers

$$\begin{aligned} I_1 &= \operatorname{Re} \left[\left(A \|y, z_2, \dots, z_n\|^2 - \langle x, y \mid z_2, \dots, z_n \rangle \right) \right. \\ &\quad \left. \times \left(\overline{\langle x, y \mid z_2, \dots, z_n \rangle} - \bar{a} \|y, z_2, \dots, z_n\|^2 \right) \right], \\ I_2 &= \|y, z_2, \dots, z_n\|^2 \operatorname{Re} \langle Ay - x, x - ay \mid z_2, \dots, z_n \rangle. \end{aligned} \quad (3.8)$$

Then we obtain, by using properties of n -inner product space,

$$\begin{aligned} I_1 &= \|y, z_2, \dots, z_n\|^2 \operatorname{Re} \left[A \overline{\langle x, y \mid z_2, \dots, z_n \rangle} + \bar{a} \langle x, y \mid z_2, \dots, z_n \rangle \right] \\ &\quad - |\langle x, y \mid z_2, \dots, z_n \rangle|^2 - \|y, z_2, \dots, z_n\|^4 \operatorname{Re}(A\bar{a}). \\ I_2 &= \|y, z_2, \dots, z_n\|^2 \operatorname{Re} \left[A \overline{\langle x, y \mid z_2, \dots, z_n \rangle} + \bar{a} \langle x, y \mid z_2, \dots, z_n \rangle \right] \\ &\quad - \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - \|y, z_2, \dots, z_n\|^4 \operatorname{Re}(A\bar{a}) \end{aligned} \quad (3.9)$$

which gives

$$I_1 - I_2 = \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - |\langle x, y \mid z_2, \dots, z_n \rangle|^2, \quad (3.10)$$

for any $x, y, z_2, \dots, z_n \in X$ and $a, A \in K$. If (3.1) holds, then $I_2 \geq 0$ and thus $I_1 - I_2 \leq I_1$

$$\begin{aligned} &\Rightarrow \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - |\langle x, y \mid z_2, \dots, z_n \rangle|^2 \\ &\leq \operatorname{Re} \left[\left(A \|y, z_2, \dots, z_n\|^2 - \langle x, y \mid z_2, \dots, z_n \rangle \right) \right. \\ &\quad \left. \times \left(\overline{\langle x, y \mid z_2, \dots, z_n \rangle} - \bar{a} \|y, z_2, \dots, z_n\|^2 \right) \right]. \end{aligned} \quad (3.11)$$

Now using the elementary inequality $\operatorname{Re}(\alpha\bar{\beta}) \leq 1/4|\alpha + \beta|^2$ for any $\alpha, \beta \in K$ ($K = R, C$), one yields

$$\begin{aligned} &\operatorname{Re} \left[\left(A \|y, z_2, \dots, z_n\|^2 - \langle x, y \mid z_2, \dots, z_n \rangle \right) \left(\overline{\langle x, y \mid z_2, \dots, z_n \rangle} - \bar{a} \|y, z_2, \dots, z_n\|^2 \right) \right] \\ &\leq \frac{1}{4} |A - a|^2 \|y, z_2, \dots, z_n\|^4. \end{aligned} \quad (3.12)$$

If we combine (3.11) and (3.12), we get the required inequality.

To prove the sharpness of the constant $1/4$, assume that (3.3) holds with a constant $C > 0$, that is,

$$\begin{aligned} & \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - |\langle x, y \mid z_2, \dots, z_n \rangle|^2 \\ & \leq C|A - a|^2 \|y, z_2, \dots, z_n\|^4, \end{aligned} \quad (3.13)$$

where x, y, z_2, \dots, z_n, A , and a satisfy the hypothesis of the theorem.

Consider $y \in X$ with $\|y, z_2, \dots, z_n\| = 1$, $a \neq A$, $m \in X$ with $\|m, z_2, \dots, z_n\| = 1$ and $\langle y, m \mid z_2, \dots, z_n \rangle = 0$ and define

$$x = \frac{A+a}{2}y + \frac{A-a}{2}m \quad (3.14)$$

then we have

$$\operatorname{Re}\langle Ay - x, x - ay \mid z_2, \dots, z_n \rangle = \frac{|A-a|^2}{4} \operatorname{Re}\langle y - m, y + m \mid z_2, \dots, z_n \rangle = 0 \quad (3.15)$$

and then the condition (3.1) is fulfilled. From (3.13) we deduce

$$\left\| \frac{A+a}{2}y + \frac{A-a}{2}m, z_2, \dots, z_n \right\|^2 - \left| \left\langle \frac{A+a}{2}y + \frac{A-a}{2}m, y \mid z_2, \dots, z_n \right\rangle \right|^2 \leq C|A-a|^2 \quad (3.16)$$

and, since

$$\begin{aligned} \left\| \frac{A+a}{2}y + \frac{A-a}{2}m, z_2, \dots, z_n \right\|^2 &= \left| \frac{A+a}{2} \right|^2 + \left| \frac{A-a}{2} \right|^2, \\ \left| \left\langle \frac{A+a}{2}y + \frac{A-a}{2}m, y \mid z_2, \dots, z_n \right\rangle \right|^2 &= \left| \frac{A+a}{2} \right|^2. \end{aligned} \quad (3.17)$$

By (3.16), we have

$$\left| \frac{A-a}{2} \right|^2 \leq C|A-a|^2, \quad (3.18)$$

for any $A, a \in K$ with $a \neq A$, which implies $C \geq 1/4$. This completes the proof. \square

Theorem 3.2. Assume that x, y, z_2, \dots, z_n, a and A are the same as in above Theorem 3.1. If $\operatorname{Re}(\bar{a}A) > 0$, then one has

$$\begin{aligned} \|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| &\leq \frac{1}{2} \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a}\right)\langle x, y \mid z_2, \dots, z_n \rangle\right]}{[\operatorname{Re}(\bar{a}A)]^{1/2}} \\ &\leq \frac{1}{2} \frac{|A + a|}{[\operatorname{Re}(\bar{a}A)]^{1/2}} |\langle x, y \mid z_2, \dots, z_n \rangle|. \end{aligned} \quad (3.19)$$

The constant $1/2$ is best possible in both inequalities in the sense that it cannot be replaced by a smaller constant.

Proof. Define

$$\begin{aligned} I &= \operatorname{Re}\langle Ay - x, x - ay \mid z_2, \dots, z_n \rangle \\ &= \operatorname{Re}\left[A\overline{\langle x, y \mid z_2, \dots, z_n \rangle} + \bar{a}\langle x, y \mid z_2, \dots, z_n \rangle\right] \\ &\quad - \|x, z_2, \dots, z_n\|^2 - \|y, z_2, \dots, z_n\|^2 \operatorname{Re}(A\bar{a}). \end{aligned} \quad (3.20)$$

We know that, for a complex number $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) = \operatorname{Re}(\bar{\alpha})$ and thus

$$\operatorname{Re}\left[A\overline{\langle x, y \mid z_2, \dots, z_n \rangle}\right] = \operatorname{Re}\left[\bar{A}\langle x, y \mid z_2, \dots, z_n \rangle\right] \quad (3.21)$$

which implies

$$I = \operatorname{Re}\left[\left(\bar{A} + \bar{a}\right)\langle x, y \mid z_2, \dots, z_n \rangle\right] - \|x, z_2, \dots, z_n\|^2 - \|y, z_2, \dots, z_n\|^2 \operatorname{Re}(A\bar{a}). \quad (3.22)$$

Since x, y, z_2, \dots, z_n, a and A are assumed to satisfy the condition (3.1), by (3.22), we deduce

$$\|x, z_2, \dots, z_n\|^2 + \|y, z_2, \dots, z_n\|^2 \operatorname{Re}(A\bar{a}) \leq \operatorname{Re}\left[\left(\bar{A} + \bar{a}\right)\langle x, y \mid z_2, \dots, z_n \rangle\right], \quad (3.23)$$

which gives

$$\begin{aligned} &\frac{1}{[\operatorname{Re}(\bar{a}A)]^{1/2}} \|x, z_2, \dots, z_n\|^2 + [\operatorname{Re}(\bar{a}A)]^{1/2} \|y, z_2, \dots, z_n\|^2 \\ &\leq \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a}\right)\langle x, y \mid z_2, \dots, z_n \rangle\right]}{[\operatorname{Re}(\bar{a}A)]^{1/2}} \end{aligned} \quad (3.24)$$

since $\operatorname{Re}(A\bar{a}) > 0$.

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq \quad (3.25)$$

for $p, q \geq 0$ and $\alpha > 0$, we have

$$\begin{aligned} & 2\|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| \\ & \leq \frac{1}{[\operatorname{Re}(\bar{a}A)]^{1/2}} \|x, z_2, \dots, z_n\|^2 + [\operatorname{Re}(\bar{a}A)]^{1/2} \|y, z_2, \dots, z_n\|^2. \end{aligned} \quad (3.26)$$

Using (3.24) and (3.26), we deduce the first inequality in (3.22). The last part is obvious by the fact, for $z \in \mathbb{C}$, $|\operatorname{Re}(z)| \leq |z|$.

To prove the sharpness of the constant $1/2$ in the first inequality in (3.22), we assume that (3.22) holds with a constant $c > 0$, that is,

$$\|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| \leq c \frac{\operatorname{Re} \left[\left(\bar{A} + \bar{a} \right) \langle x, y \mid z_2, \dots, z_n \rangle \right]}{[\operatorname{Re}(\bar{a}A)]^{1/2}} \quad (3.27)$$

provided x, y, z_2, \dots, z_n, a and A satisfy (2.1). If we take $a = A = 1$, $y = x \neq 0$, then obviously (2.1) holds and from (3.27), we obtain

$$\|x, z_2, \dots, z_n\|^2 \leq 2c \|x, z_2, \dots, z_n\|^2 \quad (3.28)$$

for any linearly independent vectors $x, z_2, \dots, z_n \in X$, which implies $c \geq 1/2$. This completes the proof. \square

When the constants involved are assumed to be positive, then we may state the following result.

Corollary 3.3. *Let $M \geq m > 0$ and assume that, for $x, y, z_2, \dots, z_n \in X$, one has*

$$\operatorname{Re} \langle My - x, x - my \mid z_2, \dots, z_n \rangle \geq 0 \quad (3.29)$$

or, equivalently,

$$\left\| x - \frac{M+m}{2}, z_2, \dots, z_n \right\| \leq \frac{1}{2} (M-m) \|y, z_2, \dots, z_n\|. \quad (3.30)$$

Then one has the following reverse of CBS inequality:

$$\begin{aligned} \|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| &\leq \frac{1}{2} \frac{M+m}{\sqrt{mM}} \operatorname{Re} \langle x, y \mid z_2, \dots, z_n \rangle \\ &\leq \frac{1}{2} \frac{M+m}{\sqrt{mM}} |\langle x, y \mid z_2, \dots, z_n \rangle|. \end{aligned} \quad (3.31)$$

The constant $1/2$ is sharp in (3.31).

Corollary 3.4. With the assumptions of the Theorem 3.2, one has

$$\begin{aligned} 0 &< \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - |\langle x, y \mid z_2, \dots, z_n \rangle|^2 \\ &\leq \frac{1}{4} \frac{|A-a|^2}{\operatorname{Re}(\bar{a}A)} |\langle x, y \mid z_2, \dots, z_n \rangle|^2. \end{aligned} \quad (3.32)$$

The constant $1/4$ is best possible in (3.32).

Corollary 3.5. With the assumption of Corollary 3.3, one has

$$\begin{aligned} 0 &\leq \|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| - |\langle x, y \mid z_2, \dots, z_n \rangle| \\ &\leq \|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| - \operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle) \\ &\leq \frac{1}{2} \frac{(\sqrt{M} - \sqrt{m})^2}{\sqrt{mM}} \operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle) \leq \frac{1}{2} \frac{(\sqrt{M} - \sqrt{m})^2}{\sqrt{mM}} |\langle x, y \mid z_2, \dots, z_n \rangle|, \end{aligned} \quad (3.33)$$

$$\begin{aligned} 0 &\leq \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - |\langle x, y \mid z_2, \dots, z_n \rangle|^2 \\ &\leq \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - [\operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle)]^2 \\ &\leq \frac{1}{4} \frac{(M-m)^2}{mM} [\operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle)]^2 \leq \frac{1}{4} \frac{(M-m)^2}{mM} |\langle x, y \mid z_2, \dots, z_n \rangle|^2. \end{aligned} \quad (3.34)$$

The constant $1/2$ in (3.33) and the constant $1/4$ in (3.34) are best possible.

3.2. Reverse of the Triangle Inequality

Corollary 3.6. Assume $x, y, z_2, \dots, z_n, m, M$ are the same as in Corollary 3.3 and $\operatorname{Re} \langle x, y \mid z_2, \dots, z_n \rangle \geq 0$. Then one has the following reverse of the triangle inequality:

$$\begin{aligned} 0 &\leq \|x, z_2, \dots, z_n\| + \|y, z_2, \dots, z_n\| - \|x + y, z_2, \dots, z_n\| \\ &\leq \frac{\sqrt{M} - \sqrt{m}}{(mM)^{1/4}} \sqrt{\operatorname{Re} \langle x, y \mid z_2, \dots, z_n \rangle}. \end{aligned} \quad (3.35)$$

Proof. It is easy to see that

$$\begin{aligned} 0 &\leq (\|x, z_2, \dots, z_n\| + \|y, z_2, \dots, z_n\|)^2 - \|x + y, z_2, \dots, z_n\|^2 \\ &= 2[\|x, z_2, \dots, z_n\|\|y, z_2, \dots, z_n\| - \operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle)] \end{aligned} \quad (3.36)$$

for any $x, y, z_2, \dots, z_n \in X$. If the assumption of Corollary 3.3 holds, then (3.33) is valid and, by (3.36), we deduce

$$\begin{aligned} 0 &\leq (\|x, z_2, \dots, z_n\| + \|y, z_2, \dots, z_n\|)^2 - \|x + y, z_2, \dots, z_n\|^2 \\ &\leq \frac{(\sqrt{M} - \sqrt{m})^2}{\sqrt{mM}} \operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle) \end{aligned} \quad (3.37)$$

which gives

$$\begin{aligned} &(\|x, z_2, \dots, z_n\| + \|y, z_2, \dots, z_n\|)^2 \\ &\leq \|x + y, z_2, \dots, z_n\|^2 + \frac{(\sqrt{M} - \sqrt{m})^2}{\sqrt{mM}} \operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle). \end{aligned} \quad (3.38)$$

Taking square root in (3.38), we have

$$\begin{aligned} &\|x, z_2, \dots, z_n\| + \|y, z_2, \dots, z_n\| \\ &\leq \sqrt{\|x + y, z_2, \dots, z_n\|^2 + \frac{(\sqrt{M} - \sqrt{m})^2}{\sqrt{mM}} \operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle)} \\ &\leq \|x + y, z_2, \dots, z_n\| + \frac{\sqrt{M} - \sqrt{m}}{(mM)^{1/4}} \sqrt{\operatorname{Re}(\langle x, y \mid z_2, \dots, z_n \rangle)}. \end{aligned} \quad (3.39)$$

From where we deduce the desire inequality (3.35). This completes the proof. \square

Let $(X, \langle \cdot, \cdot \mid \cdot, \dots \rangle)$ be a n -inner product space over real or complex number field K . If $(e_i)_{1 < i < m}$ are linearly independent vector in the n -inner product space X , and, for given $z_2, \dots, z_n \in X$, $\langle e_i, e_j \mid z_2, \dots, z_n \rangle = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, m\}$ where δ_{ij} is the Kronecker

delta, then the following inequality is the Bessel's inequality for z_2, \dots, z_n orthonormal family $(e_i)_{1 \leq i \leq m}$ in n -inner product space $(X, \langle \cdot, \cdot | \cdot, \dots \rangle)$:

$$\sum_{i=1}^m |\langle x, e_i | z_2, \dots, z_n \rangle|^2 \leq \|x, z_2, \dots, z_n\|^2 \quad \text{for any } x \in X. \tag{3.40}$$

Theorem 3.7. *Let $x_1, x_2, \dots, x_m, z_2, \dots, z_n \in X$ and $\alpha_1, \dots, \alpha_m \in K$. then one has*

$$\begin{aligned} & \left\| \sum_{i=1}^m \alpha_i x_i, z_2, \dots, z_n \right\|^2 \\ & \leq \begin{cases} \max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{i=1}^m \|x_i, z_2, \dots, z_n\|^2, \\ \left(\sum_{i=1}^m |\alpha_i|^{2a} \right)^{1/a} \left(\sum_{i=1}^m \|x_i, z_2, \dots, z_n\|^{2b} \right)^{1/b}, \quad \text{where } a > 1, \frac{1}{a} + \frac{1}{b} = 1, \\ \sum_{i=1}^m |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \dots, z_n\|^2 \end{cases} \\ & + \begin{cases} \max_{1 \leq i \neq j \leq m} \{ |\alpha_i \alpha_j| \} \sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j | z_2, \dots, z_n \rangle|, \\ \left[\left(\sum_{i=1}^m |\alpha_i|^c \right)^2 - \left(\sum_{i=1}^m |\alpha_i|^{2c} \right) \right]^{1/c} \\ \times \left(\sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j | z_2, \dots, z_n \rangle|^d \right)^{1/d}, \quad \text{where } c > 1, \frac{1}{c} + \frac{1}{d} = 1, \\ \left[\left(\sum_{i=1}^m |\alpha_i| \right)^2 - \sum_{i=1}^m |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq m} |\langle x_i, x_j | z_2, \dots, z_n \rangle|. \end{cases} \end{aligned} \tag{3.41}$$

Proof. We observe that

$$\begin{aligned} \left\| \sum_{i=1}^m \alpha_i x_i, z_2, \dots, z_n \right\|^2 &= \left\langle \sum_{i=1}^m \alpha_i x_i, \sum_{j=1}^m \alpha_j x_j, z_2, \dots, z_n \right\rangle \\ &= \sum_{i=1}^m \sum_{j=1}^m \alpha_i \bar{\alpha}_j \langle x_i, x_j | z_2, \dots, z_n \rangle \\ &= \left| \sum_{i=1}^m \sum_{j=1}^m \alpha_i \bar{\alpha}_j \langle x_i, x_j | z_2, \dots, z_n \rangle \right| \\ &\leq \sum_{i=1}^m \sum_{j=1}^m |\alpha_i| |\bar{\alpha}_j| |\langle x_i, x_j | z_2, \dots, z_n \rangle| \\ &= \sum_{i=1}^m |\alpha_i|^2 \|x_i, z_2, \dots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\alpha_i| |\alpha_j| |\langle x_i, x_j | z_2, \dots, z_n \rangle|. \end{aligned} \tag{3.42}$$

Using Holder's inequality, we may write that

$$\sum_{i=1}^m |\alpha_i|^2 \|x_i, z_2, \dots, z_n\|^2 \leq \begin{cases} \max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{i=1}^m \|x_i, z_2, \dots, z_n\|^2, \\ \left(\sum_{i=1}^m |\alpha_i|^{2a} \right)^{1/a} \left(\sum_{i=1}^m \|x_i, z_2, \dots, z_n\|^{2b} \right)^{1/b}, \text{ where } a > 1, \frac{1}{a} + \frac{1}{b} = 1, \\ \sum_{i=1}^m |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \dots, z_n\|^2. \end{cases} \quad (3.43)$$

By Holder's inequality for double sum, we also have

$$\sum_{1 \leq i \neq j \leq m} |\alpha_i| |\alpha_j| |\langle x_i, x_j \mid z_2, \dots, z_n \rangle| \leq \begin{cases} \max_{1 \leq i \neq j \leq m} |\alpha_i \alpha_j| \sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|, \\ \left(\sum_{1 \leq i \neq j \leq m} |\alpha_i|^c |\alpha_j|^c \right)^{1/c} \\ \times \left(\sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|^d \right)^{1/d}, \text{ where } c > 1, \frac{1}{c} + \frac{1}{d} = 1, \\ \sum_{1 \leq i \neq j \leq m} |\alpha_i| |\alpha_j| \max_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle| \end{cases} \quad (3.44)$$

$$= \begin{cases} \max_{1 \leq i \neq j \leq m} \{|\alpha_i \alpha_j|\} \sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|, \\ \left[\left(\sum_{i=1}^m |\alpha_i|^c \right)^2 - \left(\sum_{i=1}^m |\alpha_i|^{2c} \right) \right]^{1/c} \\ \times \left(\sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|^d \right)^{1/d}, \text{ where } c > 1, \frac{1}{c} + \frac{1}{d} = 1, \\ \left[\left(\sum_{i=1}^m |\alpha_i| \right)^2 - \sum_{i=1}^m |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|. \end{cases}$$

Using (3.43) and (3.44) in (3.42), we may deduce the desired result. \square

Corollary 3.8. *With the assumption in above Theorem 3.7, one has*

$$\begin{aligned} & \left\| \sum_{i=1}^m \alpha_i x_{i, z_2, \dots, z_n} \right\|^2 \\ & \leq \sum_{i=1}^n |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|x_{i, z_2, \dots, z_n}\|^2 + \frac{\left[\left(\sum_{i=1}^n |\alpha_i|^2 \right)^2 - \sum_{i=1}^n |\alpha_i|^4 \right]^{1/2}}{\sum_{i=1}^n |\alpha_i|^2} \right. \\ & \quad \left. \times \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|^2 \right)^{1/2} \right\} \\ & \leq \sum_{i=1}^m |\alpha_i|^2 \left\{ \max_{1 \leq i \leq m} \|x_{i, z_2, \dots, z_n}\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|^2 \right)^{1/2} \right\}. \end{aligned} \tag{3.45}$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $c = d = 2$. The second inequality in (3.45) follows by the fact that

$$\left[\left(\sum_{i=1}^m |\alpha_i|^2 \right)^2 - \sum_{i=1}^m |\alpha_i|^4 \right]^{1/2} \leq \sum_{i=1}^m |\alpha_i|^2. \tag{3.46}$$

By applying the following Cauchy-Bunyakovsky-Schwarz inequality:

$$\left(\sum_{i=1}^m a_i \right)^2 \leq m \sum_{i=1}^m a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq m. \tag{3.47}$$

one may write that

$$\left(\sum_{i=1}^m |\alpha_i|^c \right)^2 - \sum_{i=1}^m |\alpha_i|^{2c} \leq (m-1) \sum_{i=1}^m |\alpha_i|^{2c}, \quad (m \geq 1), \tag{3.48}$$

$$\left(\sum_{i=1}^m |\alpha_i| \right)^2 - \sum_{i=1}^m |\alpha_i|^2 \leq (m-1) \sum_{i=1}^m |\alpha_i|^2, \quad (m \geq 1). \tag{3.49}$$

It is obvious that

$$\max_{1 \leq i \neq j \leq m} \{ |\alpha_i \alpha_j| \} \leq \max_{1 \leq i \leq m} |\alpha_i|^2. \tag{3.50}$$

Corollary 3.9. *With the assumption in above Theorem 3.7, one has*

$$\begin{aligned}
 & \left\| \sum_{i=1}^m \alpha_i x_i, z_2, \dots, z_n \right\|^2 \\
 & \leq \begin{cases} \max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{i=1}^m \|x_i, z_2, \dots, z_n\|^2, \\ \left(\sum_{i=1}^m |\alpha_i|^{2a} \right)^{1/a} \left(\sum_{i=1}^m \|x_i, z_2, \dots, z_n\|^{2b} \right)^{1/b}, \quad \text{where } a > 1, \frac{1}{a} + \frac{1}{b} = 1, \\ \sum_{i=1}^m |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \dots, z_n\|^2 \end{cases} \\
 & + \begin{cases} \max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{1 \leq i \leq m} |\langle x_i, x_i \mid z_2, \dots, z_n \rangle|, \\ (m-1)^{1/c} \left(\sum_{i=1}^m |\alpha_i|^{2c} \right)^{1/c} \left(\sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|^d \right)^{1/d}, \quad \text{where } c > 1, \frac{1}{c} + \frac{1}{d} = 1, \\ (m-1) \sum_{i=1}^m |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \dots, z_n\|^2. \end{cases} \tag{3.51}
 \end{aligned}$$

The proof is obvious by Theorem 3.7 on applying (3.48)–(3.50).

Theorem 3.10. *Let $x, y_1, \dots, y_m, z_2, \dots, z_n$ be vectors of an n -inner product space $(X, \langle \cdot, \cdot \mid \cdot, \dots \rangle)$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in K (K = \mathbb{R}, \mathbb{C})$. Then*

$$\begin{aligned}
 & \left| \sum_{i=1}^m \alpha_i \langle x, y_i \mid z_2, \dots, z_n \rangle \right|^2 \\
 & \leq \|x, z_2, \dots, z_n\|^2 \begin{cases} \max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^2, \\ \left(\sum_{i=1}^m |\alpha_i|^{2a} \right)^{1/a} \left(\sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^{2b} \right)^{1/b}, \quad \text{where } a > 1, \frac{1}{a} + \frac{1}{b} = 1, \\ \sum_{i=1}^m |\alpha_i|^2 \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 \end{cases} \\
 & + \|x, z_2, \dots, z_n\|^2 \begin{cases} \max_{1 \leq i \neq j \leq m} \{|\alpha_i \alpha_j|\} \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|, \\ \left[\left(\sum_{i=1}^m |\alpha_i|^c \right)^2 - \left(\sum_{i=1}^m |\alpha_i|^{2c} \right) \right]^{1/c} \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^d \right)^{1/d}, \\ \left[\left(\sum_{i=1}^m |\alpha_i| \right)^2 - \left(\sum_{i=1}^m |\alpha_i|^2 \right) \right] \sum_{i=1}^m |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \dots, z_n\|^2. \end{cases} \tag{3.52}
 \end{aligned}$$

Proof. Since

$$\sum_{i=1}^m \alpha_i \langle x, y_i \mid z_2, \dots, z_n \rangle = \left\langle x, \sum_{i=1}^m \bar{\alpha}_i y_i \mid z_2, \dots, z_n \right\rangle \tag{3.53}$$

and by Schwarz's inequality in n -inner product spaces,

$$\left| \sum_{i=1}^m \alpha_i \langle x, y_i \mid z_2, \dots, z_n \rangle \right|^2 \leq \|x, z_2, \dots, z_n\|^2 \left\| \sum_{i=1}^m \bar{\alpha}_i y_i, z_2, \dots, z_n \right\|^2. \tag{3.54}$$

Using $\alpha_i = \bar{\alpha}_i$, $z_i = y_i$ ($i = 1, 2, \dots, n$) in the Theorem 3.7, we get the desired inequality (3.52). \square

Corollary 3.11. *With the assumptions in Theorem 3.10, the following holds:*

$$\begin{aligned} & \left| \sum_{i=1}^m \alpha_i \langle x, y_i \mid z_2, \dots, z_n \rangle \right|^2 \\ & \leq \|x, z_2, \dots, z_n\|^2 \\ & \times \left\{ \begin{aligned} & \sum_{i=1}^m |\alpha_i|^2 \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^2 \right)^{1/2} \right\}, \\ & \max_{1 \leq i \leq m} |\alpha_i|^2 \left\{ \sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\}, \\ & \left(\sum_{i=1}^m |\alpha_i|^{2p} \right)^{1/p} \\ & \times \left\{ \left(\sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^{2q} \right)^{1/q} + (m-1) \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^q \right)^{1/q} \right\}, \\ & \sum_{i=1}^m |\alpha_i|^2 \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + (m-1) \max_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\}. \end{aligned} \right. \end{aligned} \tag{3.55}$$

3.3. Some Boas-Bellman Type Inequalities in n -Inner Product Spaces

If we put $\alpha_i = \overline{\langle x, y_i \mid z_2, \dots, z_n \rangle}$ ($i = 1, 2, \dots, m$), in the first inequality of (3.55)

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right)^2 \\ & \leq \|x, z_2, \dots, z_n\|^2 \sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \\ & \quad \times \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^2 \right)^{1/2} \right\} \end{aligned} \quad (3.56)$$

which is equivalent to the following Boass-Bellman type inequality for n -inner products:

$$\begin{aligned} & \sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \\ & \leq \|x, z_2, \dots, z_n\|^2 \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^2 \right)^{1/2} \right\}. \end{aligned} \quad (3.57)$$

Now, if we take $\alpha_i = \overline{\langle x, y_i \mid z_2, \dots, z_n \rangle}$ ($i = 1, 2, \dots, m$), in second inequality of (3.55), we have

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right)^2 \\ & \leq \|x, z_2, \dots, z_n\|^2 \max_{1 \leq i \leq m} |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \\ & \quad \times \left\{ \sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\}. \end{aligned} \quad (3.58)$$

By taking the square root in

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right) \\ & \leq \|x, z_2, \dots, z_n\| \max_{1 \leq i \leq m} |\langle x, y_i \mid z_2, \dots, z_n \rangle| \\ & \quad \times \left\{ \sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\}^{1/2} \end{aligned} \quad (3.59)$$

for any $x, y_1, \dots, y_m, z_2, \dots, z_n$ be vectors of an n -inner product space $(X, \langle \cdot, \cdot \mid \cdot, \dots \rangle)$. If we assume that $(e_i)_{1 \leq i \leq m}$ is an orthonormal family in X with respect to the vector z_2, \dots, z_n , $\langle e_i, e_j \mid z_2, \dots, z_n \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, m\}$ then by (3.57) we deduce Bessel's inequality $\sum_{i=1}^m |\langle x, e_i \mid z_2, \dots, z_n \rangle|^2 \leq \|x, z_2, \dots, z_n\|^2$, and (3.59) implies

$$\sum_{i=1}^m |\langle x, e_i \mid z_2, \dots, z_n \rangle|^2 \leq \sqrt{m} \|x, z_2, \dots, z_n\| \max_{1 \leq i \leq m} |\langle x, e_i \mid z_2, \dots, z_n \rangle|^2. \quad (3.60)$$

For the third inequality in (3.55) $\alpha_i = \overline{\langle x, y_i \mid z_2, \dots, z_n \rangle}$ ($i = 1, 2, \dots, m$), we have

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right)^2 \\ & \leq \|x, z_2, \dots, z_n\|^2 \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^{2p} \right)^{1/p} \\ & \quad \times \left\{ \left(\sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^{2q} \right)^{1/q} + (m-1) \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^q \right)^{1/q} \right\} \end{aligned} \quad (3.61)$$

for $p > 1$, $(1/p) + (1/q) = 1$. Taking the square root in this inequality, we get

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right) \\ & \leq \|x, z_2, \dots, z_n\| \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^{2p} \right)^{1/2p} \\ & \quad \times \left\{ \left(\sum_{i=1}^m \|y_i, z_2, \dots, z_n\|^{2q} \right)^{1/q} + (m-1) \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^q \right)^{1/q} \right\}^{1/2}. \end{aligned} \quad (3.62)$$

For any $x, y_1, \dots, y_m, z_2, \dots, z_n \in X$, and $p > 1$, with $(1/p) + (1/q) = 1$, then the above inequality (3.62) becomes, for an orthonormal family with respect to the vector z_2, \dots, z_n ,

$$\left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right) \leq m^{1/q} \|x, z_2, \dots, z_n\| \left(\sum_{i=1}^m |\langle x, e_i \mid z_2, \dots, z_n \rangle|^{2p} \right)^{1/2p}. \quad (3.63)$$

We take $\alpha_i = \overline{\langle x, y_i \mid z_2, \dots, z_n \rangle}$ ($i = 1, 2, \dots, m$), in the last inequality of (3.55)

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right)^2 \\ & \leq \|x, z_2, \dots, z_n\|^2 \sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \\ & \quad \times \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + (m-1) \max_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\} \end{aligned} \quad (3.64)$$

which implies

$$\begin{aligned} & \left(\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right) \\ & \leq \|x, z_2, \dots, z_n\|^2 \left\{ \max_{1 \leq i < m} \|y_i, z_2, \dots, z_n\|^2 + (m-1) \max_{1 \leq i \neq j < m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\} \end{aligned} \quad (3.65)$$

for any $x, y_1, \dots, y_m, z_2, \dots, z_n \in X$.

4. Applications for Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a σ -algebra Σ of subsets of Ω and a countably additive measure μ on Σ with values in $R \cup \{\infty\}$. Denote by $L^2\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are n - ρ -integrable on Ω , that is, $\int_{\Omega} \rho(s) |f(s)|^n d\mu(s) < \infty$ where $\rho : \Omega \rightarrow [0, \infty)$ is a measurable function on Ω .

We can introduce the following n -inner product on $L^2\rho(\Omega)$:

$$\begin{aligned} \langle f, g \mid h_2, \dots, h_n \rangle_{\rho} &= \frac{1}{n!} \underbrace{\int_{\Omega} \int_{\Omega} \cdots \int_{\Omega}}_{n\text{-times}} \rho(s_1) \rho(s_2) \cdots \rho(s_n) \\ & \quad \times \begin{vmatrix} f(s_1) & f(s_2) & \cdots & f(s_n) \\ h_2(s_1) & h_2(s_2) & \cdots & h_2(s_n) \\ \vdots & \cdots & \ddots & \vdots \\ h_n(s_1) & h_n(s_2) & \cdots & h_n(s_n) \end{vmatrix} \\ & \quad \times \begin{vmatrix} g(s_1) & g(s_2) & \cdots & g(s_n) \\ h_2(s_1) & h_2(s_2) & \cdots & h_2(s_n) \\ \vdots & \cdots & \ddots & \vdots \\ h_n(s_1) & h_n(s_2) & \cdots & h_n(s_n) \end{vmatrix} d\mu(s_1) d\mu(s_2) \cdots d\mu(s_n), \end{aligned} \quad (4.1)$$

where by

$$\begin{vmatrix} f(s_1) & f(s_2) \dots & f(s_n) \\ h_2(s_1) & h_2(s_2) \dots & h_2(s_n) \\ \vdots & \dots & \vdots \\ h_n(s_1) & h_n(s_2) \dots & h_n(s_n) \end{vmatrix} \tag{4.2}$$

we denote the determinant of matrix

$$\begin{bmatrix} f(s_1) & f(s_2) & \dots & \dots & f(s_n) \\ h_2(s_1) & h_2(s_2) & \dots & \dots & h_2(s_n) \\ \vdots & \vdots & \dots & \dots & \vdots \\ h_n(s_1) & h_n(s_2) & \dots & \dots & h_n(s_n) \end{bmatrix}. \tag{4.3}$$

Generating the n -norm on $L^2\rho(\Omega)$ is expressed by

$$\begin{aligned} & \|f, h_2, \dots, h_n\|_p \\ &= \left(\frac{1}{n!} \underbrace{\int_{\Omega} \int_{\Omega} \dots \int_{\Omega} \rho(s_1)\rho(s_2)\dots\rho(s_n)}_{n\text{-times}} \right. \\ & \quad \times \left. \begin{vmatrix} f(s_1) & f(s_2) & \dots & \dots & f(s_n) \\ h_2(s_1) & h_2(s_2) & \dots & \dots & h_2(s_n) \\ \vdots & \vdots & \dots & \dots & \vdots \\ h_n(s_1) & h_n(s_2) & \dots & \dots & h_n(s_n) \end{vmatrix}^2 d\mu(s_1)d\mu(s_2)\dots d\mu(s_n) \right)^{1/2}. \end{aligned} \tag{4.4}$$

A simple calculation with integrals shows that

$$\langle f, g \mid h_2, \dots, h_n \rangle_{\rho} = \begin{vmatrix} \int_{\Omega} \rho f g d\mu & \int_{\Omega} \rho f h_2 d\mu & \dots & \dots & \int_{\Omega} \rho f h_n d\mu \\ \int_{\Omega} \rho f h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \dots & \dots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \dots & \dots & \vdots \\ \int_{\Omega} \rho g h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \dots & \dots & \int_{\Omega} \rho h_n^2 d\mu \end{vmatrix}, \tag{4.5}$$

$$\|f, h_2, \dots, h_n\|_p = \begin{vmatrix} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h_2 d\mu & \dots & \dots & \int_{\Omega} \rho f h_n d\mu \\ \int_{\Omega} \rho f h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \dots & \dots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \dots & \dots & \vdots \\ \int_{\Omega} \rho g h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \dots & \dots & \int_{\Omega} \rho h_n^2 d\mu \end{vmatrix}, \tag{4.6}$$

where, for simplicity, instead of $\int_{\Omega} \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_{\Omega} \rho f g d\mu$.

Proposition 4.1. Let $f, g_1, \dots, g_m, h_2, \dots, h_n \in L^2\rho(\Omega)$, where $\rho : \Omega \rightarrow [0, \infty)$ a measure function is on Ω . Then one has

$$\sum_{i=1}^m \left| \begin{matrix} \int_{\Omega} \rho f g_i d\mu & \int_{\Omega} \rho f h_2 d\mu & \cdots & \int_{\Omega} \rho f h_n d\mu \\ \int_{\Omega} \rho g_i h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho g_i h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{matrix} \right|^2$$

$$\leq \left| \begin{matrix} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h_2 d\mu & \cdots & \int_{\Omega} \rho f h_n d\mu \\ \int_{\Omega} \rho f h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho f h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{matrix} \right| \tag{4.7}$$

$$\times \left\{ \begin{matrix} \max_{1 \leq i \leq m} \left| \begin{matrix} \int_{\Omega} \rho g_i^2 d\mu & \int_{\Omega} \rho g_i h_2 d\mu & \cdots & \int_{\Omega} \rho g_i h_n d\mu \\ \int_{\Omega} \rho g_i h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho g_i h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{matrix} \right| \\ + \left(\sum_{1 \leq i \neq j \leq m} \left| \begin{matrix} \int_{\Omega} \rho g_j g_i d\mu & \int_{\Omega} \rho g_j h_2 d\mu & \cdots & \int_{\Omega} \rho g_j h_n d\mu \\ \int_{\Omega} \rho g_i h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho g_i h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{matrix} \right|^2 \right)^{1/2} \end{matrix} \right\}$$

Proof. Applying the n -inner product and n -norm defined in (4.1) and (4.4), and using (4.5) and (4.6) in (3.57)

$$\sum_{i=1}^m |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2$$

$$\leq \|x, z_2, \dots, z_n\|^2 \tag{4.8}$$

$$\times \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle|^2 \right)^{1/2} \right\},$$

we get the required proof of the proposition. □

Proposition 4.2. Let $f, g_1, \dots, g_m, h_2, \dots, h_n \in L^2 \rho(\Omega)$, where $\rho: \Omega \rightarrow [0, \infty)$ a measure function is on Ω . Then one has

$$\begin{aligned}
 & \sum_{i=1}^m \left| \begin{array}{cccc} \int_{\Omega} \rho f g_i d\mu & \int_{\Omega} \rho f h_2 d\mu & \cdots & \int_{\Omega} \rho f h_n d\mu \\ \int_{\Omega} \rho g_i h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho g_i h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{array} \right|^2 \\
 & \leq \left| \begin{array}{cccc} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h_2 d\mu & \cdots & \int_{\Omega} \rho f h_n d\mu \\ \int_{\Omega} \rho f h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho f h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{array} \right| \\
 & \times \left\{ \begin{array}{l} \max_{1 \leq i \leq m} \left| \begin{array}{cccc} \int_{\Omega} \rho g_i^2 d\mu & \int_{\Omega} \rho g_i h_2 d\mu & \cdots & \int_{\Omega} \rho g_i h_n d\mu \\ \int_{\Omega} \rho g_i h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho g_i h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{array} \right| \\ + (m-1) \max_{1 \leq i \neq j \leq m} \left| \begin{array}{cccc} \int_{\Omega} \rho g_j g_i d\mu & \int_{\Omega} \rho g_j h_2 d\mu & \cdots & \int_{\Omega} \rho g_j h_n d\mu \\ \int_{\Omega} \rho g_i h_2 d\mu & \int_{\Omega} \rho h_2^2 d\mu & \cdots & \int_{\Omega} \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\Omega} \rho g_i h_n d\mu & \int_{\Omega} \rho h_2 h_n d\mu & \cdots & \int_{\Omega} \rho h_n^2 d\mu \end{array} \right| \end{array} \right\}. \tag{4.9}
 \end{aligned}$$

Proof. By (3.65),

$$\begin{aligned}
 & \left(\sum |\langle x, y_i \mid z_2, \dots, z_n \rangle|^2 \right) \\
 & \leq \|x, z_2, \dots, z_n\|^2 \\
 & \times \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \dots, z_n\|^2 + (m-1) \max_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \dots, z_n \rangle| \right\}, \tag{4.10}
 \end{aligned}$$

and using (4.1) and (4.4), we get the proof of the proposition. □

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