Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 714147, 9 pages
doi:10.1155/2010/714147

Research Article

On Maslanka's Representation for the Riemann Zeta Function

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Received 18 October 2009; Accepted 10 May 2010

Academic Editor: Dorothy Wallace

A rigorous proof is given of the hypergeometric-like representation of the Riemann zeta function $\zeta(s)$ discovered by Maslanka as a series of Pochhamer polynomials with coefficients depending on the values of ζ at the positive even integers.

1. Introduction

In [1, 2] Maslanka introduced the following representation of $\zeta(s)$ valid for all $s \in \mathbb{C}$:

$$(s-1)\zeta(s) = \sum_{k=0}^{\infty} A_k P_k\left(\frac{s}{2}\right), \tag{1.1}$$

where the A_k are given by

$$A_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j+1)\zeta(2j+2), \tag{1.2}$$

and the P_k are the so-called *Pochhammer polynomials*, defined by

$$P_k(s) := \prod_{r=1}^{k}\left(1 - \frac{s}{r}\right) \quad (P_0(s) := 1). \tag{1.3}$$

We apologize for having changed the notation used in [1, 2], but we have found it more natural to write it as above with future applications in mind.

Maslanka indicates that he was led to this interesting expression by the desire to interpolate $\zeta(s)$ with a series of the form (1.1) treating the A_k as indeterminates to be found from the system of equations

$$(2n-1)\zeta(2n) = \sum_{k=0}^{\infty} A_k P_k(n) \quad (n = 1, 2, \ldots). \tag{1.4}$$

This system is a triangular system because $P_k(m) = 0$ when the integer $m \leq k$. Its unique solution is found by a nice combinatorial argument given in Appendix A of [1].

In [1, 2] Maslanka gave two formal proofs that the series (1.1) with the coefficients as defined by (1.2) represents $(s-1)\zeta(s)$. However, no estimate of the size of the A_k is given that would imply the convergence of the series in any region of the complex plane, and which would be needed to justify several of the formal interchanges of limits and series involved in the proofs. Professor Maslanka himself kindly indicated this gap to me in a personal communication. I thought it would be worthwhile to devote some effort, which proved quite rewarding, to provide the missing steps in the proof. An effective estimate of the rate at which $A_k \to 0$ is the crucial missing link in the proof, and to this effect we shall show below that

$$A_k \ll_p k^{-p} \tag{1.5}$$

for every positive real p.

In closing this introduction, we would like to point out an interesting paradox that could deserve some attention. Once the Maslanka representation is shown to be valid in the whole complex plane it is obvious that (1.1) will be valid for $s = 1, 0, -1, -2, \ldots$; noting however that in this case the series does not truncate, one obtains interesting identities as pointed out in [1]. If, on the other hand, one proposed at the outset the analogous problem of finding a representation of the form

$$(s-1)\zeta(s) = \sum_{k=0}^{\infty} c_k P_k(2-s), \tag{1.6}$$

then, since it is clear that the series truncates for $s = 1, 0, -1, -2, \ldots$ one gets again a triangular system leading to a surprising solution, namely,

$$(s-1)\zeta(s) = 1 + \frac{1}{2}(s-1) + \sum_{k=1}^{\infty} B_k P_k(2-s), \tag{1.7}$$

where the B_k are the Bernoulli numbers. This happens to be, in different garb, the *divergent* Euler-Maclaurin series of $\zeta(s)$. It may be objected that Dirichlet series (albeit $(s-1)\zeta(s)$ is *not* a Dirichlet series) are determined by its values at any sequence with real parts tending to $+\infty$, while that is not the case toward the left. It may seldom be the case that this question is raised at all since "all" Dirichlet series of interest have a finite abscissa of convergence so there is no way to move toward $-\infty$. On the other hand, Professor Malanska claims he has

a representation that interpolates simultaneously at points $2n$ and $3-2n$. We do not know what that series may be, perhaps he means something like $\sum c_k P_k(s/2)P_k((1-s)/2)$.

These considerations can be generalized to explore general "*p-series*" $\sum c_k P_k(s)$ which share certain characteristics with Dirichlet series. As an example we could find a p-series for $\zeta(s)^{-1}$ and arrived at the following criterion for the Riemann hypothesis that involves the Bernoulli numbers: define

$$b_k := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)} \quad (k = 0, 1, 2, \ldots). \tag{1.8}$$

Then the Riemann hypothesis is true if and only if

$$b_k \ll_\epsilon k^{-3/4+\epsilon} \quad (\forall \epsilon > 0). \tag{1.9}$$

This criterion is a discrete version the author published in [3] of the well-known Riesz criterion for the Riemann hypothesis [4]. It is well suited for numerical calculations. Plotting the results of some preliminary computations, one obtains a very smooth "curve" $b_k k^{4/3}$ that quite impresses one as tending to zero like $\log^{-2} k$.

2. Statement and Proof of the Main Theorem

We now formally state and prove the representation theorem.

Theorem 2.1. *For A_k defined as in* (1.2), *we have*

$$(s-1)\zeta(s) = \sum_{k=1}^{\infty} A_k P_k\left(\frac{s}{2}\right), \tag{2.1}$$

for all $s \in \mathbb{C}$. The convergence of the series is uniform and absolute in every compact set of the complex plane.

Throughout, it shall be very important to bear in mind the simple estimate for the size of the Pochhammer polynomials contained in the following lemma already proved in [3, Lemma 2.3].

Lemma 2.2. *For every compact set $H \subset \mathbb{C}$, there is a positive constant C_H, not depending on k, such that*

$$|P_k(s)| \le C_H k^{-\Re s} \quad (|s| \in H, k = 1, 2, \ldots). \tag{2.2}$$

Do note also the equally trivial facts contained in the following lemma

Lemma 2.3.

$$P_k(s) = (-1)^k \binom{s}{k} = \frac{\Gamma(k+1-s)}{k!\Gamma(1-s)}. \tag{2.3}$$

Proof of Theorem 2.1. The key estimate (1.5) for the coefficients A_k shall be proved in the next section. We accept it for the moment and show now how the informal steps in Maslanka's second proof can be made rigorous. We shall first show that with no assumption on the size of the A_k one can prove equality in (1.1) for $\Re s > 4$. We start with Maslanka's clever idea of expressing $s-1$ as the derivative of $-\alpha^{1-s}$ at $\alpha = 1$. We indicate by D_α the operation of differentiating with respect to α and proceed as follows:

$$\begin{aligned}
\alpha^{-s}(s-1)\zeta(s) &= -D_\alpha \alpha^{1-s} \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= -D_\alpha \alpha \sum_{n=1}^{\infty} \frac{1}{(\alpha n)^s} \\
&= -D_\alpha \alpha \sum_{n=1}^{\infty} \frac{1}{(\alpha n)^2} \left(\frac{1}{(\alpha n)^2} \right)^{s/2-1} \\
&= -D_\alpha \alpha \sum_{n=1}^{\infty} \frac{1}{(\alpha n)^2} \left(1 - \left(1 - \frac{1}{(\alpha n)^2} \right) \right)^{s/2-1} \\
&= -D_\alpha \alpha \sum_{n=1}^{\infty} \frac{1}{(\alpha n)^2} \sum_{k=0}^{\infty} (-1)^k \binom{\frac{s}{2}-1}{k} \left(1 - \frac{1}{(\alpha n)^2} \right)^k \\
&= -D_\alpha \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \frac{1}{\alpha n^2} \left(1 - \frac{1}{(\alpha n)^2} \right)^k,
\end{aligned} \tag{2.4}$$

where we have applied the binomial theorem assuming as we may that the derivative is calculated from the right so $\alpha > 1$. Consider now the formally differentiated double series on the right-hand side above, namely,

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \frac{1}{(\alpha n)^2} \left(1 - \frac{2k+1}{(\alpha n)^2} \right) \left(1 - \frac{1}{(\alpha n)^2} \right)^{k-1}. \tag{2.5}$$

Using Lemma 2.2, we see that this double series is compact-uniformly termwise majorized by a convergent double series of positive terms in $\Re s \geq 4 + 2\epsilon$ since

$$\left| P_k\left(\frac{s}{2}\right) \right| \frac{1}{\alpha n^2} \left(1 - \frac{2k+1}{(\alpha n)^2} \right) \left(1 - \frac{1}{(\alpha n)^2} \right)^k \ll k^{-1-\epsilon} \frac{1}{n^2}. \tag{2.6}$$

Thus we can both invert the sums at the end of (2.4) and do the differentiation termwise at $\alpha = 1$ obtaining

$$(s-1)\zeta(s) = \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - \frac{2k+1}{n^2} \right) \left(1 - \frac{1}{n^2} \right)^{k-1}. \tag{2.7}$$

We easily calculate the last series on the right as

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left(\zeta(2j+2)-(2k+1)\zeta(2j+4)\right) \tag{2.8}$$

which, after separating in two sums and shifting indices in the second, becomes

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(2j+1)\zeta(2j+2), \tag{2.9}$$

which is none other than A_k. One could even argue this would be a direct way to find the coefficients. We have thus shown the desired relation (1.1) at least for $\Re s > 4$. However, the strong estimate (1.5), to be proved below, providing that $A_k \ll k^{-p}$ for every $p > 0$, together with Lemma 2.2, immediately shows that the series $\sum A_k P_k(s/2)$ converges uniformly on any compact subset of the plane, thus defining an entire function that must be equal to $(s-1)\zeta(s)$ by analytic continuation. □

3. The Size of the Coefficients

We shall establish here this Proposition which was announced as the essential estimate (1.5) in the introduction.

Proposition 3.1. *For any $p > 0$ there is a constant $C_p > 0$ such that*

$$|A_k| \le C_p k^{-p}, \quad k = 1, 2, \ldots. \tag{3.1}$$

To lighten up the proof, we first prove some lemmas. Define the crucial sequence of rational functions $\phi_k(x)$ by

$$\phi_k(x) := \left(1-\frac{1}{x^2}\right)^k \frac{1}{x} \quad (k \in \mathbb{Z}^+). \tag{3.2}$$

We now record a very simple while extremely important property of $\phi(x)$.

Lemma 3.2. *For $0 \le a \le k$, denote*

$$\phi_k^{(a)}(x) := \frac{d^a}{dx^a}\phi_k(x). \tag{3.3}$$

Then we have

$$\phi_k^{(a)}(1) = \phi_k^{(a)}(\infty) = 0, \tag{3.4}$$

where the second equality is unrestricted on a. All the $\phi_k^{(a)}(x)$ for $a \ge 1$ are integrable.

The following lemma is the essential and quite elementary tool to get a grip on the size of A_k.

Lemma 3.3. *Define a double sequence of polynomials $p_{a,j}$, with integral coefficients as follows: let $p_{0,0} = 1$, $p_{a,j} = 0$ when $j < 0$ and when $j > a$, and specify the recurrence equation*

$$p_{a,j} = -(2j + a)p_{a-1,j} + (2k + 2j - a)p_{a-1,j-1}. \tag{3.5}$$

Each $p_{a,j}$ is a polynomial of degree j. For any integers $k \geq a \geq 0$,

$$\phi_k^{(a)}(x) = \left(1 - \frac{1}{x^2}\right)^{k-a} \sum_{j=0}^{a} \frac{p_{a,j}(k)}{x^{a+2j+1}}. \tag{3.6}$$

The simple proof is by (double) induction. The recurrence relation (3.5) will not be used other than the fact that it is part of the proof. It is nice to record it explicitly for further use if need be.

Having straightened some notation, as all the above was essentially, we are ready for a first preliminary estimate.

Lemma 3.4. *For any fixed integer $a \geq 1$ and $\epsilon > 0$, there is a constant $C = C(a, \epsilon)$ such that*

$$\int_1^\infty \left|\phi_k^{(a)}(x)\right| dx \leq Ck^{-(a/2)(1-\epsilon)}. \tag{3.7}$$

Proof. Take $k \geq a \geq 1$. We split the interval of integration at the point $x = k^{1/2-\epsilon/3}$. We express the ath derivative according to Lemma 3.3 and equation (3.6). For the finite range of the integral, we have

$$\begin{aligned}
\int_1^{k^{1/2-\epsilon/3}} \left|\phi_k^{(a)}(x)\right| dx &\leq \int_1^{k^{1/2-\epsilon/3}} \left|\left(1 - \frac{1}{x^2}\right)^{k-a} \sum_{j=0}^{a} \frac{p_{a,j}(k)}{x^{a+2j+1}}\right| dx \\
&\leq \left|\sum_{j=1}^{a} p_{a,j}(k)\right| \int_1^{k^{1/2-\epsilon/3}} \left(1 - \frac{1}{x^2}\right)^{k-a} dx \\
&\leq O_a(k^a)\left(1 - \frac{1}{k^{1-2\epsilon/3}}\right)^{k-a} k^{1/2-\epsilon/3} \\
&= O_a\left(e^{-k^{\epsilon/3}}\right).
\end{aligned} \tag{3.8}$$

Now for the infinite range of integration we have

$$\begin{aligned}
\int_{k^{1/2-\epsilon/3}}^{\infty}\left|\phi_k^{(a)}(x)\right|dx &\le \int_{k^{1/2-\epsilon/3}}^{\infty}\left|\left(1-\frac{1}{x^2}\right)^{k-a}\sum_{j=0}^{a}\frac{p_{a,j}(k)}{x^{a+2j+1}}\right|dx \\
&\le \sum_{j=0}^{a}\left|p_{a,j}(k)\right|\int_{k^{1/2-\epsilon/3}}^{\infty}\frac{1}{x^{a+2j+1}}dx \\
&\le \sum_{j=0}^{a}\frac{1}{a+2j}\frac{\left|p_{a,j}(k)\right|}{k^{(1/2-\epsilon/3)(a+2j)}} \\
&\le \frac{1}{k^{a(1/2-\epsilon/3)}}\sum_{j=0}^{a}\frac{1}{a+2j}\frac{\left|p_{a,j}(k)\right|}{k^{j(1-2\epsilon/3)}} \\
&\le \frac{1}{k^{a(1/2-\epsilon/3)}}\sum_{j=0}^{a}\frac{1}{a+2j}O_a\left(k^{(2\epsilon/3)j}\right) \\
&\le \frac{1}{k^{a(1/2-\epsilon/3)}}O_a\left(k^{(2\epsilon/3)a}\right) = O_a\left(k^{-(a/2)(1-\epsilon/2)}\right).
\end{aligned} \tag{3.9}$$

Since this last estimate (3.9) surely dominates the smaller one in (3.8), we can allow the substitution of $\epsilon/2$ by ϵ to arrive at the desired (3.7). □

Remark 3.5. Actually there is no need for ϵ to be arbitraily small for the use this estimate (3.7) is destined for. If one takes, say, $\epsilon = 1$ we see that the order obtained is $O_a(k^{-a/4})$, which is plenty, since we are planning to differentiate a large but fixed number of times, while $k \to \infty$. The idea perhaps was to see how much we can eke out of this method, which is as close as $k^{-a/2}$ as desired. Perhaps a more painstaking analysis, taking into account the actual nature of the polynomials $p_{a,j}$, could yield an essentially faster order of convergence to zero, which is desirable for the effectiveness of the Maslanka representation.

At this point it is high time to connect the coefficients A_k with the function ϕ_k. So we have first a high school triviality, namely, the following lemma.

Lemma 3.6. *For any $x \neq 0$*

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(2j+1)\frac{1}{x^{2j+2}} = -\phi_k'(x). \tag{3.10}$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{k}(-1)^j\binom{k}{j}(2j+1)\frac{1}{x^{2j+2}} &= -\frac{d}{dx}\left(\frac{1}{x}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{x^{2j}}\right) \\
&= -\frac{d}{dx}\left(\frac{1}{x}\left(1-\frac{1}{x^2}\right)^k\right).
\end{aligned} \tag{3.11}$$

□

It should now be clear why the work on the derivatives of $\phi_k(x)$is carried out if the reader notices that the sum below should be estimated by Euler-Maclaurin with a large number of terms.

Lemma 3.7.

$$A_k = -\sum_{n=1}^{\infty} \phi_k'(n). \tag{3.12}$$

Proof. The following interchange of sums is totally elementary to justify:

$$\begin{aligned} A_k &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j+1)\zeta(2j+2) \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j+1) \sum_{n=1}^{\infty} \frac{1}{n^{2j+2}} \\ &= \sum_{n=1}^{\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j+1) \frac{1}{n^{2j+2}} \\ &= -\sum_{n=1}^{\infty} \phi_k'(n), \end{aligned} \tag{3.13}$$

where, of course, we applied Lemma 3.6 in the last equality. □

We are finally ready to prove the estimate for the A_k.

Proof of Proposition 3.1. Take an integer $a > 4p$ and assume that $k > a$; naturally one will make $k \to \infty$. The crucial elementary properties of ϕ_k and its derivatives expressed in Lemma 3.3 imply the remarkable fact that the application of the Euler-Maclaurin summation formula to a depth of a steps to the sum in (3.12), that is, to

$$A_k = -\sum_{n\geq 1} \phi_k'(n) \tag{3.14}$$

results in the sum being equal to the remainder term! Therefore,

$$A_k = -\frac{(-1)^a}{a!} \int_1^{\infty} \overline{B_a}(x) \phi_k^{(a)}(x)\, dx, \tag{3.15}$$

where $\overline{B_a}(x)$ is the ath periodified Bernoulli polynomial. Now apply Lemma 3.4 and equation (3.7) with $\epsilon = 1/2$ to get

$$\begin{aligned} |A_k| &\leq \frac{\left\|\overline{B_a}\right\|_\infty}{a!} \int_1^\infty \left|\phi_k^{(a)}(x)\right| dx \\ &\leq O_a\left(k^{-a/4}\right) = O_a\left(k^{-p}\right). \end{aligned} \tag{3.16}$$

□

Remark 3.8. We believe there is a room for improvement in estimating the smallness of the A_k. This should have a bearing on how useful the Maslanka formula may turn out to be in the end.

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