Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 713563, 33 pages
doi:10.1155/2010/713563

Research Article

Extension of Spectral Scales to Unbounded Operators

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Received 30 May 2010; Accepted 14 June 2010

Academic Editor: Palle E. Jorgensen

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We extend the notion of a spectral scale to n-tuples of unbounded operators affiliated with a finite von Neumann Algebra. We focus primarily on the single-variable case and show that many of the results from the bounded theory go through in the unbounded situation. We present the currently available material on the unbounded multivariable situation. Sufficient conditions for a set to be a spectral scale are established. The relationship between convergence of operators and the convergence of the corresponding spectral scales is investigated. We establish a connection between the Akemann et al. spectral scale (1999) and that of Petz (1985).

1. Introduction and Preliminaries

The notion of the spectrum of a self-adjoint operator has proved to be of great interest and use in various branches of mathematics. It is natural to try and extend the notion to n-tuples of operators. In 1999, Akemann et al. came up with the notion of a spectral scale [1, page 277]. The setting is as follows. Let M be a finite von Neumann algebra equipped with a normal, faithful tracial state, τ. Elements of M can be thought of as bounded operators on some Hilbert Space, H, [2, page 308]. For a given self-adjoint $b \in M$ the corresponding spectral scale, B, which we will define below, yields information about the spectrum of b in a nice geometric way. Many of the results can be extended to n-tuples of self-adjoint operators in M. The primary aim of this paper is to explain several of the results on spectral scales, and show how they can be extended when, instead of considering $b \in M$, we consider $g \in M_*$.

In Section 2, we consider the single-variable case which is fairly well developed. A sequence of technical lemmas culminating in Lemma 2.10 are required before we can make significant progress in the single-variable case. We illustrate with examples. Finally, we establish sufficient conditions to guarantee that a subset of $\mathbb{R}^2$ is a spectral scale.

In Section 3, we consider the geometric structure of the n-dimensional spectral scale. It turns out that there is little difficulty in generalizing from the bounded situation.

In Section 4, we discuss certain invariance properties of the spectral scale. Significant difficulties arise in the unbounded situation although we believe that, if Conjecture 4.5 is correct, many of the difficulties would be removed.

Section 5 addresses some miscellaneous results. First, we address the natural question of whether the convergence of a sequence of operators implies the convergence of the corresponding spectral scales. Second, we establish a relationship between two logically distinct objects [1, 3] which were both defined by their authors as "spectral scales".

Finally, in Section 6 we outline some possible future directions of research.

Let us start with some preliminary definitions.

Definition 1.1. Let H be a Hilbert space. Let M be a subalgebra of $B(H)$. If M is closed in the weak operator topology, self-adjoint, and contains 1, then M is a *von Neumann algebra* [2, page 308].

Let M^+ denote the set of positive elements of M.

Definition 1.2. Let $\tau : M^+ \to [0,\infty]$ be a function such that for $a, b, a_\alpha \in M^+$ and $\lambda, \mu \in [0,\infty]$ we have:

$$\begin{gathered}\tau(\lambda a+\mu b)=\lambda\tau(a)+\mu\tau(b),\\ \tau(a^*a)=\tau(aa^*),\\ a_\alpha\uparrow a\Longrightarrow\tau(a_\alpha)\uparrow\tau(a)\quad[\text{normal}],\\ \tau(a)=0\Longrightarrow a=0\quad[\text{faithful}],\\ \tau(1)<\infty\quad[\text{finite}].\end{gathered}\tag{1.1}$$

(The last equation implies that $\infty \notin \operatorname{Im}(\tau)$).

Then τ is a *faithful, finite, normal trace* on M^+ [4, pages 504-5].

Theorem 1.3. *Let τ be a faithful, finite, normal trace on M^+. Since any element of M can be written as a finite linear combination of positive elements of M, τ can be extended to a linear functional on all of M[5, page 309].*

Two projections p, q in M are *equivalent* ($p \sim q$) if there exists $u \in M$ such that $uu^* = p$ and $u^*u = q$. A projection p is *finite* if $p \sim q \le p \Rightarrow p = q$. M is *finite* if the projection 1 is finite [5, page 296]. Throughout, we will assume that M is finite.

Further, we will assume that there exists a faithful, finite, normal trace τ of M, with $\tau(1) = 1$; that is, τ is a *faithful, normal, tracial state* on M.

A crucial property of τ is that "things" commute in trace—that is, although, in general $ab \neq ba$, for $a, b \in M$, we do have the equality $\tau(ab) = \tau(ba)$ [4, page 517].

Let $M_1^+ = \{a \in M \mid 0 \le a \le 1\}$. Let $(b_1, b_2, \ldots, b_n)$ be an n-tuple of self-adjoint operators in M. Let

$$\begin{gathered}\Psi : M \longrightarrow \mathbb{C}^{n+1}\\ a \longmapsto (\tau(a), \tau(b_1a), \tau(b_2a), \ldots, \tau(b_na)).\end{gathered}\tag{1.2}$$

Definition 1.4 (see [1, page 260]). $\Psi(M_1^+) := B$is called the *spectral scale* of$(b_1, b_2, \ldots, b_n)$ with respect to τ.

Now τ is normal. Further Ψ is linear and continuous with respect to the weak operator topology. Moreover, M_1^+ is convex and compact in the weak operator topology: $\tau(M_1^+) \in \mathbb{R}$ and $\tau(b_i a) = \tau(a^{1/2} b_i a^{1/2}) \in \mathbb{R}$ for $i = 1, \ldots, n$. Therefore B is a compact, convex subset of $\mathbb{R}^{n+1}$.

There have been a large number of results concerning spectral scale. Some papers on the subject include those in [1, 6, 7].

In 2004, Akemann and David Sherman conjectured that, if we replace B with the set

$$\{(\tau(a), g_1(a), g_2(a), \ldots, g_n(a)) \mid a \in M_1^+\}, \tag{1.3}$$

where each $g_i \in M_*$ is self-adjoint, we will yield similar results. This paper verifies this, and generalizes much of the first paper on spectral scales [1].

Some results on "noncommutative integration" will prove useful in our exposition. We will use Nelson's 1972 [8] paper on the subject with specific theorem and page references as appropriate.

In his paper, Nelson defines $L^1(M)$, the predual of M [8, Section 3, pages 112 ff.]. The duality is given by the bilinear form $(a, b) \mapsto \tau(ab) = \tau(ba)$ [8, Section 3, page 112] for $a \in M$ and $b \in L^1(M)$. Now $ba \in L^1(M)$ [8, page 112 ff.], and Nelson shows that elements of $L^1(M)$ are closed, densely defined operators affiliated with M [8, Theorem 1, page 107, and Theorem 5, page 114]. It follows that a bounded linear functional, $g \in M_*$ can be represented by a (possibly unbounded) linear operator b affiliated with M and we get the equality $g(a) = \tau(ba)$ for every $a \in M$.

2. Spectral Scale Theory for Unbounded Operators—the Single-Variable Case

We are now prepared to discuss how the spectral scale theory generalizes. We start with the single-variable situation.

Definition 2.1. Let $g \in M_*$ be a self-adjoint linear functional. Let

$$B(g) = \{(\tau(a), g(a)) \mid a \in M_1^+\}. \tag{2.1}$$

Then $B(g)$ is the *spectral scale* of g with respect to τ.

From the theory of noncommutative integration, we see that $B(g) = \{(\tau(a), \tau(ba)) \mid a \in M_1^+\}$ for some operator b affiliated with M. Since g is self-adjoint, b too will be self-adjoint, and hence, as with the original spectral scale, our generalized spectral scale is a compact, convex subset of $\mathbb{R}^2$.

Notation 1. We will often write B for $B(g)$.

The following definition was suggested to the author in conversation by Akemann.

Definition 2.2 (Akemann). If b is bounded, we will call g an *operator functional*.

Our main goal in this section is to show that g is an operator functional if and only if the slopes of the lower boundary function of B are all finite. We remark that Akemann et al. have already shown the "only if" part of this statement [1, Section 1, pages 261–274]. For this reason, we may assume throughout that b is unbounded, and show that the lower boundary curve of B has, as a consequence, an infinite slope. To get there, we will need a number of preliminary results.

Proposition 2.3. *B is mapped onto itself by a reflection through the point $(1/2, \tau(b)/2)$.*

Proof. Let $a \in M_1^+$. Then $(\tau(a), \tau(ba)) \in B$. Therefore

$$(\tau(1-a), \tau(b(1-a))) = (1-\tau(a), \tau(b)-\tau(ba)) \in B. \tag{2.2}$$

Thus the map $\upsilon : (x_0, x_1) \mapsto (1-x_0, \tau(b)-x_1)$ takes $\Psi(a)$ to $\Psi(1-a)$ and hence takes B onto itself. The fixed point of υ is $(1/2, \tau(b)/2)$, and the points (x_0, x_1) and $(1-x_0, \tau(b)-x_1)$ lie on a straight line that passes through $(1/2, \tau(b)/2)$. The straight line is given by the equation

$$y - x_1 = \left(\frac{\tau(b)-2x_1}{1-2x_0}\right)(x-x_0). \tag{2.3}$$

Note also that υ^2 is the identity map on $\mathbb{R}^2$. Hence, $\upsilon(B) = B$ and υ reflects B through the point $(1/2, \tau(b)/2)$. □

For the next several results, we will need the unbounded spectral theorem for self-adjoint operators. We state it here in the functional calculus form.

Theorem 2.4 (see von Neumann in [9, page 562]). *Let b be a (densely defined) self-adjoint operator in H with domain $D(b)$. Then $\exists!$ algebraic $*$-homomorphism ϕ takes bounded Borel functions on $\mathbb{R}$ into $B(H)$ such that the following hold.*

(a) *ϕ is norm continuous.*

(b) *Let $\{h_n(x)\}_{n\in\mathbb{N}}$ be a sequence of bounded Borel functions with $h_n(x) \to x$ as $n \to \infty$ for each x and $|h_n(x)| \le |x|$ for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for $\psi \in D(b)$, $\phi(h_n)\psi \to b\psi$ as $n \to \infty$. The convergence is in norm.*

(c) *If $h_n(x) \to h(x)$ pointwisely, and the sequence $\{\|h_n\|_\infty\}_{n\in\mathbb{N}}$ is bounded, then $\phi(h_n) \to \phi(h)$ strongly.*

(d) *If $b\psi = \lambda\psi$, then $\phi(h)\psi = h(\lambda)\psi$.*

(e) *If $h \ge 0$, then $\phi(h) \ge 0$.*

For a given h, a bounded Borel function on $\mathbb{R}$, it is customary to write $\phi(h)$ as $h(b)$. In other words, the "ϕ" is understood. For now it is more convenient to write ϕ explicitly.

Definition 2.5. For $s \in \mathbb{R}$ let $\phi(\chi_{(-\infty,s)}) = p_s^-$ and $\phi(\chi_{(-\infty,s]}) = p_s^+$. More generally, if h is a characteristic function on a Borel subset of $\mathbb{R}$, then $\phi(h)$ is a projection; such projections are referred to as *spectral projections* [9, pages 234, 267].

For the most part, we will only need spectral projections obtained from intervals. Note that, for $s \in \mathbb{R}$, $p_s^+ - p_s^-$ is nonzero on the domain of b (and hence all of H) if and only if s is an eigenvalue of b. Also, since Borel functions commute with respect to multiplication and ϕ is a homomorphism, $\mathrm{Im}(\phi)$ is an Abelian subalgebra of $B(H)$. Assume now that b is affiliated with our finite von Neumann algebra, M. In this case it turns out that $\mathrm{Im}(\phi)$ is an Abelian subalgebra of M. This follows from the way that ϕ is constructed.

Lemma 2.6. *Let $c \in [p_s^-, p_s^+]$. Then*

$$\begin{aligned} (b - s1)(1 - c) &= (b - s1)\left(1 - p_s^+\right) \geq 0, \\ (s1 - b)c &= (s1 - b)p_s^- \geq 0. \end{aligned} \tag{2.4}$$

Proof. Using the decomposition $1 = p_s^- + (p_s^+ - p_s^-) + (1 - p_s^+)$, we can write

$$b = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & b_2 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

Hence,

$$(b - s1)(1 - c) = \begin{pmatrix} b_1 - s1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 - s1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - c' & 0 \\ 0 & 0 & 1 \end{pmatrix} = (b - s1)\left(1 - p_s^+\right). \tag{2.6}$$

(Of course, these equalities only make sense on the domain of b.)

For every $N > s$ we get:

$$\begin{aligned} x\chi_{(s,N]} &\geq s\chi_{(s,N]}, \\ \phi\left(x\chi_{(s,N]}\right) &\geq s\phi\left(\chi_{(s,N]}\right), \\ \lim_{N \to \infty} \phi\left(x\chi_{(s,N]}\right) &\geq \lim_{N \to \infty} s\phi\left(\chi_{(s,N]}\right), \\ b\left(1 - p_s^+\right) &\geq s\left(1 - p_s^+\right). \end{aligned} \tag{2.7}$$

Therefore $b(1 - p_s^+) \geq s(1 - p_s^+)$ and hence $(b - s1)(1 - p_s^+) \geq 0$. The other statement in this lemma follows via a similar argument. □

Lemma 2.7. *Let h be a characteristic function of a bounded Borel subset of $\mathbb{R}$. Then $\mathrm{Im}(\phi(h)) \subset D(b)$.*

Proof. Let h_n be a sequence of bounded Borel functions such that $\lim_{n\to\infty} h_n(x) = x$ for all $x \in \mathbb{R}$ and $|h_n(x)| \le |x|$. By the Spectral theorem,

$$\phi(h_n)\psi \longrightarrow b\psi \tag{2.8}$$

for every $\psi \in D(b)$. Note that $h_n(x)h(x)$ converges to $xh(x)$ for every x. Since $xh(x) := k(x)$ is a bounded Borel function, $\phi(k) \in B(H)$. For $\psi \in D(b)$, $\xi \in H$, we have

$$\langle b\psi, \phi(h)\xi\rangle = \lim_{n\to\infty}\langle \phi(h_n)\psi, \phi(h)\xi\rangle = \lim_{n\to\infty}\langle \phi(hh_n)\psi, \xi\rangle = \langle \psi, (\phi(k))^*\xi\rangle. \tag{2.9}$$

Hence $\phi(h)\xi \in D(b^*)$ and $b^*\phi(h)\xi = (\phi(k))^*\xi$. Since b is self-adjoint, we have the desired result. ☐

Corollary 2.8. *The following set relations hold:*

$$\begin{gathered}\bigcup_{t>s}(p_t^- - p_s^+)H \subset (D(b) \cap (1 - p_s^+)H),\\ \bigcup_{t<s}(p_s^- - p_t^+)H \subset (D(b) \cap (p_s^-)H).\end{gathered} \tag{2.10}$$

Lemma 2.9. *The range projection of $(b - s1)(1 - p_s^+)$ is $1 - p_s^+$. The range projection of $(s1 - b)p_s^-$ is p_s^-.*

Proof. Let q be the range projection of $(b - s1)(1 - p_s^+)$. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of bounded Borel functions on $\mathbb{R}$ such that $\lim_{n\to\infty} h_n(x) = x$for all $x \in \mathbb{R}, n \in \mathbb{N}$. Let $h(x) = \chi_{(s,\infty)}(x)$. Then $\phi(h) = 1 - p_s^+$. From the Spectral theorem, we have

$$\begin{gathered}\phi(h_n)\psi \longrightarrow b\psi \quad \forall\psi \in D(b),\\ \phi(h_n)\phi(h)\psi \longrightarrow b(1 - p_s^+)\psi \quad \text{(by Lemma 2.7)},\\ \phi(h_n)(\phi(h))^2\psi \longrightarrow b(1 - p_s^+)\psi,\\ \phi(h)\phi(h_n)\phi(h)\psi \longrightarrow b(1 - p_s^+)\psi.\end{gathered} \tag{2.11}$$

Taking the limit as $n \to \infty$ on the left side, we get

$$(1 - p_s^+)b(1 - p_s^+)\psi = b(1 - p_s^+)\psi, \tag{2.12}$$

and therefore

$$(1 - p_s^+)(b - s1)(1 - p_s^+)\psi = (b - s1)(1 - p_s^+)\psi \tag{2.13}$$

for every $\psi \in D(b)$.

We have shown that $q \leq 1-p_s^+$. For $t > s$ let q_t be the range projection of $(b-s1)(p_t^- - p_s^+)$. By the same reasoning as above, $q_t \leq p_t^- - p_s^+$. Also,

$$qb(p_t^- - p_s^+) = qb(1-p_s^+)(p_t^- - p_s^+) = b(1-p_s^+)(p_t^- - p_s^+) = b(p_t^- - p_s^+). \tag{2.14}$$

Hence, $q_t \leq q$. Similarly, for $t_1 \geq t_2 > s$, we have $q_{t_1} \geq q_{t_2}$.

Now $b(p_t^- - p_s^+) = \phi(h\chi_{(s,t)}) \in B(H)$ by Lemma 2.7. Therefore, $(b-s1)(p_t^- - p_s^+)$ is a bounded operator. In fact, $(b-s1)(p_t^- - p_s^+) \geq 0$ on $(b-s1)(p_t^- - p_s^+)H := H_t$. We show that H_t is an invariant subspace of H under $(b-s1)(p_t^- - p_s^+)$. Suppose that $\eta \in H_t$ and $\eta \perp \operatorname{Im}(((b-s1)(p_t^- - p_s^+))H_t)$. Then for every $\xi \in H_t$,

$$\begin{gathered}\left\langle (b-s1)(p_t^- - p_s^+)\xi, \eta \right\rangle = 0,\\ \left\langle (b-s1)\xi, \eta \right\rangle = 0,\\ \left\langle \xi, (b-s1)\eta \right\rangle = 0,\\ (b-s1)\eta = 0,\end{gathered} \tag{2.15}$$

since $b \neq s1$ in H_t, and $\eta = 0$.

Hence, $\overline{\operatorname{Im}((b-s1)(p_t^- - p_s^+))} = \operatorname{Im}(q_t)$. Thus,

$$q_t = p_t^- - p_s^+ \longrightarrow 1 - p_s^+ \tag{2.16}$$

as $t \to \infty$. Since $q_t \leq q$ for every t,

$$1 - p_s^+ = \lim_{t\to\infty} q_t \leq q. \tag{2.17}$$

But we already know that $(1-p_s^+) \geq q$ and so equality holds.

The second statement in the lemma follows from an analogous proof. □

Lemma 2.10. *Let $a \in M_1^+$. If $a^{1/2}(b-s1)(1-p_s^+)a^{1/2} = 0$, then $a \leq p_s^+$.*

If $(1-a)^{1/2}(s1-b)p_s^-(1-a)^{1/2} = 0$, then $a \geq p_s^-$.

Proof. Write $a^{1/2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12}^* & a_{22} & a_{23} \\ a_{13}^* & a_{23}^* & a_{33} \end{pmatrix}$, using the decomposition

$$1 = p_s^- + (p_s^+ - p_s^-) + (1 - p_s^+). \tag{2.18}$$

Note that $(1-p_s^+)a_{i3}^* = a_{i3}^*$ for $i = 1,2,3$. Assume that

$$a^{1/2}(b-s1)(1-p_s^+)a^{1/2} = 0. \tag{2.19}$$

The diagonal entries of $a^{1/2}(b-s1)(1-p_s^+)a^{1/2}$ are $a_{i3}(b_2-s1)a_{i3}^*$ for $i=1,2,3$. Hence, $a_{i3}(b_2-s1)a_{i3}^*=0$. Thus, for all ψ, such that $a_{i3}^*\psi\in D(b_2)$, $a_{i3}(b_2-s1)a_{i3}^*\psi=0$. Since $b_2-s1\geq 0$ on $(1-p_s^+)H\cap D(b_2)$ which contains $a_{i3}^*\psi$, we get

$$\begin{gathered}\left\langle a_{i3}(b_2-s1)a_{i3}^*\psi,\psi\right\rangle=0,\\ \left\langle (b_2-s1)^{1/2}a_{i3}^*\psi,(b_2-s1)^{1/2}a_{i3}^*\psi\right\rangle=0.\end{gathered}\tag{2.20}$$

Hence $(b_2-s1)^{1/2}a_{i3}^*\psi=0$ and so $(b_2-s1)a_{i3}^*\psi=0$. Since s is not an eigenvalue of b_2, $a_{13}^*\psi=0$. Therefore

$$\left(1-p_s^+\right)(b_2-s1)a_{i3}^*\psi=0,\tag{2.21}$$

and so

$$\left(p_t^- - p_s^+\right)(b_2-s1)a_{i3}^*\psi=0\tag{2.22}$$

for $t>s$. From the Spectral theorem we can then conclude that

$$(b_2-s1)\left(p_t^- - p_s^+\right)a_{i3}^*\psi=0.\tag{2.23}$$

Thus,

$$\left(p_t^- - p_s^+\right)a_{i3}^*\psi=0\tag{2.24}$$

for every $a_{i3}^*\psi\in D(b_2)$. But

$$\lim_{t\to\infty}\left(p_t^- - p_s^+\right)a_{i3}^*\psi=\left(1-p_s^+\right)a_{i3}^*\psi=a_{i3}^*\psi,\tag{2.25}$$

and $(p_t^- - p_s^+)a_{i3}^*\psi\in D(b_2)$ for every $\psi\in H$ and every $t>s$. Hence, $D(b_2)$ is dense in $\operatorname{Im}(a_{i3}^*)$. Since $D(b_2)\cap\operatorname{Im}(a_{i3}^*)=\{0\}$, we have $a_{i3}^*\equiv 0$ for $i=1,2,3$. Thus,

$$a^{1/2}=\begin{pmatrix}a_{11}&a_{12}&0\\ a_{12}^*&a_{22}&0\\ 0&0&0\end{pmatrix},\tag{2.26}$$

so

$$a=\left(a^{1/2}\right)^2\leq p_s^+.\tag{2.27}$$

The other statement in the lemma follows from an analogous argument. □

We remark that in the original paper on spectral scales [1, Lemma 1.2, pages 262, 263], the above conclusion was obtained with a little less work, since, in that situation, b was bounded and so we did not have to worry about the domain of b. The proofs of the next several results, however, are virtually identical to the original proofs. In other words, much of the hard work has now been done.

Lemma 2.11. *Fix $s \in \mathbb{R}$, $c \in [p_s^-, p_s^+]$, and $a \in M_1^+$. Suppose that $\tau(a) = \tau(c)$. Then the following hold:*

(1) $\tau(ba) \geq \tau(bc)$.

(2) *If $\tau(ba) = \tau(bc)$, then $a \in [p_s^-, p_s^+]$.*

(3) *If $c = p_s^{\pm}$, then $\tau(ba) = \tau(bc) \Rightarrow a = c$.*

Proof. Note that $b(1-c) \geq s(1-c)$ and $bc(1-a) \leq sc(1-a)$ from Lemma 2.6. Hence

$$\tau(b(1-c)a) = \tau\left(a^{1/2}b(1-c)a^{1/2}\right) \geq \tau\left(a^{1/2}s(1-c)a^{1/2}\right) = \tau(s(1-c)a), \tag{2.28}$$

since τ is faithful. Similarly, $\tau(bc(1-a)) \geq \tau(sc(1-a))$.

(1) We compute

$$\begin{aligned} \tau(ba) - \tau(bc) &= \tau(ba) - \tau(bca) - \tau(bc) + \tau(bca) \\ &= \tau(b(1-c)a) - \tau(bc(1-a)) \\ &\geq \tau(s(1-c)a) - \tau(sc(1-a)) \\ &= \tau(sa) - \tau(sc) = 0, \end{aligned} \tag{2.29}$$

and so $\tau(ba) \geq \tau(bc)$.

(2) Suppose that $\tau(ba) = \tau(bc)$. Then

$$\tau((1-c)a) = \tau(a-ca) = \tau(c-ca) = \tau(c(1-a)). \tag{2.30}$$

Similarly,

$$\tau(b(1-c)a) = \tau(bc(1-a)). \tag{2.31}$$

Therefore

$$\begin{aligned} \tau(bc(1-a)) = \tau(b(1-c)a) &\geq s\tau((1-c)a) \\ &= s\tau(c(1-a)) \geq \tau(bc(1-a)), \end{aligned} \tag{2.32}$$

and hence equality holds throughout. Thus

$$\tau(b(1-c)a) = s\tau((1-c)a), \tag{2.33}$$

$$\tau(bc(1-a)) = s\tau(c(1-a)). \tag{2.34}$$

From (2.33),

$$\tau\left(a^{1/2}(b-s1)(1-c)a^{1/2}\right)=0, \tag{2.35}$$

while from (2.34),

$$\tau\left((1-a)^{1/2}(b-s1)c(1-a)^{1/2}\right)=0. \tag{2.36}$$

Since τ is faithful and the arguments are positive, the arguments are in fact equal to zero. By Lemma 2.10, $p_s^- \le a \le p_s^+$.

(3) Suppose that $c=p_s^{\pm}$ and $\tau(ba)=\tau(bc)$. Then $\tau(a)=\tau(p_s^{\pm})$. Since a is comparable to $p_s^{\pm}$, and τ is faithful, $a=p_s^{\pm}$. □

We next state a theorem proved by Akemann and Pedersen [10, Theorem 2.2].

Theorem 2.12. *If M and N are von Neumann algebras, Ψ is a normal linear map from M to N, and F a face of $\Psi(M_1^+)$, then there are unique projections p and q in M with $p\le q$ such that $\Psi^{-1}(F)\cap M_1^+=[p,q]$ and $F=\Psi([p,q])$.*

The following results are generalizations of the main theorems for the $n=1$ case from the first paper on spectral scales [1, Theorems 1.5–1.7, pages 266–274]. We will introduce some new notation at this time.

Notation 2. Recall that we are assuming that $M\subset B(H)$ is a finite von Neumann algebra equipped with a faithful, normal, tracial state τ. The operator b is unbounded and self-adjoint on H affiliated with M obtained from a linear functional $g\in M_*$ (i.e., $g(a)=\tau(ba)$ for each $a\in M$). $B:=\{(\tau(a),\tau(ba))\mid a\in M_1^+\}$ is the spectral scale of b. The *lower boundary* of B is given by

$$\{(x,y)\in B\mid (x,y')\in B\Longrightarrow y'\ge y\}. \tag{2.37}$$

The *upper boundary* of B is given by

$$\{(x,y)\in B\mid (x,y')\in B\Longrightarrow y'\le y\}. \tag{2.38}$$

The endpoints of the lower boundary are $f(0)$ and $f(1)$.

Let $\sigma(b)$ denote the spectrum of b, and let σ_p be the point spectrum of b. Let f be the function on $[0,1]$ whose graph is the lower boundary. For $s,\alpha\in\mathbb{R}$, let

$$L(s,\alpha)=\{(x_0,x_1)\mid x_1=sx_0+\alpha\}. \tag{2.39}$$

Let $L^{\uparrow}(s,\alpha)$ be the positive half-plane determined by $L(s,\alpha)$.

Our next result describes the faces of the lower boundary of B. We do not include the endpoints at this time.

Theorem 2.13. (1) *The zero-dimensional faces in the lower boundary of B are precisely the points of the form $\Psi(p_s^{\pm})$ for $s \in \sigma(b)$. Also,*

$$\Psi^{-1}\left(\Psi\left(p_s^{\pm}\right)\right) \cap M_1^{+} = \left\{p_s^{\pm}\right\}. \tag{2.40}$$

(2) *The one-dimensional faces in the lower boundary of B are the sets of the form $F = \Psi([p_s^{-}, p_s^{+}])$ for $s \in \sigma(b)$. For each face F,*

$$\Psi^{-1}(F) \cap M_1^{+} = \left[p_s^{-}, p_s^{+}\right]. \tag{2.41}$$

The slope of F is s.

Proof. We have the following steps.

Step 1. We show that $\Psi(p_s^{\pm})$ are zero-dimensional faces.

Fix $s \in \sigma(b)$. If $a \in M_1^{+}$ and $\tau(a) = \tau(p_s^{\pm})$, then $\tau(ba) \geq \tau(bp_s^{\pm})$ by Lemma 2.11. Hence,

$$\Psi\left(p_s^{\pm}\right) = \left(\tau\left(p_s^{\pm}\right), \tau\left(bp_s^{\pm}\right)\right) = \left(\tau(a), \tau\left(bp_s^{\pm}\right)\right) \tag{2.42}$$

is on the lower boundary of B.

But

$$\begin{gathered}\Psi\left(p_s^{\pm}\right) = \Psi(a) \Longleftrightarrow \tau\left(p_s^{\pm}\right) = \tau(a),\\ \tau\left(bp_s^{\pm}\right) = \tau(ba) \Longleftrightarrow p_s^{\pm} = a,\end{gathered} \tag{2.43}$$

again by Lemma 2.11. Hence,

$$\Psi^{-1}\left(\Psi\left(p_s^{\pm}\right)\right) \cap M_1^{+} = \left\{p_s^{\pm}\right\}. \tag{2.44}$$

We now show that $\Psi(p_s^{\pm})$ is an extreme point of B. Suppose that

$$\Psi\left(p_s^{\pm}\right) = \lambda\Psi(a_1) + (1-\lambda)\Psi(a_2) = \Psi(\lambda a_1 + (1-\lambda)a_2) \tag{2.45}$$

for $a_1, a_2 \in M_1^{+},\ 0 < \lambda < 1$. Since

$$\begin{gathered}\Psi^{-1}\left(\Psi\left(p_s^{\pm}\right)\right) \cap M_1^{+} = \left\{p_s^{\pm}\right\},\\ p_s^{\pm} = \lambda a_1 + (1-\lambda)a_2.\end{gathered} \tag{2.46}$$

Since projections are extreme points in M_1^{+}, $a_1 = a_2 = p_s^{\pm}$.

Step 2. For $s \in \sigma_p(b)$, $\Psi([p_s^{-}, p_s^{+}])$ are faces of B.

Fix $s \in \sigma_p(b)$ so that

$$p_s^{-} < p_s^{+} \Longrightarrow \Psi\left(p_s^{-}\right) \neq \Psi\left(p_s^{+}\right). \tag{2.47}$$

Then

$$b(p_s^+ - p_s^-) = s(p_s^+ - p_s^-). \tag{2.48}$$

If $c \in [p_s^-, p_s^+]$, then $\tau(bc) \leq \tau(ba)$ for each a such that $\tau(a) = \tau(c)$. Hence $\Psi(c)$ is on the graph of f which is the lower boundary curve.

Write $p_\lambda = \lambda p_s^- + (1-\lambda)p_s^+$ for all $\lambda \in (0,1)$. Then $p_\lambda \in [p_s^-, p_s^+]$, and $\Psi(p_\lambda)$ is a typical point on the line segment connecting $\Psi(p_s^-)$ and $\Psi(p_s^+)$. Hence graph(f) contains this line segment. The slope is

$$\frac{\tau(bp_s^+) - \tau(bp_s^-)}{\tau(p_s^+) - \tau(p_s^-)} = \frac{\tau(b(p_s^+ - bp_s^-))}{\tau(p_s^+ - p_s^-)} = \frac{\tau(s(p_s^+ - p_s^-))}{\tau(p_s^+ - p_s^-)} = s. \tag{2.49}$$

Let F denote the line segment in the graph of f that contains $\Psi([p_s^-, p_s^+])$ and consider the endpoints of F. By Theorem 2.12, there are projections $p < q$ such that $[p,q] = \Psi^{-1}(F) \cap M_1^+$, and hence

$$p \leq p_s^- < p_s^+ \leq q. \tag{2.50}$$

If $p < p_s^-$, then, since $p_s^- < p_s^+ \leq q$, $\Psi(p_s^-)$ is in the interior of F, contradicting Step 1. Hence $p = p_s^-$ and similarly $q = p_s^+$. Thus, $\Psi([p_s^-, p_s^+])$ is a line segment in graph(f) with slope s and

$$\Psi^{-1}([p_s^-, p_s^+]) \cap M_1^+ = [p_s^-, p_s^+]. \tag{2.51}$$

Step 3. We show that we have accounted for all of the graph of f, except possibly the endpoints.

Fix a point

$$(x_0, x_1) = \Psi(a) = (\tau(a), \tau(ba)), \tag{2.52}$$

with $0 < x_0 < 1$, and assume that $\Psi(a) \neq \Psi(p_s^\pm)$ for every $s \in \sigma(b)$. Write

$$\begin{aligned} r_1 &= \sup\{s \in \sigma(b) \mid \tau(p_s^-) \leq x_0\}, \\ r_2 &= \inf\{s \in \sigma(b) \mid \tau(p_s^+) \geq x_0\}. \end{aligned} \tag{2.53}$$

We would like to show that $r_1 = r_2$. Since $\sigma(b)$ is closed, $r_i \in \sigma(b)$ for $i = 1, 2$.

By definition,

$$(p_{r_1}^-) \leq x_0 \leq \tau(p_{r_2}^+). \tag{2.54}$$

Since $(x_0, x_1) \neq \Psi(p_s^\pm)$ for $s \in \sigma(b)$,

$$\tau(p_{r_1}^-) < x_0 < \tau(p_{r_2}^+). \tag{2.55}$$

If $s < r_2$, $s \in \sigma(b)$, then by definition

$$\tau(p_s^+) < x_0 < \tau(p_{r_2}^+). \tag{2.56}$$

Similarly, $r_1 < s$ and

$$s \in \sigma(b) \Longrightarrow \tau(p_{r_1}^-) < x_0 < \tau(p_s^-). \tag{2.57}$$

Suppose that $r_1 < r_2$. Then $\tau(p_{r_1}^+) \leq x_0 \leq \tau(p_{r_2}^-)$. If

$$r_1 < s < r_2 \tag{2.58}$$

for some $s \in \sigma(b)$, we would have

$$\tau(p_s^+) < x_0 < \tau(p_s^-), \tag{2.59}$$

which is clearly false. Hence,

$$(r_1, r_2) \cap \sigma(b) = \emptyset. \tag{2.60}$$

But then $p_{r_1}^+ = p_{r_2}^-$ and $\tau(p_{r_1}^+) = \tau(p_{r_2}^-)$, which again is a contradiction. Hence,

$$r_1 = r_2 := r. \tag{2.61}$$

Since

$$\tau(p_r^-) < x_0 < \tau(p_r^+), \tag{2.62}$$

$p_r^- < p_r^+$ so r is an eigenvalue of b. From Step 2, $\Psi([p_r^-, p_r^+])$ is a line segment in graph(f). Hence (x_0, x_1) is on the interior of that line segment. □

Corollary 2.14. (1) *The extreme points on the upper boundary (excluding the endpoints) are precisely the points of the form $F = \Psi(1 - p_s^{\pm})$ for $s \in \sigma(b)$ and*

$$\Psi^{-1}(1 - p_s^{\pm}) \cap M_1^+ = \{1 - p_s^{\pm}\}. \tag{2.63}$$

(2) *The line segments on the upper boundary are precisely the sets of the form $F = \Psi[1-p_s^+, 1-p_s^-]$, for $s \in \sigma_p(b)$. The slope of F is s and*

$$\Psi^{-1}(F) \cap M_1^+ = [1 - p_s^+, 1 - p_s^-]. \tag{2.64}$$

Proof. This result is a direct consequence of applying Proposition 2.3 to Theorem 2.13. □

Let $s_{\min}$ be the left endpoint of $\sigma(b)$. Note that $s_{\min}$ may be $-\infty$. Let $s_{\max}$ be the right endpoint of $\sigma(b)$. Note that $s_{\max}$ may be ∞.

Proposition 2.15. *If $s > s_{\min}$, then the left derivative of the lower boundary function f at $\tau(p_s^-)$ exists and is given by the formula*

$$f'_-(\tau(p_s^-)) = \sup\{s' \in \sigma(b) \mid s' < s\}. \tag{2.65}$$

If $s < s_{\max}$, then the right derivative of f at $\tau(p_s^+)$ exists and is given by the formula

$$f'_+(\tau(p_s^+)) = \inf\{s' \in \sigma(b) \mid s' > s\}. \tag{2.66}$$

Proof. Since B is convex, f is a convex function, and so the left and right derivatives exist. Fix $s \in \mathbb{R}$ with $s > s_{\min}$. Then $\sigma(b) \cap (-\infty, s) \neq \emptyset$. Define $r = \sup\{s' \in \sigma(b) \mid s' < s\}$. Since $\sigma(b)$ is closed, $r \in \sigma(b)$.

Case 1 ($(r-\epsilon, r) \cap \sigma(b) = \emptyset$ for some $\epsilon > 0$). If $r = s$, $(r-\epsilon, r) \cap \sigma(b) = (s-\epsilon, s) = \emptyset$ which contradicts the choice of r. Thus $r < s$, and hence r is an isolated point in the spectrum, that is, r is an eigenvalue of b. Moreover, $p_r^+ = p_s^-$ and so $\Psi(p_s^-) = \Psi(p_r^+)$ is the right-hand endpoint of a line segment in graph(f) with slope r by Theorem 2.13. Hence $f'_-(\tau(p_s^-)) = r$.

Case 2 ($(r-\epsilon, r) \cap \sigma(b) \neq \emptyset$ for every $\epsilon > 0$). Choose $r_n \in \sigma(b)$ such that $r_n \uparrow r$ and $r_n \neq r$ for $n \in \mathbb{N}$. Then $\Psi(p_{r_n}^-)$ is on the graph of f. Furthermore, $p_{r_n}^- \to p_r^-$ in the weak-$*$ topology. Since τ is normal, $\tau(p_{r_n}^-) \to \tau(p_r^-)$. Since τ is faithful, $\tau(p_{r_n}^-) \neq \tau(p_r^-)$. We have

$$r_n(p_r^- - p_{r_n}^-) \leq b(p_r^- - p_{r_n}^-) \leq r(p_r^- - p_{r_n}^-) \tag{2.67}$$

for every n. Hence,

$$\tau(r_n(p_r^- - p_{r_n}^-)) \leq \tau(b(p_r^- - p_{r_n}^-)) \leq \tau(r(p_r^- - p_{r_n}^-)) \tag{2.68}$$

for every n. Thus,

$$r_n \leq \frac{f(\tau(p_r^-)) - f(\tau(p_{r_n}^-))}{f(p_r^-) - f(p_{r_n}^-)} \leq r. \tag{2.69}$$

Letting $n \to \infty$ gives the desired result.

The statement regarding right derivatives is proved in a similar way. □

Proposition 2.16. *The corners of f are in one-to-one correspondence with the gaps of $\sigma(b)$, that is, the maximal bounded intervals in the complement of the spectrum. (One is not currently concerned with unbounded maximal intervals in the complement of the spectrum, that is, those which take the form $(-\infty, s)$ or (s, ∞).)*

Proof. Let (r,t) be an interior gap of the spectrum. Then for every $s_1, s_2 \in (r,t)$ we have $p_{s_1}^+ = p_{s_2}^-$. Fix $s \in (r,t)$. Then

$$\begin{aligned} f'_-(\tau(p_s^-)) &= \sup\{s' \in \sigma(b) \mid s' < s\} < \inf\{r' \in \sigma(b) \mid r' > s\} \\ &= f'_+(\tau(p_s^+)) = f'_+(\tau(p_s^-)). \end{aligned} \tag{2.70}$$

Hence, f is not differentiable at $\tau(p_s^-)$, and so a gap in the spectrum corresponds to a corner. Conversely, we have already seen that f is differentiable at $p_s^\pm$ for $s \in \sigma(b)$. □

Proposition 2.17. *For each* $s \in \mathbb{R}$,

$$\tau((b - s1)p_s^-) = \tau((b - s1)p_s^+). \tag{2.71}$$

The line $L(s,\alpha)$ *is a line of support for* B *such that*

$$B \subset L^\uparrow(s,\alpha) \Longleftrightarrow \alpha = \tau((b - s1)p_s^\pm). \tag{2.72}$$

In this case, $L(s,\alpha)$ *passes through* $\Psi(p_s^\pm)$. *Moreover, one has*

$$\begin{aligned} \Psi^{-1}(L(s,\alpha)) \cap M_1^+ &= [p_s^-, p_s^+], \\ \Psi([p_s^-, p_s^+]) &= L(s,\alpha) \cap B. \end{aligned} \tag{2.73}$$

Proof. Fix $s \in \mathbb{R}$. If s is an eigenvalue, then

$$b(p_s^+ - p_s^-) = s(p_s^+ - p_s^-). \tag{2.74}$$

Otherwise, $p_s^- = p_s^+$. Either way,

$$(b - s1)(p_s^+ - p_s^-) = 0, \tag{2.75}$$

and so

$$\tau(b - s1)p_s^+ = \tau(b - s1)p_s^-. \tag{2.76}$$

Let $\alpha = \tau(b - s1)p_s^\pm$. Then

$$-s\tau(p_s^\pm) + \tau(bp_s^\pm) = \alpha, \tag{2.77}$$

so $\Psi(p_s^\pm)$ lies in $L(s,\alpha)$. We now wish to show that $L(s,\alpha)$ is a line of support for B. There are several cases to consider.

Case 1 ($s \in \sigma_p(b)$). In this situation, $\Psi(p_s^-)$ and $\Psi(p_s^+)$ are endpoints of a line segment in graph(f) whose slope is s. $L(s,\alpha)$ passes through both points and has slope s. Thus, $L(s,\alpha)$ contains this line segment and is tangent to f. Hence $B \subset L^\uparrow(s,\alpha)$.

Case 2 ($s \in \sigma(b) \setminus \sigma_p(b)$). Note that s is not an isolated point in $\sigma(b)$. Moreover, $p_s^- = p_s^+ = p_s$. At least one of the one-sided derivatives of f takes the value s at $\tau(p_s)$. Hence, B admits a line of support at $\Psi(p_s)$ with slope s. As with Case 1, the line is $L(s,\alpha)$ and $B \subset L^{\uparrow}(s,\alpha)$.

Case 3 ((s_1, s_2) is an interior gap in the spectrum). In this case $p_{s_1}^+ = p_{s_2}^-$. Let $\alpha_1 = \tau((b-s_1)p_{s_1}^+)$ and $\alpha_2 = \tau((b-s_2)p_{s_2}^-)$. Then $L(s_1,\alpha_1)$ and $L(s_2,\alpha_2)$ are lines of support passing through $\Psi(p_{s_1}^+)$ whose slope lies between s_1 and s_2. If L is a line of support for B whose slope lies between s_1 and s_2, then $L^{\uparrow} \supset B$. But $L(s,\alpha)$ is such a line for $s \in (s_1,s_2)$. Hence, the statement is true for any $s \in (s_1,s_2)$.

Case 4 ($s < s_{\min}$ or $s > s_{\max}$). Since b is unbounded, at least one of $s_{\min}$ and $s_{\max}$ has infinite magnitude. Suppose that $s_{\min}$ is finite (and so $s_{\max}$ must be infinite). Then $s_{\min} \in \sigma(b)$. Moreover, $p_{s_{\min}}^- = 0$. $L(s_{\min},0)$ is a line of support for B at $\Psi(0)$ and $b \subset L^{\uparrow}(s_{\min},0)$ by Case 1. Suppose $s < s_{\min}$. Then $L(s,0)$ is also a line of support for B at $\Psi(0)$ and $B \subset L^{\uparrow}(s,0)$.

The case for $s > s_{\max}$ is dealt with similarly. Hence, for every $s \in \mathbb{R}$, $L(s,\alpha)$ is a line of support for B and $B \subset L^{\uparrow}(s,\alpha)$.

Conversely, for fixed s, the lines $L(s,\beta)$ are all parallel as β varies over $\mathbb{R}$. Hence, there exists a unique β_0 for which $L(s,\beta_0)$ is a line of support and $B \subset L^{\uparrow}(s,\beta_0)$. But $L(s,\alpha)$ has these properties and hence $\alpha = \beta_0$.

For the last statement, consider $F = L(s,\alpha) \cap B$. Then F is a face of B. Hence, F is an extreme point or a line segment on graph(f).

If F is an extreme point, then $F = \{\Psi(p_s^{\pm})\}$, then $s \in \sigma(b)$. Since F is an extreme point, then $F = \{\varphi(p_s^-)\} = \{\Psi(p_s^+)\}$, and so

$$\Psi^{-1}(F) \cap M_1^+ = \Psi^{-1}(L(s,\alpha)) \cap M_1^+ = \{p_s^{\pm}\} = [p_s^-, p_s^+]. \tag{2.78}$$

Similarly, if F is a line segment, then $F = \Psi([p_s^-, p_s^+])$ for some $s \in \sigma_p(b)$, and so

$$\Psi^{-1}(F) \cap M_1^+ = \Psi^{-1}(L(s,\alpha)) \cap M_1^+ = [p_s^-, p_s^+]. \tag{2.79}$$

□

From the above results, if $s_{\min} = -\infty$, then the right derivative of $f(x)$ approaches $-\infty$ as $x \downarrow 0$. If in addition $s_{\max} = \infty$, then the left derivative of $f(x)$ approaches ∞ as $x \uparrow 1$. By Proposition 2.3, the graph of the upper boundary curve of B is vertical at $x = 0$. Hence, the only line of support at $\Psi(0)$ is vertical. Similarly, the only line of support at $\Psi(1)$ is vertical. Therefore, if both $s_{\min}$ and $s_{\max}$ are nonreal, then $\Psi(0)$ and $\Psi(1)$ are not corners of B.

Conversely, if one of $s_{\min}$ and $s_{\max}$ is finite, then $\Psi(0)$ and $\Psi(1)$ are corners of B.

Here the bounded and unbounded spectral scale theories do not coincide, since, in the bounded situation, $\Psi(0)$ and $\Psi(1)$ are always corners.

In both situations, we can read spectral data of the lower boundary curve as follows

(1) 1-dimensional faces correspond to eigenvalues of b. The slope of each face is the corresponding eigenvalue.

(2) Other places where the lower boundary curve is differentiable correspond to elements of the continuous spectrum. The slope at such a given point is the corresponding element of the spectrum.

(3) Corners on the lower boundary curve correspond to gaps in the spectrum.

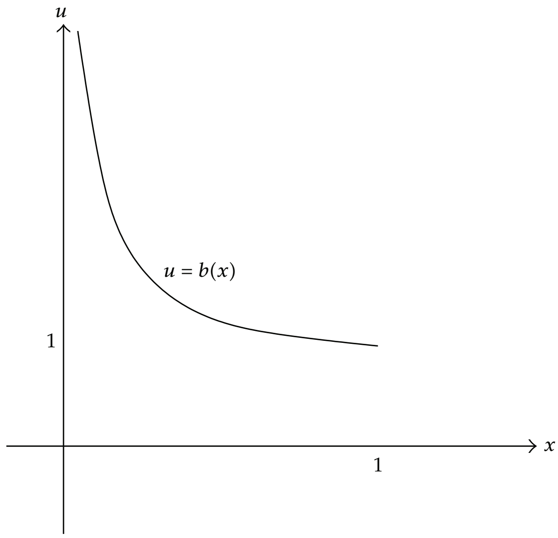

Figure 1: Graph of b for Example 2.18.

We now exhibit two examples. In both examples, we will take $H = L^2[0,1]$ and $M = L^\infty[0,1]$. The trace τ is integration with respect to Lebesgue measure and $a(\psi)(x) := a(x)\psi(x)$ for $a \in M$, $\psi \in H$, and $x \in [0,1]$. Then $M_* = L^1[0,1]$ and τ makes sense on M_*.

Example 2.18. Define $b(x) = \sqrt{1/x}$ almost everywhere. Then $b \in M_*$ is densely defined and self-adjoint on H. It turns out that the equation of the lower boundary function is $w = f(y) = 2 - 2\sqrt{(1-y)}$. This was obtained by integrating b multiplied by appropriate characteristic functions. Observe that $1 \le f'(y) \le \infty$ and $\tau(b) = 2$. Hence the center of B is $P = (0.5, 1)$ and we get Figures 1 and 2.

Example 2.19. Define $b(x) = \sqrt{1/x} - \sqrt{2}$ for $0 < x \le 0.5$ and $b(x) = \sqrt{2} - \sqrt{1/(1-x)}$ for $0.5 < x < 1$. Then the lower boundary curve for $0 \le y \le 0.5$ is given by $w = f(y) = -2\sqrt{y} + \sqrt{2}y$. And b was chosen so that we would get a spectral scale that is invariant under the reflection $y = 0.5$. Note that $f'(0) = -\infty$, $f'(0.5) = 0$, $f(0.5) = -\sqrt{2} + 1/\sqrt{2}$, $\tau(b) = 0$, and $f'(1) = \infty$. The resulting pictures are shown in Figures 3 and 4.

We now examine a question posed to the author by Crandall. We start by stating the necessary properties that $U \subset \mathbb{R}^2$ must have in order for it to be a spectral scale for an operator functional.

Definition 2.20. A *prespectral scale* is a set U contained in $\mathbb{R}^2$ which satisfies the following properties.

(i) U is compact and convex.

(ii) $(0,0) \in U$ and there are no other points of the form $(0, y)$ in U.

(iii) $U \subset \{(x,y) \mid 0 \le x \le 1\}$.

Further, if $u = \inf\{y \mid (1/2, y) \in U\}$ and $v = \sup\{y \mid (1/2, y) \in U\}$, then the following are given.

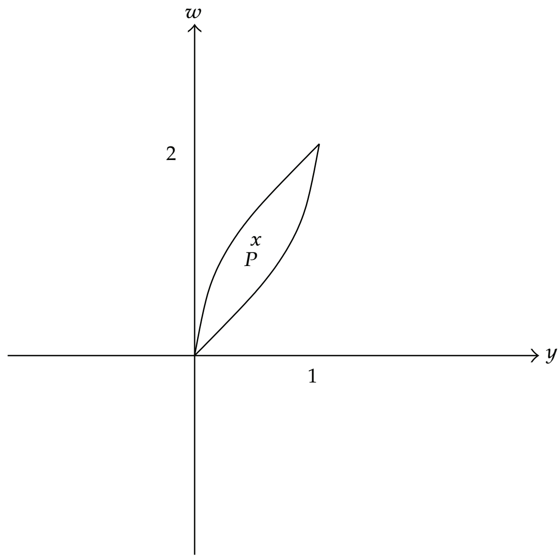

Figure 2: Spectral scale for b in Example 2.18.

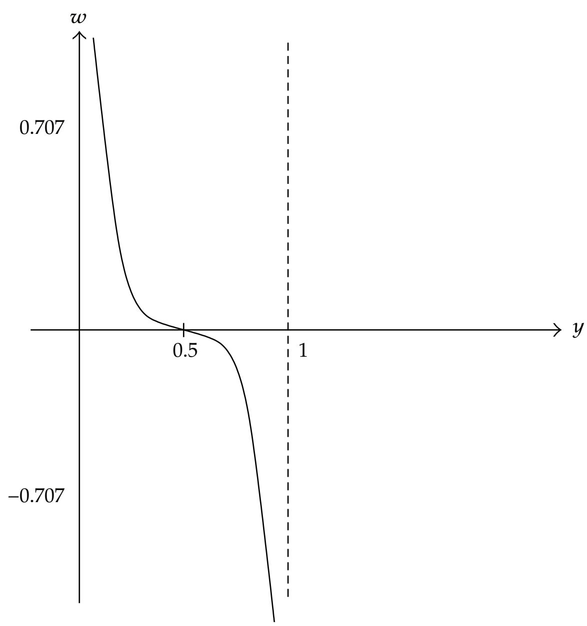

Figure 3: Graph of b for Example 2.19.

(iv) U is invariant under the reflection $(a,b) \mapsto (1-a, u+v-b)$.

(v) The set $G := \{(x,z) \in U \mid (x,y) \in U \Rightarrow y \geq z\}$ is the graph of a function f on $[0,1]$, which we will call the *lower boundary curve* of U.

Lemma 2.21. *Let U be a prespectral scale with lower boundary curve f. Then f is a continuous convex function.*

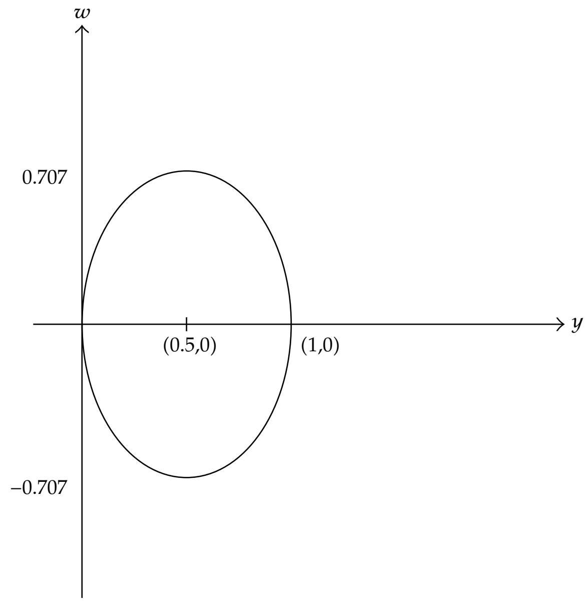

Figure 4: Spectral scale for b in Example 2.19.

Proof. Since U is closed, $G \subset U$. Let $0 < a < b < 1$. Since U is convex, the line segment

$$\left\{ w \in \mathbb{R}^2 \mid w = t\big(a, f(a)\big) + (1-t)\big(b, f(b)\big) \right\} \tag{2.80}$$

is a subset of U. Now

$$\big(ta + (1-t)b, f(ta + (1-t)b)\big) \in G \subset U. \tag{2.81}$$

From the definition of G,

$$tf(a) + (1-t)f(b) \geq f(ta + (1-t)b). \tag{2.82}$$

Hence, f is convex on $(0,1)$, and therefore continuous on $(0,1)$ [11, pages 61, 62]. Then $\lim_{x\downarrow 0} f(x)$ and $\lim_{x\uparrow 1} f(x)$ exist as extended real numbers [12, page 116]. Since U is compact, then $\lim_{x\downarrow 0} f(x)$ is finite and $\lim_{x\downarrow 0}(x, f(x)) \in U$. Therefore, $(0, \lim_{x\downarrow 0} f(x)) \in U$. By (ii) in Definition 2.20, $\lim_{x\downarrow 0} f(x) = 0 = f(0)$. Applying (iv) from Definition 2.20, $\lim_{x\uparrow 1} f(x) = f(1)$. Thus, f is continuous and convex on $[0,1]$. □

Since f is convex on $[0,1]$, the left and right derivatives of f exist for all $x \in [0,1]$ as extended real numbers, and f is differentiable almost everywhere [12, pages 113, 114].

It is easy to see that a spectral scale must be a prespectral scale: condition (i) is noted on page 3 of this paper, condition (ii) follows from Definition 2.1, condition (iii) follows from the fact that τ is a state, condition (iv) follows from Proposition 2.3, and condition (v) follows from the definition of the lower boundary (Notation 2.14).

Crandall asked whether a prespectral scale is automatically a spectral scale. In the next theorem, we show that the answer is yes.

Theorem 2.22. *Let $M = L^\infty[0,1]$ and let τ be the Lebesgue integral on $[0,1]$. Given a prespectral scale U, there exists $g \in L^1[0,1] = M_*$ self-adjoint such that $U = B(g)$.*

Proof. From the symmetry required for U (condition (v)) it is sufficient to examine the lower boundary curve, f, of U. The function f has the following properties (as noted in Lemma 2.21):

(i) $f : [0,1] \to \mathbb{R}$ is continuous and convex,

(ii) $f(0) = 0$.

Let us denote $f'_R(t)$ as the right derivative of f at $t \in [0,1)$. Similarly, denote $f'_L(t)$ as the left derivative of f at $t \in (0,1]$. Let $g(t) = f'_R(t)$ on [0,1) and $g(1) = f'_L(1)$. Since f is convex, then g is nondecreasing. Hence, g has at most a countable number of discontinuities. Since f is convex, then f is of bounded variation. By Exercise 14.H in [13, page 244], f is absolutely continuous. By Theorem 7.20 in [11, page 148], $g \in L^1[0,1] = M_*$, and the fundamental theorem of calculus holds. Let $a \in M_1^+$, with $\int_0^1 a(s)ds = t \in [0,1]$. Since g is increasing,

$$\int_0^1 g(s)a(s)ds \geq \int_0^t g(s)ds = f(t) - f(0) = f(t). \tag{2.83}$$

Hence, f is the lower boundary curve of $B(g)$. □

3. The Geometry of Spectral Scales in Higher Dimensions

This section is devoted to further generalizations of results from the original paper on spectral scales by Akemann et al. [1, Section 2, pages 276–280]. Often, with some modifications, the proofs are the same as in the original paper. Recall that H is a Hilbert space, $M \subset B(H)$ is a finite von Neumann algebra equipped with faithful, normal, tracial state, τ.

Notation 3. In Section 2 of this paper, we considered $g \in M_*$. We now consider an n-tuple of self-adjoint linear functionals, $(g_1, \ldots, g_n) \in M_*^n$. Let $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n \setminus \{0\}$. Let $g_\mathbf{t} = \sum_{i=1}^n t_i g_i$. Then $g_\mathbf{t}$ is also self-adjoint since each g_i is self-adjoint and each t_i is real. For each g_k $(k = i, \mathbf{t})$ there is an associated self-adjoint, densely defined operator in H, b_k, and for every $a \in M$ we have $g_k(a) = \tau(b_k a)$.

Definition 3.1. Let $\Psi(a) = (\tau(a), g_1(a), \ldots, g_n(a))$ for every $a \in M$. Let $\Psi_\mathbf{t}(a) = (\tau(a), g_\mathbf{t}(a))$ for each $a \in M$. Then $B := \Psi(M_1^+)$ is the *spectral scale* of $(b_1, \ldots, b_n)$ with respect to τ and $B_\mathbf{t} := \Psi_\mathbf{t}(M_1^+)$ is the *spectral scale* of $g_\mathbf{t}$ with respect to τ.

Essentially the motivation for the introduction of $g_\mathbf{t}$ is it allows us to reduce the n-dimensional case to the 1-dimensional case by studying "2-dimensional cross-sections" of the spectral scale. We note that $b_\mathbf{t} \neq \sum_{i=1}^n t_i b_i$. Indeed, the right hand side may have trivial domain. However, as we will see, equality "almost" holds; that is, equality holds in trace.

Define $\pi_\mathbf{t}(x_0, x_1, \ldots, x_n) = (x_0, \sum_{i=1}^n t_i x_i)$, where $x_0, x_i \in \mathbb{R}$.

Proposition 3.2. *The equality $\Psi_\mathbf{t} = \pi_\mathbf{t} \circ \Psi$ holds.*

Proof. For $a \in M$ we have

$$\begin{aligned}
\pi_{\mathbf{t}}(\Psi(a)) &= \pi_{\mathbf{t}}(\tau(a), \tau(b_1 a) \cdots \tau(b_n a)) = \pi_{\mathbf{t}}\big(\tau(a), g_1(a), \ldots, g_n(a)\big) \\
&= \left(\tau(a), \sum_{i=1}^{n} t_i g_i(a)\right) = \big(\tau(a), g_{\mathbf{t}}(a)\big) = (\tau(a), \tau(b_{\mathbf{t}} a)) = \Psi_{\mathbf{t}}(a).
\end{aligned} \tag{3.1}$$

□

Corollary 3.3. *As a consequence of this calculation, $B_{\mathbf{t}} = \pi_{\mathbf{t}}(B)$.*

We next introduce some additional notation.

Notation 4. Let $p^{+}_{\mathbf{t},s}$ be the spectral projection of $b_{\mathbf{t}}$ determined by $(-\infty, s]$.
Let $p^{-}_{\mathbf{t},s}$ be the spectral projection of $b_{\mathbf{t}}$ determined by $(-\infty, s)$.
Let $P(\mathbf{t}, s, \alpha) = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid -s x_0 + \sum_{i=1}^{n} t_i x_i = \alpha\}$.
Let $P^{\downarrow}(\mathbf{t}, s, \alpha) = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid -s x_0 + \sum_{i=1}^{n} t_i x_i \leq \alpha\}$.
Let $P^{\uparrow}(\mathbf{t}, s, \alpha) = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid -s x_0 + \sum_{i=1}^{n} t_i x_i \geq \alpha\}$.

The following results discuss the geometrical properties of B.

Proposition 3.4. *If $\mathbf{x}$ is an extreme point of B, the n there exists a projection $p \in M$, such that $\Psi(p) = \mathbf{x}$ and $\Psi^{-1}(\mathbf{x}) \cap M_1^+ = \{p\}$. Further, $\Psi(p^{\pm}_{\mathbf{t},s})$ is an extreme point of B.*

Proof. Fix an extreme point $\mathbf{x} \in B$. Since $\{\mathbf{x}\}$ is a face of B, by Theorem 2.12 there are unique projections, $p \leq q$ in M, such that $\Psi^{-1}(\mathbf{x}) \cap M_1^+ = [p, q]$. Thus, $\Psi(p) = \Psi(q) = \mathbf{x}$ and so $\tau(p) = \tau(q)$. Since τ is faithful, we have that $p = q$, and so $\Psi^{-1}(\mathbf{x}) \cap M_1^+ = \{p\}$.

Next, suppose that $\Psi(p^{\pm}_{\mathbf{t},s}) = \Psi(\lambda a_1 + (1-\lambda) a_2)$ for some $\lambda \in (0,1)$, $a_1, a_2 \in M_1^+$. Then

$$\Psi_{\mathbf{t}}\big(p^{\pm}_{\mathbf{t},s}\big) = \pi_{\mathbf{t}}\big(\Psi\big(p^{\pm}_{\mathbf{t},s}\big)\big) = \pi_{\mathbf{t}}(\Psi(\lambda a_1 + (1-\lambda) a_2)) = \Psi_{\mathbf{t}}(\lambda a_1 + (1-\lambda) a_2). \tag{3.2}$$

Thus,

$$\begin{aligned}
\tau\big(b_{\mathbf{t}}\big(p^{\pm}_{\mathbf{t},s}\big)\big) &= \tau(b_{\mathbf{t}}(\lambda a_1 + (1-\lambda) a_2)), \\
\tau\big(p^{\pm}_{\mathbf{t},s}\big) &= \tau(\lambda a_1 + (1-\lambda) a_2).
\end{aligned} \tag{3.3}$$

By Lemma 2.11, $p^{\pm}_{\mathbf{t},s} = \lambda a_1 + (1-\lambda) a_2$. Since projections are extreme points in M_1^+, then $a_1 = a_2 = p^{\pm}_{\mathbf{t},s}$, and hence $\Psi(p^{\pm}_{\mathbf{t},s})$ is an extreme point of B. □

Proposition 3.5. *Suppose that $\tau((b_t - s1) p^{+}_{\mathbf{t},s}) = \tau((b_{\mathbf{t}} - s1) p^{-}_{\mathbf{t},s})$. Then $P(\mathbf{t}, s, \alpha)$ is a hyperplane of support for B with*

$$B \subset P^{\uparrow}(\mathbf{t}, s, \alpha) \Longleftrightarrow \alpha = \tau\big((b_{\mathbf{t}} - s1) p^{\pm}_{\mathbf{t},s}\big). \tag{3.4}$$

In this case, $\Psi(p^{\pm}_{\mathbf{t},s}) \in P(\mathbf{t}, s, \alpha)$.

Proof. $\Leftarrow$

Let $\alpha = \tau((b_{\mathbf{t}} - s1)p^{\pm}_{\mathbf{t},s})$. Then

$$\alpha = \tau\left(b_{\mathbf{t}}p^{\pm}_{\mathbf{t},s}\right) - s\tau\left(p^{\pm}_{\mathbf{t},s}\right) = \left(\sum_{i=1}^{n} t_i\tau\left(b_i\left(p^{\pm}_{\mathbf{t},s}\right)\right)\right) - s\tau\left(p_{\mathbf{t},s}\right)^{\pm}. \tag{3.5}$$

Therefore $\Psi(p^{\pm}_{\mathbf{t},s}) \in P(\mathbf{t},s,\alpha)$. Let $L_{\mathbf{t}}(s,\alpha) = \{(x_0,x_1) \mid -sx_0 + x_1 = \alpha\}$. By Proposition 2.17, $L_{\mathbf{t}}(s,\alpha)$ is a line of support for $B_{\mathbf{t}}$ and $B_{\mathbf{t}} \subset L^{\uparrow}_{\mathbf{t}}(s,\alpha)$. Fix $a \in M^{+}_{1}$. Then $\Psi_{\mathbf{t}}(a) \in L^{\uparrow}_{\mathbf{t}}(s,\alpha)$. Hence,

$$\alpha \leq -s\tau(a) + \tau(b_{\mathbf{t}}a) = -s\tau(a) + \sum_{i=1}^{n} t_i\tau(b_ia) \tag{3.6}$$

and $\Psi(a) \in P^{\uparrow}(\mathbf{t},s,\alpha)$.

$\Rightarrow$

Fix $\mathbf{t}$ and s, and let β vary over $\mathbb{R}$. The hyperplanes $P(\mathbf{t},s,\beta)$ are all parallel and hence there exists a unique β_0 such that $P(\mathbf{t},s,\beta_0)$ supports B and $B \subset P^{\uparrow}(\mathbf{t},s,\beta_0)$. But we have seen that α satisfies these conditions and so $\alpha = \beta_0$. □

Proposition 3.6. *If $\alpha = \tau((b_{\mathbf{t}} - s1)p^{\pm}_{\mathbf{t},s})$, then $F = P(\mathbf{t},s,\alpha) \cap B$ is a face of B. Further,*

$$\Psi^{-1}(P(\mathbf{t},s,\alpha)) \cap M^{+}_{1} = \left[p^{-}_{\mathbf{t},s}, p^{+}_{\mathbf{t},s}\right], \tag{3.7}$$

and $F = \Psi([p^{-}_{\mathbf{t},s}, p^{+}_{\mathbf{t},s}])$.

Proof. Let $\alpha = \tau((b_{\mathbf{t}} - s1)p^{\pm}_{\mathbf{t},s})$. By our previous result, $P(\mathbf{t},s,\alpha)$ is a supporting hyperplane for B. Hence, $F := P(\mathbf{t},s,\alpha) \cap B$ is a face of B. By Theorem 2.12, there are unique projections $p \leq q$ in M such that $\Psi^{-1}(F) \cap M^{+}_{1} = [p,q]$ and $\Psi([p,q]) = F$. If $a \in M^{+}_{1}$ and $\Psi(a) \in P(\mathbf{t},s,\alpha)$, then $\Psi(a) \in P(\mathbf{t},s,\alpha) \cap B$, and therefore

$$\left[p,q\right] = \Psi^{-1}(F) = M^{+}_{1} = \Psi^{-1}(P(\mathbf{t},s,\alpha) \cap B) \cap M^{+}_{1} = \Psi^{-1}(P(\mathbf{t},s,\alpha)) \cap M^{+}_{1}. \tag{3.8}$$

We would like to show that $p = p^{-}_{\mathbf{t},s}$ and $q = p^{+}_{\mathbf{t},s}$.

Since $\Psi(p^{\pm}_{\mathbf{t},s}) \in F$, we have

$$p \leq p^{-}_{\mathbf{t},s} \leq p^{+}_{\mathbf{t},s} \leq q. \tag{3.9}$$

But $\pi_{\mathbf{t}}(P(\mathbf{t},s,\alpha)) = L_{\mathbf{t}}(s,\alpha)$ and $\pi_{\mathbf{t}}(B) = B_{\mathbf{t}}$. Hence,

$$\Psi\left(\left[p,q\right]\right) = \pi_{\mathbf{t}}\left(\Psi\left(\left[p,q\right]\right)\right) = \pi_{\mathbf{t}}(F) = \pi_{\mathbf{t}}(\Psi(P(\mathbf{t},s,\alpha)) \cap B) \subset L_{\mathbf{t}}(s,\alpha) \cap B_{\mathbf{t}} = \Psi_{\mathbf{t}}\left(\left[p^{-}_{\mathbf{t},s}, p^{+}_{\mathbf{t},s}\right]\right). \tag{3.10}$$

Therefore, $p^{-}_{\mathbf{t},s} \leq p \leq q \leq p^{+}_{\mathbf{t},s}$, and so $p^{-}_{\mathbf{t},s} = p$ and $p^{+}_{\mathbf{t},s} = q$. □

Proposition 3.7. *Let* $\beta = \tau((b_{\mathbf{t}} - s1)(1 - p^{\pm}_{\mathbf{t},s}))$*. Then* $P(\mathbf{t}, s, \beta)$ *supports* B*, and passes through* $\Psi(1 - p^{\pm}_{\mathbf{t},s})$ *and* $B \subset P^{\downarrow}(\mathbf{t}, s, \beta)$*.*

Proof. If $\gamma = \tau((b_{-\mathbf{t}} - (-s)1)p^{\pm}_{-\mathbf{t},-s})$, then $P(-\mathbf{t}, -s, \gamma)$ is a hyperplane of support for B that contains $\Psi(p^{\pm}_{-\mathbf{t},-s})$ and $B \subset P^{\uparrow}(-\mathbf{t}, -s, \gamma)$. But $g_{-\mathbf{t}} = -g_{\mathbf{t}}$, so $\tau(b_{-\mathbf{t}}\cdot) = -\tau(b_{\mathbf{t}}\cdot)$ and $p^{\pm}_{-\mathbf{t},-s} = 1 - p^{\mp}_{-\mathbf{t},-s}$. Therefore,

$$\gamma = \tau\big((b_{-\mathbf{t}} - (-s)1)p^{\pm}_{-\mathbf{t},s}\big) = \tau\big((-b_{\mathbf{t}} + s1)\big(1 - p^{\mp}_{\mathbf{t},s}\big)\big) = -\tau\big((b_{\mathbf{t}} - s1)\big(1 - p^{\pm}_{\mathbf{t},s}\big)\big) = -\beta. \tag{3.11}$$

Therefore,

$$\begin{gathered} P(-\mathbf{t}, -s, \gamma) = P(-\mathbf{t}, -s, -\beta) = P(\mathbf{t}, s, \beta), \\ P^{\uparrow}(-\mathbf{t}, -s, -\beta) = P^{\downarrow}(\mathbf{t}, s, \beta). \end{gathered} \tag{3.12}$$

□

4. Invariance Properties of the Spectral Scale

The main goal in this section is to establish the circumstances required for the spectral scale to determine (up to equivalence of tracial representations) the algebra and the n-tuple, $(g_1, \ldots, g_n)$. Let N be the algebra generated by 1 and $b_i\chi_w(b_i)$ where w ranges over the bounded Borel subsets of $\mathbb{R}$ and i ranges from 1 to n.

Observe that $\lim_{k\to\infty}(\tau(b_i\chi_{[-k,k]}(b_i)a) = \tau(b_i a))$. To see this, note that $\chi_{[-k,k]}(b_i) \to 1$ strongly. Hence, for a fixed $a \in M$, $\chi_{[-k,k]}(b_i)a \to a$ strongly and so $\chi_{[-k,k]}(b_i)a \to a$ weakly. Since $\tau(b_i\cdot) = g_i(\cdot)$ is a bounded linear functional on M, we have the desired convergence.

We now show that we only need N to generate the spectral scale for the n-tuple $(g_1, \ldots, g_n)$ with respect to τ. By Proposition 2.35 from [5, page 232], there exists a faithful normal projection $E : M \to N$, with $\|E\| = 1$ such that $\tau = \tau \circ E$. Hence for $a \in M$,

$$\begin{aligned} \Psi(a) &= (\tau(a), \tau(b_1 a), \ldots, \tau(b_n a)) \\ &= \lim_{k\to\infty}\big(\tau(a), \tau\big(b_1\chi_{[-k,k]}(b_1)a\big), \ldots, \tau\big(b_n\chi_{[-k,k]}(b_n)a\big)\big) \\ &= \lim_{k\to\infty}\big(\tau(E(a)), \tau\big(b_1\chi_{[-k,k]}(b_1)E(a)\big), \ldots, \tau\big(b_n\chi_{[-k,k]}(b_n)E(a)\big)\big) \\ &= (\tau(E(a)), \tau(b_1 E(a)), \ldots, \tau(b_n E(a))) \\ &= \Psi(E(a)). \end{aligned} \tag{4.1}$$

Since E is faithful and normal, $\Psi(M_1^+) = \Psi(N_1^+)$ as desired.

We now introduce additional notation and change some of the old notation.

Notation 5. Let U and V be finite von Neumann algebras equipped with faithful, normal, tracial states τ_U and τ_V, respectively. Let H_U and H_V be the associated Hilbert spaces obtained by the tracial Gelfand-Naimark-Segal (GNS) construction [2, pages 278, 279]. Let $g_1, \ldots, g_n \in U_*$ and $h_1, \ldots, h_n \in V_*$ be self-adjoint. Then there exist $b_1, \ldots, b_n$ closed, densely defined, self-adjoint operators affiliated with U such that $\tau_U(b_i u) = g_i(u)$ for all $u \in U$. Similarly, there exist $c_1, \ldots, c_n$ closed, densely-defined, self-adjoint operators affiliated with V such that

$\tau_V(c_i v) = h_i(v)$ for all $v \in V$. Let M be the von Neumann algebra generated by 1 and $b_i \chi_\omega(b_i)$, where $i = 1, \ldots, n$ and ω ranges over the bounded Borel subsets of $\mathbb{R}$. Similarly, let N be the von Neumann algebra generated by 1 and $c_i \chi_\omega(c_i)$. Note that $g_1, \ldots, g_n \in M_*$ and $h_1, \ldots, h_n \in N_*$. When we are concerned only with objects restricted to M and N, we will write $\cdot_M$ and $\cdot_N$, respectively.

Let B be the spectral scale for $b_1, \ldots, b_n$ relative to τ_M determined by Ψ_M and C the spectral scale for $c_1, \ldots, c_n$ relative to τ_N determined by Ψ_N. Let π_M and π_N be the GNS representations of M and N. Let ξ_M and ξ_N be the canonical cyclic vectors that arise from this tracial GNS construction.

Definition 4.1. Suppose that there exists a surjective unitary transformation $u : H_M \to H_N$ such that $u\xi_M = \xi_N$ and $u\pi_M(b_i\chi_\omega(b_i)) = \pi_N(c_i\chi_\omega(c_i))u$ for $i = 1, \ldots, n$ and all bounded Borel subsets ω of $\mathbb{R}$. Then the tracial representations of M and N are said to be *equivalent*.

This definition is unsatisfying since it requires uncountably many conditions. We believe that there exists a more satisfactory definition of equivalence using the g_i's and the h_i's. We have not to date been able to formulate such a definition.

Proposition 4.2. *Suppose that $B = C$. Then there exists an isometry, Φ, from M_* to N_* such that $\Phi(g_i) = h_i$ for $i = 1, \ldots, n$ and $\Phi(\tau_M) = \tau_N$.*

Proof. Let us temporarily denote $\tau_M = g_{n+1}$ and $\tau_N = h_{n+1}$. For $i = 1, \ldots, n+1$, define $\Phi(g_i) = h_i$.

We would first like to show that Φ is well defined and can be extended linearly to the span of the g_i's. Suppose one of the g_i's is a linear combination of the others. Without loss of generality, $g_1 = \sum_{n=2}^{\infty} \alpha_i g_i$. Let $v \in N_1^+$. Since $B = C$, there exists $u \in M_1^+$ such that $g_i(u) = h_i(v)$ for every i. Thus,

$$h_1(v) = g_1(u) = \sum_{n=2}^{\infty} \alpha_i g_i(u) = \sum_{n=2}^{\infty} \alpha_i h_i(v). \tag{4.2}$$

If $v \in N^+$ is not zero, then $w = (1/\|v\|)v \in N_1^+$. There exists $u \in M_1^+$ such that $g_i(u) = h_i(w)$ for every i. Therefore,

$$h_1(v) = \|v\| h_1(w) = \|v\| g_1(u) = \|v\| \sum_{n=2}^{\infty} \alpha_i g_i(u) = \|v\| \sum_{n=2}^{\infty} \alpha_i h_i(w) = \sum_{n=2}^{\infty} \alpha_i h_i(v). \tag{4.3}$$

Since any element in N is a finite linear combination of elements in N^+, it follows that if $g_1 = \sum_{n=2}^{\infty} \alpha_i g_i$, then $h_1 = \sum_{n=2}^{\infty} \alpha_i h_i$. Hence, Φ is well-defined and we can therefore extend it linearly to linear combinations of the g_i's and hence to all of M_*.

We now show that Φ is an isometry. For $v \in M_*$ let us denote $\Phi(v) = \Phi_v$. Let M_1 be the set of points in M with norm 1, and let N_1 be the set of points in N with norm 1. We need to show that

$$\|v\| = \sup_{x \in M_1} |v(x)| = \sup_{y \in N_1} \left|\Phi_v(y)\right| = \|\Phi_v\|. \tag{4.4}$$

Consider $x \geq 0$. Then $x \in M_1^+$. There exists $y \in N_1^+$ such that $g_i(x) = h_i(y)$ for $i = 1, \dots, n+1$, and $y \neq 0$. Let $w = y/\|y\|$, and let $v = \sum_{i=1}^{n+1} \alpha_i g_i$. Hence,

$$\begin{gathered}
\sum_{i=1}^{n+1} \alpha_i g_i(x) = \sum_{i=1}^{n+1} \alpha_i h_i(w) \|w\|, \\
\left| \sum_{i=1}^{n+1} \alpha_i g_i(x) \right| = \left| \sum_{i=1}^{n+1} \alpha_i h_i(w) \right| \|w\| \leq \left| \sum_{i=1}^{n+1} \alpha_i h_i(w) \right|, \\
\sup_{x \in M_1 \cap M_1^+ |v(x)|} \leq \sup_{w \in N_1 \cap N_1^+} |\Phi_v(w)|.
\end{gathered} \tag{4.5}$$

A similar calculation shows the reverse inequality and therefore

$$\sup_{x \in M_1 \cap M_1^+} |v(x)| = \sup_{w \in N_1 \cap N_1^+} |\Phi_v(w)|. \tag{4.6}$$

□

Notation 6. Recall that $p_{\mathbf{t},s}^+$ is the spectral projection of $b_{\mathbf{t}}$ corresponding to $(-\infty, s]$ and $p_{\mathbf{t},s}^-$ is the spectral projection of $b_{\mathbf{t}}$ corresponding to $(-\infty, s)$. Let $q_{\mathbf{t},s}^{\pm}$ denote the spectral projections of $c_{\mathbf{t}}$ on the same intervals.

Proposition 4.3. *The following are equivalent:*

(1) $B = C$,

(2) $B_{\mathbf{t}} = C_{\mathbf{t}}$ *for* $\mathbf{t} \in \mathbb{R}^n \setminus \{0\}$,

(3) $\tau_M(p_{\mathbf{t},s}^{\pm}) = \tau_N(q_{\mathbf{t},s}^{\pm})$ *for* $s \in \mathbb{R}$ *and* $\mathbf{t} \in \mathbb{R}^n \setminus \{0\}$,

(4) $\tau_M(f(b_{\mathbf{t}})) = \tau_N(f(c_{\mathbf{t}}))$ *for* $\mathbf{t} \in \mathbb{R}^n \setminus \{0\}$ *and f a bounded Borel function on* $\mathbb{R}$,

(5) $\tau_M((b_{\mathbf{t}}\chi_\omega(b_{\mathbf{t}}))^k) = \tau_N((c_{\mathbf{t}}\chi_\omega(c_{\mathbf{t}}))^k)$ *for every* $k \in \mathbb{N}, \mathbf{t} \in \mathbb{R}^n \setminus \{0\}$, *and ω is a bounded Borel subset of* $\mathbb{R}$.

Proof. (1)⇒(2). Consider

$$B_{\mathbf{t}} = \pi_{\mathbf{t}}(B) = \pi_{\mathbf{t}}(C) = C_{\mathbf{t}}. \tag{4.7}$$

(2)⇒(1).

Suppose that $B \neq C$. Then (relabeling if necessary) there exists a vector $\mathbf{x} = (x_0, x_1, \dots, x_n) \in B \setminus C$. Since C is compact and convex and $\mathbf{x} \notin C$, there exists a hyperplane that strictly separates C from x. Thus, there exists $\mathbf{t} = (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}$, and $\beta \in \mathbb{R}$ such that for $\mathbf{y} = (y_0, y_1, \dots, y_n) \in C$, we have

$$\sum_{k=0}^{n} t_k x_k < \beta < \sum_{k=0}^{n} t_k y_k. \tag{4.8}$$

Hence, $(x_0, \sum_{k=1}^{n} t_k x_k) \neq (y_0, \sum_{k=1}^{n} t_k y_k)$. Therefore, $\pi_{\mathbf{t}}(\mathbf{x}) \neq \pi_{\mathbf{t}}(\mathbf{y})$ for every $\mathbf{y} \in C$, and so $\pi_{\mathbf{t}}(\mathbf{x}) \in B_{\mathbf{t}} \setminus C_{\mathbf{t}}$.

(1)⇒(3).

Fix $s, \mathbf{t}$. There exists unique α such that $P(\mathbf{t}, s, \alpha)$ is a hyperplane of support for B and $B \subset P^{\uparrow}(\mathbf{t}, s, \alpha)$. Since $B = C$, then α has the same properties with respect to C. Hence, $P(\mathbf{t}, s, \alpha) \cap B = P(\mathbf{t}, s, \alpha) \cap C$. Therefore,

$$\Psi_M([p_{\mathbf{t},s}^-, p_{\mathbf{t},s}^+]) = P(\mathbf{t}, s, \alpha) \cap B = P(\mathbf{t}, s, \alpha) \cap C = \Psi_N([q_{\mathbf{t},s}^-, q_{\mathbf{t},s}^+]), \tag{4.9}$$

and so $\tau_M(p_{\mathbf{t},s}^{\pm}) = \tau_N(q_{\mathbf{t},s}^{\pm})$.

(3)⇒(4).

Given (3), (4) holds when f is a characteristic function of an interval $(-\infty, s)$ or $(-\infty, s]$. These intervals generate the Borel structure of $\mathbb{R}$. Now τ_M and τ_N are normal and linear. Since any bounded Borel function, f, is uniformly approximated by linear combinations of characteristic functions, (4) holds for all such f.

(4)⇒(3).

Since characteristic functions on intervals are bounded and Borel, this is immediate.

(4)⇒(2).

Take $f^- = \chi_{(-\infty,s)}$ and $f^+ = \chi_{(-\infty,s]}$. Then $f^{\pm}(b_{\mathbf{t}}) = p_{\mathbf{t},s}^{\pm}$. Therefore, $\tau_U(p_{\mathbf{t},s}^{\pm}) = \tau_V(q_{\mathbf{t},s}^{\pm})$. Define $g^{\pm}$ on $\mathbb{R}$ by $g^{\pm}(a) = af^{\pm}(a)$ for $a \in \mathbb{R}$. $g^{\pm}$ is a bounded Borel function, and $g^{\pm}(b_{\mathbf{t}}) = b_{\mathbf{t}}p_{\mathbf{t},s}^{\pm}$. Hence, $\tau_M(b_{\mathbf{t}}p_{\mathbf{t},s}^{\pm}) = \tau_N(c_{\mathbf{t}}q_{\mathbf{t},s}^{\pm})$ for every nonzero $\mathbf{t} \in \mathbb{R}^n$, $s \in \mathbb{R}$. We have

$$\left(\tau_M(p_{\mathbf{t},s}^{\pm}), \tau_M(b_{\mathbf{t}}p_{\mathbf{t},s}^{\pm})\right) = \left(\tau_N(q_{\mathbf{t},s}^{\pm}), \tau_N(c_{\mathbf{t}}q_{\mathbf{t},s}^{\pm})\right). \tag{4.10}$$

These are the extreme points of the lower boundaries of $B_{\mathbf{t}}$ and $C_{\mathbf{t}}$, and so the lower boundaries coincide; hence the upper boundaries coincide by Proposition 2.3, and so $B_{\mathbf{t}} = C_{\mathbf{t}}$.

(4)⇒(5).

Define $f(s) = (s\chi_\omega(s))^k$ for $s \in \mathbb{R}$. Then f is a bounded Borel function, and hence, by assumption, $\tau_M(f(b_{\mathbf{t}})) = \tau_N(f(c_{\mathbf{t}}))$.

(5)⇒(4).

Define $h(s) = (s\chi_\omega(s))^k$ for $s \in \mathbb{R}$. By assumption, $\tau_M(h(b_{\mathbf{t}})) = \tau_N(h(c_{\mathbf{t}}))$, for every ω, a bounded Borel subset of $\mathbb{R}$, and every $k \in \mathbb{N}$. But the $h(b_{\mathbf{t}})$'s are weakly dense in M, and the $h(c_{\mathbf{t}})$'s are weakly dense in N. Also, τ_M and τ_N are normal, and so $\tau_M(f(b_{\mathbf{t}})) = \tau_N(f(c_{\mathbf{t}}))$, for f being a bounded Borel function on $\mathbb{R}$. □

Lemma 4.4. *The tracial representations of M and N are equivalent if and only if*

$$\tau_M(\phi(b_1\chi_\omega(b_1), \ldots, b_n\chi_\omega(b_n))) = \tau_N(\phi(c_1\chi_\omega(c_1), \ldots, c_n\chi_\omega(c_n))) \tag{4.11}$$

for every ϕ, a monomial in n variables, and ω, a bounded Borel subset of $\mathbb{R}$.

Proof. We begin by making the notation a little less cumbersome. Let $\mathbf{b}_\omega = (b_1\chi_\omega(b_1), \ldots, b_n\chi_\omega(b_n))$, and let $\mathbf{c}_\omega = (c_1\chi_\omega(c_1), \ldots, c_n\chi_\omega(c_n))$.

⇐

Suppose that $\tau_M(\phi(\mathbf{b}_\omega)) = \tau_N(\phi(\mathbf{c}_\omega))$ for every ϕ and ω.

Define $u\pi_M(\phi(\mathbf{b}_\omega))\xi_M = \pi_N(\phi(\mathbf{c}_\omega))\xi_N$. Extend this definition by linearity to polynomials. Let ϕ_1, ϕ_2 be two such polynomials. Then for $i = 1, \dots, n$ we have

$$\begin{aligned}
&\langle \pi_M(b_i\chi_{\omega_i}(b_i))\pi_M(\phi_1(\mathbf{b}_\omega))\xi_M, \pi_M(\phi(\mathbf{b}_\omega))\xi_M\rangle \\
&\quad = \left\langle \left(\pi_M(\phi_2(\mathbf{b}_\omega))^*\pi_M(b_i\chi_{\omega_i}(b_i))\pi_M(\phi_1(\mathbf{b}_\omega))\right)\xi_M, \xi_M\right\rangle \\
&\quad = \left\langle \left(\pi_M\left((\phi_2(\mathbf{b}_\omega))^* b_i\chi_{\omega_i}(b_i)\phi_1(\mathbf{b}_\omega)\right)\right)\xi_M, \xi_M\right\rangle \\
&\quad = \tau_M\left((\phi_2(\mathbf{b}_\omega))^* b_i\chi_{\omega_i}(b_i)\phi_1(\mathbf{b}_\omega)\right) \\
&\quad = \tau_N\left((\phi_2(\mathbf{c}_\omega))^* c_i\chi_{\omega_i}(c_i)\phi_1(\mathbf{c}_\omega)\right) \\
&\quad = \langle \pi_N(c_i\chi_{\omega_i}(c_i))\pi_N(\phi_1(\mathbf{c}_\omega))\xi_N, \pi_N(\phi(\mathbf{c}_\omega))\xi_N\rangle \\
&\quad = \langle \pi_N(c_i\chi_{\omega_i}(c_i))u\pi_M(\phi_1(\mathbf{b}_\omega))\xi_M, u\pi_M(\phi(\mathbf{b}_\omega))\xi_M\rangle.
\end{aligned} \tag{4.12}$$

Such polynomials are dense in H_M and H_N, so u extends to a unitary transformation from H_M to H_N with the desired properties.

$\Rightarrow$

Suppose that the tracial representations of M and N are equivalent. Then

$$\begin{aligned}
\tau_N(\phi(\mathbf{c}_\omega)) &= \langle \pi_N(\phi(\mathbf{c}_\omega))\xi_N, \xi_N\rangle = \langle u\pi_M(\phi(\mathbf{b}_\omega))u^*\xi_N, \xi_N\rangle \\
&= \langle \pi_M(\phi(\mathbf{b}_\omega))\xi_M, \xi_M\rangle = \tau_M(\phi(\mathbf{b}_\omega)).
\end{aligned} \tag{4.13}$$

□

Suppose that M is Abelian and d, e are closed, densely defined operators affiliated with M. Then $d+e$ and de are closable, densely defined operators whose closures are affiliated with M and M', the set of operators affiliated with M is an Abelian $*$-algebra [2, pages 351, 352]. If in addition d and e are self-adjoint, then $d + e$ and de are also self-adjoint and hence closed [14, page 536]. Further, we have

$$\tau(b_{\mathbf{t}}a) = g_{\mathbf{t}}(a) = \sum_{i=1}^{n} t_i g_i(a) = \sum_{i=1}^{n} t_i\tau(b_i a) = \tau\left(\sum_{i=1}^{n} t_i b_i a\right) \tag{4.14}$$

for every $a \in M$. Thus,

$$\tau\left(\left(b_{\mathbf{t}} - \sum_{i=1}^{n} t_i b_i\right)a\right) = 0, \tag{4.15}$$

for every $a \in M$. Let b^+ be the positive part of $b_{\mathbf{t}} - \sum_{i=1}^{n} t_i b_i$ and b^- the negative part. Choose $a = b^+\chi_\omega(b^+)$, where ω is any bounded Borel set. Then a is positive since M' is commutative and

$$\left(b_{\mathbf{t}} - \sum_{i=1}^{n} t_i b_i\right)a = (b^+)^2\chi_\omega(b^+) \geq 0. \tag{4.16}$$

But $\tau((b^+)^2\chi_\omega(b^+)) = 0$, since τ is faithful $(b^+)^2\chi_\omega(b^+) = 0$. Thus $b^+ = 0$ on its domain. Similarly $b^- = 0$, and so $b_{\mathbf{t}} - \sum_{i=1}^n t_i b_i = 0$. Thus $\sum_{i=1}^n t_i b_i = b_{\mathbf{t}}$.

To proceed with the theory as given in [1, Section 3, page 281 ff.], it would be convenient if the following conjecture were true.

Conjecture 4.5. *If M and N are Abelian and $B = C$, then the tracial representations of M and N are equivalent.*

By Lemma 4.4, it is enough to show that, if $\phi(x_1,\dots,x_n) = x_1^{k_1}\cdots x_n^{k_n}$ denotes a monomial in the commuting variables $x_1,\dots,x_n$, then

$$\tau_M(\phi(b_1\chi_\omega(b_1),\dots,b_n\chi_\omega(b_n))) = \tau_N(\phi(c_1\chi_\omega(c_1),\dots,c_n\chi_\omega(c_n))) \tag{4.17}$$

for every ω, a bounded Borel subset of $\mathbb{R}$. By part (5) of Proposition 4.3, we know that $\tau_M((b_{\mathbf{t}}\chi_\omega(b_{\mathbf{t}}))^k) = \tau_N((c_{\mathbf{t}}\chi_\omega(c_{\mathbf{t}}))^k)$ for every $k \in \mathbb{N}, \mathbf{t} \in \mathbb{R}^n \setminus \{0\}$, with ω being a bounded Borel subset of $\mathbb{R}$. Let us fix k and ω, and let P be the set of all monomials ϕ in n commuting variables such that $\sum_{i=1}^n k_i = k$. Routine computations show that

$$\tau_M\Big((b_{\mathbf{t}}\chi_\omega(b_{\mathbf{t}}))^k\Big) = \sum_{\phi\in P} t_1^{k_1}\cdots t_n^{k_n}\tau_M(\phi(b_1,\dots,b_n)\chi_\omega(b_{\mathbf{t}})). \tag{4.18}$$

Similarly,

$$\tau_N\Big((c_{\mathbf{t}}\chi_\omega(c_{\mathbf{t}}))^k\Big) = \sum_{\phi\in P} t_1^{k_1}\cdots t_n^{k_n}\tau_N(\phi(c_1,\dots,c_n)\chi_\omega(c_{\mathbf{t}})). \tag{4.19}$$

Since

$$\tau_M\Big((b_{\mathbf{t}}\chi_\omega(b_{\mathbf{t}}))^k\Big) = \tau_N\Big((c_{\mathbf{t}}\chi_\omega(c_{\mathbf{t}}))^k\Big), \tag{4.20}$$

then we have

$$\sum_{\phi\in P} t_1^{k_1}\cdots t_n^{k_n}\tau_M(\phi(b_1,\dots,b_n)\chi_\omega(b_{\mathbf{t}})) = \sum_{\phi\in P} t_1^{k_1}\cdots t_n^{k_n}\tau_N(\phi(c_1,\dots,c_n)\chi_\omega(c_{\mathbf{t}})). \tag{4.21}$$

In the bounded case, no characteristic functions are present and so we can equate coefficients of the polynomials. Even if we could do that here, we still do not have the desired result since we want something independent of $\mathbf{t}$.

5. Miscellaneous Results

A natural question is to ask whether convergence of n-tuples of self-adjoint operators implies convergence of the corresponding spectral scales. Since spectral scales are compact and convex, the Hausdorff Metric is a natural metric to work with. The following definition is taken from [15, page 274].

Definition 5.1. Let (X, d) be a metric space, with A and B being nonempty subsets of X. Define $d(A, B) = \inf\{d(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \in A, \mathbf{b} \in B\}$. For $\gamma > 0$, let us define

$$A_\gamma = \{\mathbf{x} \in X \mid d(\{\mathbf{x}\}, A) < \gamma\},$$
$$B_\gamma = \{\mathbf{x} \in X \mid d(\{\mathbf{x}\}, B) < \gamma\}. \tag{5.1}$$

Define

$$d_H(A, B) = \inf\{\gamma > 0 \mid A \subset B_\gamma, B \subset A_\gamma\}. \tag{5.2}$$

Then $d_H(A, B)$ is the *Hausdorff distance* between A and B.

We first establish a result for the original definition of a spectral scale, that is the spectral scale from Definition 1.4.

Theorem 5.2. *Let $m \in \mathbb{N}$. Suppose that $\mathbf{b}_n \to \mathbf{b}$ strongly in each coordinate for $\mathbf{b}_n, \mathbf{b} \in M^m$ where $\mathbf{b}_n$ and $\mathbf{b}$ are self-adjoint (in each coordinate). If $B(\mathbf{b}_n)$ and $B(\mathbf{b})$ are the corresponding spectral scales, then $B(\mathbf{b}_n) \to B(\mathbf{b})$ in the Hausdorff metric induced by the usual topology on $\mathbb{R}^{m+1}$.*

Proof. Write b_{jn} for the jth coordinate of $\mathbf{b}_n$ and b_{j0} for the jth coordinate of $\mathbf{b}$. Let $\epsilon > 0$. Now $B(\mathbf{b}_n) = \{(\tau(a), \tau(b_{1n}a), \ldots, \tau(b_{mn}a)) \mid a \in M_1^+\}$ and $B(\mathbf{b}) = \{(\tau(a), \tau(b_{10}a), \ldots, \tau(b_{m0}a)) \mid a \in M_1^+\}$. Fix a j. Define $c_{jn} = b_{jn} - b_{j0}$ and note that c_{jn} is self-adjoint. Further, $c_{jn} \to 0$ strongly, and so $c_{jn}^2 \to 0$ strongly as well. Since τ is normal, $\tau(c_{jn}^2) \to 0$. Therefore, there exists $N_j \in \mathbb{N}$ such that $n \geq N_j \Rightarrow |\tau(c_{jn}^2)| < \epsilon^2$. Since τ is a weight defined on all of M [4, page 486], the map $(a, b) \to \tau(b^* a)$ $(a, b \in M)$ is a positive-definite inner product on M [4, page 489]. (The map is definite because τ is faithful.) Hence, the Cauchy-Schwarz inequality applies. For $a \in M_1^+$ and $n \geq N_j$, we have

$$\begin{aligned}
\left|\tau(c_{jn}a)\right| &= \langle a, c_{jn}\rangle \\
&\leq \left[\tau\left(a^2\right)\right]^{1/2}\left[\tau\left(c_{jn}^2\right)\right]^{1/2} \quad \text{[Cauchy-Schwarz]} \\
&< \epsilon\left[\tau\left(a^2\right)\right]^{1/2} \\
&\leq \epsilon\left(\left\|a^2\right\|\tau(1)\right)^{1/2} \quad \text{[Hölder]} \\
&\leq \epsilon\left(\left\|a^2\right\| \cdot 1\right)^{1/2} \\
&\leq \epsilon.
\end{aligned} \tag{5.3}$$

Let $N = \max_{j=1,\ldots,m} N_j$, and fix $n \geq N$. For $a \in M_1^+$, let

$$\alpha(a) = (\tau(a), \tau(b_{1n}a), \ldots, \tau(b_{mn}a)),$$
$$\beta(a) = (\tau(a), \tau(b_{10}a), \ldots, \tau(b_{m0}a)). \tag{5.4}$$

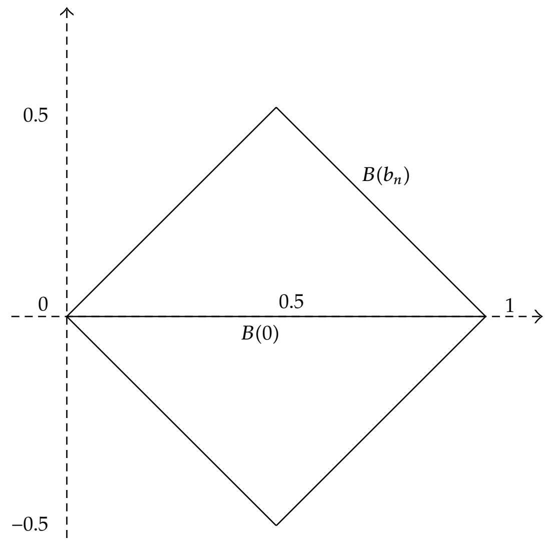

Figure 5: Spectral scales for Example 5.3.

Note that all points in $B(\mathbf{b}_n)$ are of the form $\alpha(a)$ and all points in $B(\mathbf{b})$ are of the form $\beta(a)$. Then by the inequality in (5.3) we have for every a

$$\|(\alpha-\beta)(a)\|_\infty = \sup_{j=1,\ldots,m} \left|\tau(b_{jn}a) - \tau(b_{i0}a)\right| \le \epsilon. \tag{5.5}$$

Hence $d_H(B(b), B(b_n)) \le 2\epsilon$ for $n \ge N$. Since ϵ was arbitrary, the result follows. □

Theorem 5.2 is false if we replace strong convergence with weak convergence. Sherman came up with the following example in conversation with the author of this paper, and kindly gave permission for it to be included here.

Example 5.3. Let $H = L^2[0,1]$ and $M = L^\infty[0,1]$. Then there exist $b_n,\ b \in M$, self-adjoint, such that $b_n \to b$ weakly, but $B(b_n) \nrightarrow B(b)$ in the Hausdorff metric.

Proof. Let $U_n = \{x \in [0,1] \mid \text{the } n\text{th binary digit of } x \text{ is } 0\}$. Let

$$b_n = T_{U_n} - T_{(U_n)^c}, \tag{5.6}$$

where T_W is the characteristic function on the set W. Each b_n is clearly self-adjoint, and standard analysis arguments show that the sequence $\{b_n\}_{n\in\mathbb{N}}$ converges weakly to 0. Now $\sigma(b_n) = \{1,-1\}$ for every n. Since $\mu(U_n) = \mu((U_n)^c) = 0.5$ for every n, $B(b_n)$ is the ball in $(\mathbb{R}^2, \|\cdot\|_1)$ centered at $(0.5, 0)$ with radius 0.5 (see Figure 5). On the other hand, the spectral scale for 0 is simply the line segment $\{(x,0) \mid 0 \le x \le 1\}$. Hence, $d_H(B(b_n), B(0)) = 0.5$ for every $n \in \mathbb{N}$. (The base metric for d_H is the Euclidean norm.) □

The next result concerns the unbounded situation that we have been dealing with for most of this paper.

Theorem 5.4. *Let* $\mathbf{g}_n = (g_{1n}, \ldots, g_{mn})$, $\mathbf{g} = (g_{10}, \ldots, g_{m0})$ *be self-adjoint* m*-tuples of bounded linear functionals on our finite von Neumann algebra* M*. Suppose that* $g_{jn} \to g_{j0}$ *in the dual norm. Then* $B(\mathbf{g}_n) \to B(\mathbf{g})$ *in the Hausdorff metric.*

Proof. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow \sup_{j=1,\ldots,m} \left\| g_{jn} - g_{j0} \right\| < \epsilon. \tag{5.7}$$

Let $\alpha \in B(\mathbf{g})$. We want to find that $\beta \in B(\mathbf{g}_n)$ (for $n \geq N$) such that $\|\alpha - \beta\| \leq \epsilon$.

Since $\alpha \in B(\mathbf{g})$, then there exists $a \in M_1^+$ such that $\alpha = \{(\tau(a), g_{10}(a), \ldots, g_{m0}(a))\}$. Define $\beta = \{(\tau(a), g_{1n}(a), \ldots, g_{mn}(a))\}$. Then

$$\left| g_{jn}(a) - g_{j0}(a) \right| = \left| (g_{jn} - g_{j0})(a) \right| \leq \left\| g_{jn} - h_{j0} \right\| < \epsilon. \tag{5.8}$$

Note that ϵ does not depend on j. Hence taking supremums over all the j's, we have that $\|\alpha - \beta\|_\infty < \epsilon$. Hence, $B(\mathbf{g}) \subset (B(\mathbf{g}_n))_{2\epsilon}$. Similarly, $B(\mathbf{g}_n) \subset (B(\mathbf{g}))_{2\epsilon}$. Since ϵ was arbitrary, we have convergence in the Hausdorff metric as desired. □

The spectral scale as given in Definition 1.4 is not the only object that has been called a spectral scale. The following definition of a spectral scale was formulated by Petz.

Definition 5.5 (see [3, page 74]). Let A be a finite von Neumann algebra equipped with a faithful, normal, tracial state, τ. Let b be a self-adjoint operator affiliated with A. The *Spectral Scale* is defined for $t \in [0,1]$ as follows:

$$\lambda_t(b) = \inf\left\{ s \mid 1 - \tau\left(p_s^+\right) \leq t \right\}, \tag{5.9}$$

where $p_s^+ = \chi_{(-\infty,s]}(b)$ as before.

Notation 7. We shall call this spectral scale the *Petz spectral scale*. We will call the spectral scale from Definition 2.1 the *AAW spectral scale*.

We now show how the two notions are related. To this end, we first find what values $\lambda_t(b)$ can take for a given t. To this end, fix t and note that we can write

$$\lambda_t(b) = \inf\left\{ s \mid \tau\left(p_s^+\right) \geq 1 - t \right\}. \tag{5.10}$$

There are 3 cases to consider.

Case 1. There exists $s_0 \in \sigma(b)$ such that $\tau(p_{s_0}^+) = 1 - t$.

In this case, $s < s_0 \Rightarrow \tau(p_s^+) < 1 - t$ and hence, $\lambda_t(b) = s_0$.

Case 2. For every $s \in \sigma(b)$, $\tau(p_s^+) \neq 1 - t$ but there exists $s_0 \in \sigma(b)$ such that $\tau(p_{s_0}^+) > 1 - t$.

Since $\sigma(b)$ is closed, and τ is weak-$*$ continuous, we may choose $s_0 \in \sigma(b)$ so that $s_0 > s_1 \Rightarrow \tau(p_{s_1}^+) < 1 - t$. Hence, s_0 is the smallest real value such that $\tau(p_{s_0}^+) > 1 - t$, and so $s_0 = \lambda_t(b)$.

Case 3. For every $s \in \sigma(b)$, $\tau(p_s^+) < 1 - t$.

Note that, in this case, $t = 0$ and b must be an unbounded operator, with unbounded spectrum on the right. In this case, $\lambda_0(b) = \infty$.

The following result was proposed by Pavone in conversation with the author of this paper.

Proposition 5.6. *Let f be the function whose graph is the upper boundary curve of the AAW spectral scale. Then, for $t > 0$, $f'_-(t) = \lambda_t(b)$.*

Proof. By the rotational symmetry of the AAW spectral scale, $f'_-(t) = g'_+(1-t)$ where g is the function whose graph is the lower boundary curve of the AAW spectral scale. If $s_0 \in \sigma(b)$ and $\tau(p_{s_0}^+) = 1 - t$, then $g'_+(\tau(p_{s_0}^+)) = s_0 = \lambda_t(b)$. If $t \in (s_1, s_0)$, a gap in the spectrum, with $s_0, s_1 \in \sigma(b)$ then the right-hand derivative of g at t is $s_0 = \lambda_t(b)$. □

At $t = 0$, the slope of f is $\sup \sigma(b)$. If this is a finite number $s_{\max}$, then $\lambda_0(b) = s_{\max}$. If $\sup \sigma(b) = \infty$, then, as we saw in Case 3 above, $\lambda_0(b) = \infty$. This completes our discussion of the relationship between the AAW spectral scale and the Petz spectral scale.

6. Future Research

A great deal of further work has been done with the spectral scale in the bounded situation. For us, the first question to ask is whether Conjecture 4.5 is in fact true. If so, we believe that many of the remaining results in [1] can be extended to the unbounded case fairly readily.

Additionally, we believe that the idea of a spectral scale of an unbounded operator can be used in the discussion of numerical range.

Definition 6.1. Let b be a (bounded) linear operator on H. Define

$$W_k(b) = \left\{ \frac{1}{k} \sum_{i=1}^{k} \langle bx_i, x_i \rangle \mid i \neq j \Longrightarrow \langle x_i, x_j \rangle = 0, \langle x_i, x_i \rangle = 1 \right\}. \tag{6.1}$$

Then $W_k(b)$ is the *k-numerical range* of b. When $k = 1$, we simply write $W(b)$, and refer to it as the *numerical range* [6, page 226].

We can write $b = b_1 + ib_2$, where b_1 and b_2 are self-adjoint and so we can define the spectral scale of b to be $B(b) := B(b_1, b_2)$. It turns out that the boundary of $W(b)$ is exactly the set of radial complex slopes on $B(b)$ at the origin.

In the unbounded situation, we start with $g = g_1 + ig_2 \in M_*$, where M is a finite von Neumann algebra equipped with τ, a finite, faithful, normal, tracial state. We can certainly find b, b_i such that $g(a) = \tau(ba)$ and $g_i(a) = \tau(b_i a)$ for every $a \in M$, but (except in the Abelian case) it is not obvious that there is any relationship between b and the b_i's. We define the numerical range for g by making the additional assumption in Definition 6.1 that the x_i's are in the domain of B. At the moment, it is not clear that there is any relationship between $W(g)$ and $B(g_1, g_2)$. However, if we can establish some kind of relationship, it is natural to ask how much of the theory developed in [6, 7] can be extended in this context.

Finally, we ask whether Theorem 2.22 can be extended beyond the single-variable situation?

Acknowledgments

The author wishes to thank Christopher Pavone, Roger Roybal, David Sherman, and especially Charles Akemann for several valuable conversations in connection with the material presented here.

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