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Research Article

On the Spezialschar of Maass

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Let $M_k^{(n)}$ be the space of Siegel modular forms of degree n and even weight k. In this paper firstly a certain subspace $\mathrm{Spez}(M_k^{(2n)})$, the Spezialschar of $M_k^{(2n)}$, is introduced. In the setting of the Siegel threefold, it is proven that this Spezialschar is the Maass Spezialschar. Secondly, an embedding of $M_k^{(2)}$ into a direct sum $\bigoplus_{\upsilon=0}^{[k/10]} \mathrm{Sym}^2 M_{k+2\upsilon}$ is given. This leads to a basic characterization of the Spezialschar property. The results of this paper are directly related to the nonvanishing of certain special values of L-functions related to the Gross-Prasad conjecture. This is illustrated by a significant example in the paper.

1. Introduction

Maass introduced and applied in a series of papers [1–3] the concept of a Spezialschar to prove the Saito-Kurokawa conjecture [4, 5]. Let $M_k^{(2)}$ be the space of Siegel modular forms of degree 2 and weight k. Let $\mathbb{A}$ be the set of positive semidefinite half-integral matrices of degree 2. Hence $T \in \mathbb{A}$ can be identified with the quadratic form $T = [n, r, m]$. A modular form $F \in M_k^{(2)}$ is in the Spezialschar if the Fourier coefficients $A(T)$ of F satisfy the relation

$$A([n,r,m]) = \sum_{d|(n,r,m)} d^{k-1} A\left(\left[\frac{nm}{d^2}, \frac{r}{d}, 1\right]\right), \tag{1.1}$$

for all $T \in \mathbb{A}$. The space of special forms is called the Maass Spezialschar M_k^{Maass}.

The purpose of this paper is twofold. First we introduce the concept of the Spezialschar $\mathrm{Spez}(M_k^{(2n)})$ for Siegel modular forms of even degree $2n$. This is done in terms of the Hecke algebra $\mathscr{H}^n$ attached to Siegel modular forms of degree n. Let us fix the embedding

$$Sp_n \times Sp_m \longrightarrow Sp_{n+m},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & \tilde{a} & 0 & \tilde{b} \\ c & 0 & d & 0 \\ 0 & \tilde{c} & 0 & \tilde{d} \end{pmatrix}. \tag{1.2}$$

Let $|_k$ be the Petersson slash operator and let $\widetilde{T}$ be the normalized Heckeoperator $T \in \mathscr{H}^n$ (see (4.21)). Let $\bowtie_T = (\widetilde{T} \times 1_{2n}) - (1_{2n} \times \widetilde{T})$ and

$$\mathrm{Spez}\left(M_k^{(2n)}\right) := \left\{F \in M_k^{(2n)} | F|_k \bowtie_T = 0,\ \forall T \in \mathscr{H}^n\right\}. \tag{1.3}$$

Then we have the following theorem.

Theorem 1.1. *The Spezialschar introduced in this paper is the Maass Spezialschar in the case of the Siegel threefold.*

$$\mathrm{Spez}\left(M_k^{(2)}\right) = M_k^{\mathrm{Maass}}. \tag{1.4}$$

The second topic of this paper is the characterization of the space of Siegel modular forms of degree two and the corresponding Spezialschar in terms of Taylor coefficients and certain differential operators:

$$\mathfrak{D}_{k,\underline{2\nu}} : M_k^{(2)} \longrightarrow M_{k+2\nu}^{\mathrm{Sym}}, \tag{1.5}$$

here $\nu \in \mathbb{N}_0$ and $M_{k+2\nu}^{\mathrm{Sym}} = \mathrm{Sym}^2(M_{k+2\nu})$. To simplify the notation, let $M_k := M_k^{(1)}$. It would be very interesting to generalize this approach also to other situations, as to the Hermitian modular forms [6].

Before we summarize the main results, we give an example which also serves as an application. Let F_1, F_2, and F_3 be a Hecke eigenbasis of the space of Siegel cusp forms $S_{20}^{(2)}$ of weight 20 and degree 2. Let F_1 and F_2 generate the Maass Spezialschar. Let f_1 and f_2 be the normalized Hecke eigenbasis of $S_{24}^{(1)}$. Then we have

$$\mathfrak{D}_{20,\underline{4}}\, F_j = \alpha_j f_1 \otimes f_1 + \beta_j \left(f_1 \otimes f_2 + f_2 \otimes f_1\right) + \gamma_j f_2 \otimes f_2. \tag{1.6}$$

It it conjectured by Gross and Prasad [7] that the coefficients α_j, β_j, and γ_j are related to special values of certain automorphic L-functions. The Gross-Prasad conjecture has been proven by Ichino [8] for the Maass Spezialschar and $\nu = 0$. Moreover, we show in this paper that the vanishing at such special values has interesting consequences. We have $F_j \in S_{20}^{\mathrm{Maass}}$ if and only if the special value β_j is zero.

Theorem 1.2. *Let $k \in \mathbb{N}_0$ be even. Then we have the embedding*

$$\mathbb{D}_k = \bigoplus_{\nu=0}^{\lfloor k/10 \rfloor} \mathfrak{D}_{k,\underline{k+2\nu}} : M_k^{(2)} \longrightarrow M_k^{\mathrm{Sym}} \oplus S_{k+2}^{\mathrm{Sym}} \oplus \cdots \oplus S_{k+2\lfloor k/10 \rfloor}^{\mathrm{Sym}}. \tag{1.7}$$

For $F \in S_k^{(2)}$, we have $\mathfrak{D}_{k,\underline{0}} F \in S_k^{\mathrm{Sym}}$.

Surprisingly the Maass Spezialschar property can be recovered in $M_k^{\mathrm{Sym}} \oplus S_{k+2}^{\mathrm{Sym}} \oplus \cdots \oplus S_{k+2\lfloor k/10 \rfloor}^{\mathrm{Sym}}$ in the following transparent way. Let (f_j) be the normalized Hecke eigenbasis of M_k. Let us define the diagonal subspaces $M_k^D = \{\sum_j \alpha_j f_j \otimes f_j \in M_k^{\mathrm{Sym}}\}$ and $S_k^D = S_k^{\mathrm{Sym}} \bigcap M_k^D$. Then we can state the following theorem.

Theorem 1.3. *Let $F \in M_k^{(2)}$. Then we have*

$$F \in M_k^{\mathrm{Maass}} \Longleftrightarrow \mathbb{D}_k F \in M_k^D \oplus S_{k+2}^D \oplus \cdots \oplus S_{k+2\lfloor k/10 \rfloor}^D, \tag{1.8}$$

and similarly

$$F \in S_k^{\mathrm{Maass}} \Longleftrightarrow \mathbb{D}_k F \in \bigoplus_{\nu=0}^{\lfloor k/10 \rfloor} S_{k+2\nu}^D. \tag{1.9}$$

These two theorems give a transparent explanation of our example from a general point of view. In Section 5, we deduce an application related to a multiplicity one theorem of SL_2 instead of $GL(2)$.

Notation. Let $Z \in \mathbb{C}^{n,n}$ and tr the trace of a matrix; then we put $e\{Z\} = e^{2\pi i (\mathrm{tr}\, Z)}$. For $l \in \mathbb{Z}$, we define $\pi_l = (2\pi i)^l$. Let $x \in \mathbb{R}$; then we use Knuth's notation $\lfloor x \rfloor$ to denote the greatest integer smaller or equal to x. Let $\mathbb{A}_2$ denote the set of half-integral positive semidefinite matrices. We parametrize the elements $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ with $T = [n, r, m]$. The subset of positive-definite matrices we denote with $\mathbb{A}_2^+$.

2. Ultraspherical Differential Operators

The first two sections of this paper follow the strategy of Eichler and Zagier [9]. Nevertheless, there are several topics which are different (e.g., divisors of Jacobi forms and Siegel modular forms).

Let us start with the notation of the ultraspherical polynomial $p_{k,\underline{2\nu}}$. Let k and ν be elements of $\mathbb{N}_0$. Let a and b be elements of a commutative ring. Then we put

$$p_{k,\underline{2\nu}}(a,b) = \sum_{\mu=0}^{\nu} (-1)^{\mu} \frac{(2\nu)!}{\mu!(2\nu-2\mu)!} \frac{(k+2\nu-\mu-2)!}{(k+\nu-2)!} a^{2\nu-2\mu} b^{\mu}. \tag{2.1}$$

If we specialize the parameters, we have $p_{k,\underline{0}}(a,b) = 1$ and $p_{k,\underline{2\nu}}(0,0) = 0$ for $\nu \in \mathbb{N}$.

Let $\mathbb{H}_n$ be the Siegel upper half-space of degree n. Let $\mathtt{M}_k^{(n)}$ be the vector space of Siegel modular forms on $\mathbb{H}_n$ with respect to the full modular group $\Gamma_n = \mathtt{Sp}_n(\mathbb{Z})$. Moreover, let $\mathtt{S}_k^{(n)}$ denote the subspace of cusp forms. If $n = 1$, we drop the index to simplify notation. We denote the coordinates of the threefold $\mathbb{H}_2$ by $(\tau, z, \widetilde{\tau})$ for $\begin{pmatrix} \tau & z \\ z & \widetilde{\tau} \end{pmatrix} \in \mathbb{H}_2$ and put $q = e\{\tau\}$, $\xi = e\{z\}$, and $\widetilde{q} = e\{\widetilde{\tau}\}$. Let $\widehat{d_k}$ be the dimension of S_k.

Definition 2.1. Let $k, \nu \in \mathbb{N}_0$ and let k be even. Then we define the ultraspherical differential operator $\mathfrak{D}$ on the space of holomorphic functions F on $\mathbb{H}_2$ in the following way:

$$\mathfrak{D}_{k,\underline{2\nu}} F(\tau, \widetilde{\tau}) = p_{k,\underline{2\nu}} \left(\frac{1}{2\pi i} \frac{\partial}{\partial z}, \left(\frac{1}{2\pi i}\right)^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \widetilde{\tau}} \right) F \bigg|_{z=0} (\tau, \widetilde{\tau}). \tag{2.2}$$

In the case $\nu = 0$, we get the pullback $F(\tau, 0, \widetilde{\tau})$ of F on $\mathbb{H} \times \mathbb{H}$.

Let $F \in \mathtt{M}_k^{(2)}$ with Tth Fourier coefficient $A^F(n, r, m)$ for $T = [n, r, m] \in \mathbb{A}_2$. Then we have

$$\begin{aligned} \mathfrak{D}_{k,\underline{2\nu}} F\ (\tau, \widetilde{\tau}) &= \sum_{n,m=0}^{\infty} A_{\underline{2\nu}}^F(n, m) q^n \widetilde{q}^m \quad \text{with} \\ A_{\underline{2\nu}}^F(n, m) &= \sum_{r \in \mathbb{Z}, r^2 \leqslant 4nm} p_{k,\underline{2\nu}}(r, nm) A^F(n, r, m). \end{aligned} \tag{2.3}$$

Let $M_k^{\mathrm{Sym}} = \mathrm{Sym}^2 M_k$, $S_k^{\mathrm{Sym}} = \mathrm{Sym}^2 S_k$, and $S_k^{\mathrm{Sym}} = (S_k \otimes S_k)^{\mathrm{Sym}}$. Let us further introduce a related Jacobi differential operator $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m}$. This is given by exchanging $\pi_{-1}(\partial/\partial\widetilde{\tau})$ with m in the definition of the ultraspherical differential operator given in (2.2). Applying the operator $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m}$ on Jacobi forms $\Phi \in J_{k,m}$ of weight k and index m on $\mathbb{H} \times \mathbb{C}$ matches with the effect of the operator $\mathfrak{D}_{2\nu}$ introduced in [9, Section 3 formula (2)] on Φ.

Since $F \in M_k^{(2)}$ has a Fourier-Jacobi expansion of the form

$$F(\tau, z, \widetilde{\tau}) = \sum_{m=0}^{\infty} \Phi_m^F(\tau, z) \widetilde{q}^m, \quad \text{with } \Phi_m^F \in J_{k,m}, \tag{2.4}$$

it makes sense to consider $\mathfrak{D}_{k,\underline{2\nu}}$ with respect to this decomposition in a Fourier-Jacobi expansion

$$\mathfrak{D}_{k,\underline{2\nu}} = \bigoplus_{m=0}^{\infty} \mathfrak{D}_{k,\underline{2\nu}}^{J,m}. \tag{2.5}$$

Lemma 2.2. *Let $k, \nu \in \mathbb{N}_0$ and let k be even. Then $\mathfrak{D}_{k,\underline{2\nu}}$ maps $M_k^{(2)}$ to $M_{k+2\nu}^{\mathrm{Sym}}$ and to $S_{k+2\nu}^{\mathrm{Sym}}$ if $\nu \neq 0$. Moreover, the subspace $S_k^{(2)}$ of cusp forms is always mapped to $S_{k+2\nu}^{\mathrm{Sym}}$.*

Proof. We know from the work of Eichler and Zagier [9] since $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m} = \mathfrak{D}_{k,2\nu}$ that $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m} \Phi_m^F \in M_{k+2\nu}$. Let $\nu > 0$; then $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m} \Phi_m^F \in S_{k+2\nu}$ and for $F \in S_k^{(2)}$ we have $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m} \Phi_m^F \in S_{k+2\nu}$ for all

$\nu \in \mathbb{N}_0$. We are now ready to act with the ultraspherical differential operator with respect to its Fourier-Jacobi expansion directly on the Fourier-Jacobi expansion of F in a canonical way

$$\mathfrak{D}_{k,\underline{2\nu}} F\ (\tau, \tilde{\tau}) = \sum_{m=0}^{\infty} \left(\mathfrak{D}_{k,\underline{2\nu}}^{J,m} \Phi_m^F \right)(\tau) \tilde{q}^m, \tag{2.6}$$

where all "coefficients" $a_m^F(\tau) = \mathfrak{D}_{k,\underline{2\nu}}^{J,m} \Phi_m^F(\tau)$ are modular forms. This shows us, that if we apply the Peterson slash operator $|_{k+2\nu}\gamma$ here $\gamma \in \Gamma$ to this function with respect to the variable τ, the function is invariant. The same argument also works for the Fourier-Jacobi expansion with respect to τ. From this we deduce that $\mathfrak{D}_{k,\underline{2\nu}} F(\tau, \tilde{\tau}) = \sum_{i,j} \alpha_{i,j} f_i(\tau) f_j(\tilde{\tau})$. Here $(f_i)_i$ is a basis of $M_{k+2\nu}$. Finally the cuspidal conditions in the lemma also follow from symmetry arguments. □

Remark 2.3. Let $F : \mathbb{H}_2 \to \mathbb{C}$ be holomorphic. Let $g \in SL_2(\mathbb{R})$ and let $S = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$, where $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have

$$\begin{aligned} \mathfrak{D}_{k,\underline{2\nu}}\left(F|_k(g \times 1_2)\right) &= \left(\mathfrak{D}_{k,\underline{2\nu}}F\right)|_{k+2\nu}(g \times 1_2), \\ \mathfrak{D}_{k,\underline{2\nu}}\left(F|_k(1_2 \times g)\right) &= \left(\mathfrak{D}_{k,\underline{2\nu}}F\right)|_{k+2\nu}(1_2 \times g), \\ \mathfrak{D}_{k,\underline{2\nu}}(F|_k S)(\tau, \tilde{\tau}) &= \left(\mathfrak{D}_{k,\underline{2\nu}}F\right)|_{k+2\nu}(\tilde{\tau}, \tau). \end{aligned} \tag{2.7}$$

Remark 2.4. There are other possibilities for construction of differential operators as used in this section (see Ibukiyama for an overview [10]). But since the connection between our approach and the theory developed of Eichler and Zagier [9] is so useful we decided to do it this way. We also wanted to introduce the concept of Fourier-Jacobi expansion of differential operators, which is interesting in its own right.

3. Taylor Expansion of Siegel Modular Forms

The operators $\mathfrak{D}_{k,\underline{2\nu}}$ can be seen at this point as somewhat artificial. If we apply $\mathfrak{D}_{k,\underline{2\nu}}$ to Siegel modular forms F, we lose information. For example, we know that $\dim S_{20}^{(2)} = 3$ and contains a two-dimensional subspace of Saito-Kurokawa lifts. Since $\dim S_{20}^{\mathrm{Sym}} = 1$, we obviously lose information if we apply $\mathfrak{D}_{20,\underline{0}}$. But even worse let F_1 and F_2 be a Hecke eigenbasis of the space of Saito-Kurokawa lifts and F_3 a Hecke eigenform of the orthogonal complement; then we have $\mathfrak{D}_{20,\underline{0}} F_j \neq 0$ for $j = 1, 2, 3$. The general case seems to be even worse, since for example, $\dim M_k^{(2)} \sim k^3$ and $\dim M_k^{\mathrm{Sym}} \sim k^2$. On the other hand, from an optimistic viewpoint we may find about k pieces $\mathfrak{D}_{k,\underline{2\nu}} F$ which code all the relevant information needed to characterize the Siegel modular forms F.

Garrett in his papers [11, 12] introduced the method of calculating pullbacks of modular forms to study automorphic L-functions. We also would like to mention the work of Gelbart et al. at this point ([13]). And recently Ichino in his paper: "Pullbacks of Saito-Kurokawa lifts" [8] extended Garrett's ideas in a brilliant way to prove the Gross-Prasad conjecture [7] for Saito-Kurokawa lifts. In the new language we have introduced, it is obvious to consider Garrett's pullbacks as the 0th Taylor coefficients of F around $z = 0$. Hence it seems to be very lucrative to study also the higher Taylor coefficients and hopefully get some transparent link.

Let $k \in \mathbb{N}_0$ be even. Let $F \in M_k^{(2)}$ and $\Phi \in J_{k,m}$. Then we denote by

$$F(\tau, z, \widetilde{\tau}) = \sum_{\nu=0}^{\infty} \chi_{2\nu}^{F}(\tau, \widetilde{\tau}) z^{2\nu}, \qquad \Phi(\tau, z) = \sum_{\nu=0}^{\infty} \chi_{2\nu}^{\Phi}(\tau) z^{2\nu}, \tag{3.1}$$

the corresponding Taylor expansions with respect to z around $z = 0$. Here we already used the invariance of F and Φ with respect to the transformation $z \mapsto (-z)$ since k is even. Suppose $\chi_{2\nu_0}$ is the first nonvanishing Taylor coefficient; then we denote $2\nu_0$ the vanishing order of the underlying form. If the form is identically zero, we define the vanishing order to be ∞. To simplify our notation, we introduce normalizing factor

$$\gamma_{k,\underline{2\nu}} = \left(\frac{1}{2\pi i}\right)^{2\nu} \frac{(k+2\nu-2)!(2\nu)!}{(k+2\nu/2-2)!}. \tag{3.2}$$

Further we put

$$\chi_{2\nu}^{\mu,\mu} = \frac{\partial^{2\mu}}{\partial\tau^{\mu}\partial\widetilde{\tau}^{\mu}} \chi_{2\nu}^{F}, \qquad \xi_{2\nu}^{\mu,\mu} = \left(\gamma_{k,\underline{2\nu}}\right)^{-1} \frac{\partial^{2\mu}}{\partial\tau^{\mu}\partial\widetilde{\tau}^{\mu}} \mathfrak{D}_{k,\underline{2\nu}} F. \tag{3.3}$$

Then a straightforward calculation leads to the following useful formula.

Lemma 3.1. *Let $k, \nu \in \mathbb{N}_0$ and let k be even. Let $F \in M_k^{(2)}$. Then we have*

$$\left(\mathfrak{D}_{k,\underline{2\nu}} F\right)(\tau, \widetilde{\tau}) = \gamma_{k,\underline{2\nu}} \sum_{\mu=0}^{\nu} (-1)^{\mu} \frac{(k+2\nu-\mu-2)!}{(k+2\nu-2)!\mu!} \left(\frac{\partial^{2\mu} \chi_{2\nu-2\mu}^{F}}{\partial\tau^{\mu}\partial\widetilde{\tau}^{\mu}}\right). \tag{3.4}$$

A similar formula is valid for Jacobi forms with normalizing factor $\gamma_{k,\underline{2\nu}}^{\mathrm{J},m} = \gamma_{k,\underline{2\nu}}$.

Corollary 3.2. *Let $2\nu_0$ be the vanishing order of $F \in M_k^{(2)}$. Then we have $\mathfrak{D}_{k,\underline{2\nu}} F = 0$ for $\nu < \nu_0$ and*

$$\mathfrak{D}_{k,\underline{2\nu_0}} F(\tau, \widetilde{\tau}) = \gamma_{k,\underline{2\nu}} \chi_{2\nu_0}^{F}(\tau, \widetilde{\tau}) \in M_{k+2\nu_0}^{\mathrm{Sym}} \setminus \{0\}. \tag{3.5}$$

Similarly, we have for $\Phi \in J_{k,m}$ with vanishing order $2\nu_0$ the properties $\mathfrak{D}_{k,\underline{2\nu}}^{\mathrm{J},m} \Phi = 0$ for $\nu < \nu_0$ and $\mathfrak{D}_{k,\underline{2\nu_0}}^{\mathrm{J},m} \Phi = \gamma_{k,\underline{2\nu}} \chi_{2\nu_0}^{\Phi} \in M_{k+2\nu_0}$.

Example 3.3. It is well known that $\dim S_{10}^{(2)} = 1$. Let $\Phi = \Phi_{10} \in S_{10}^{(2)}$ be normalized in such a way that $A^{\Phi}(1,1,1) = 1$. Then it follows from $\mathfrak{D}_{10,\underline{0}} \Phi = 0$ that $A^{\Phi}(1,0,1) = -2$ since $\dim\ S_{10}^{\mathrm{Sym}} = 0$. Then Φ has the Taylor expansion

$$\Phi_{10}(\tau, z, \widetilde{\tau}) = \frac{3}{5}\pi_2 \Delta(\tau)\Delta(\widetilde{\tau}) z^2 + \Delta'(\tau)\Delta'(\widetilde{\tau}) z^4 + O\left(z^6\right), \tag{3.6}$$

we can also express the Taylor coefficients $\chi_{2\nu}^F$ in terms of the modular forms $\mathfrak{D}_{k,\underline{2\nu}}F$. This can be done by inverting the formula (3.4). Finally we get

$$\chi_{2\nu} = \sum_{\mu=0}^{\nu} \frac{(k+2\nu-2\mu-1)!}{(k+2\nu-\mu-1)!\mu!} \, \xi_{2\nu-2\mu}^{\mu,\mu}. \tag{3.7}$$

Before we state our first main result about the entropy of the family $\mathfrak{D}_{k,\underline{0}}F$, $\mathfrak{D}_{k,\underline{2}}F$, $\mathfrak{D}_{k,\underline{4}}F, \ldots$ we introduce some further notation.

$$\begin{aligned} \mathbb{W}_k &= S_k^{\mathrm{Sym}} \oplus \bigoplus_{j=1}^{\lfloor k/10 \rfloor} S_{k+2j}^{\mathrm{Sym}}, \\ \mathbb{W}_k^{\mathrm{cusp}} &= S_k^{\mathrm{Sym}} \oplus \bigoplus_{j=1}^{\lfloor k/10 \rfloor} S_{k+2j}^{\mathrm{Sym}}. \end{aligned} \tag{3.8}$$

These spaces will be the target of our next consideration. More precisely we define a linear map from the space of Siegel modular forms of degree 2 into these spaces with remarkable properties. The following result is equivalent to Theorem 1.2 in the introduction, given with a slightly different notation.

Theorem 3.4. *Let $k \in \mathbb{N}_0$ be even. Then we have the linear embedding*

$$\mathbb{D}_k : \begin{cases} M_k^{(2)} \hookrightarrow \mathbb{W}_k, \\ F \longmapsto \bigoplus_{\nu=0}^{\lfloor k/10 \rfloor} \mathfrak{D}_{k,\underline{2\nu}}F. \end{cases} \tag{3.9}$$

Since $\mathfrak{D}_{k,\underline{0}}S_k^{(2)}$ is cuspidal we have the embedding of $S_k^{(2)}$ into $\mathbb{W}_k^{\mathrm{cusp}}$.

Remark 3.5. It can be deduced from [14] that $\mathfrak{D}_{k,\underline{0}} \oplus \mathfrak{D}_{k,\underline{2}}$ is surjective. Hence for $k < 20$ we have

(i) $M_k^{(2)}$ is isomorphic to M_k for $k < 10$;

(ii) $M_k^{(2)}$ is isomorphic to $M_k^{\mathrm{Sym}} \oplus S_k^{\mathrm{Sym}}$ for $10 \leqslant k < 20$ and $S_k^{(2)} \simeq S_k \oplus S_{k+2}$.

Proof. First of all we recall that we have already shown that $\mathfrak{D}_{k,\underline{0}}M_k^{(2)} \subseteq M_k^{\mathrm{Sym}}$ and $\mathfrak{D}_{k,\underline{2\nu}}M_k^{(2)} \subseteq S_{k+2\nu}^{\mathrm{Sym}}$ for $\nu > 0$. Let $F \in M_k^{(2)}$ and suppose that $\mathbb{D}_kF$ is identically zero. Then it would follow from our inversion formula (3.7) that

$$F(\tau, z, \widetilde{\tau}) = \sum_{\nu=\lfloor k/10 \rfloor+1}^{\infty} \chi_{2\nu}^F(\tau, \widetilde{\tau}) z^{2\nu}. \tag{3.10}$$

For such F the general theory of Siegel modular forms of degree 2 says that the special function $\Phi_{10} \in S_k^{(2)}$, which we already studied in one of our examples, divides F in the $\mathbb{C}$-algebra of modular forms. And this is fulfilled at least with a power of $\lfloor k/10 \rfloor + 1 = t_k > 0$.

Hence there exists a Siegel modular form G of weight $k-10t_k$. But since this weight is negative and nontrivial Siegel modular forms of negative weight do not exist, the form G has to be identically zero. Hence we have shown that if $\mathbb{D}_k F \equiv 0$ then $F \equiv 0$. And this proves the statement of the theorem. □

Remark 3.6. The number $\lfloor k/10 \rfloor$ in the Theorem is optimal. This follows directly from properties of Φ_{10}.

Remark 3.7. Let $E_k^{2,1}(f)$ be a Klingen Eisenstein series attached to $f \in S_k$. Let E_k denote an elliptic Eisenstein series of weight k. Then it can be deduced from [12] that $\mathfrak{D}_{k,\underline{0}} E_k^{2,1}(f) = f \otimes E_k + E_k \otimes f \bmod S_k^{\mathrm{Sym}}$.

Remark 3.8. It would be interesting to have a different proof of the Theorem 3.4 independent of the special properties of Φ_{10}.

Remark 3.9. The asymptotic limit of the dimension of the quotient of $\mathbb{W}_k / M_k^{(2)}$ is equal to 91/25. Let us put $d_k = \dim M_k$.

(i) The dimension of the target space $\mathbb{W}_k$ is given by

$$\begin{aligned} \dim \mathbb{W}_k &\sim \frac{1}{288} \int_0^{k/10} (k+2x)^2 dx \\ &\sim \frac{1}{288} \frac{1}{2 \cdot 3} \frac{91}{5^3} k^3. \end{aligned} \tag{3.11}$$

(ii) The asymptotic dimension formula of $M_k^{(2)}$ is given by

$$\dim M_k^{(2)} \sim \frac{1}{288} \frac{1}{2 \cdot 3 \cdot 5} k^3. \tag{3.12}$$

(see, [1, Introduction]).

4. The Spezialschar

In this section, we first recall some basic facts on the Maass Spezialschar [5]. Then we determine the image of the Spezialschar in the space $\mathbb{W}_k$ for all even weights k. Then finally we introduce a Spezialschar as a certain subspace of the space of Siegel modular forms of degree $2n$ and weight k. Then we show that in the case $n = 1$ this Spezialschar coincides with the Maass Spezialschar.

4.1. Basics of the Maass Spezialschar

Let $J_{k,m}$ be the space of Jacobi forms of weight k and index m. We denote the subspace of cusp forms with $J_{k,m}^{\mathrm{cusp}}$. Let $|_{k,m}$ be the slash operator for Jacobi forms and V_l ($l \in \mathbb{N}_0$) the operator, which maps $J_{k,m}$ to $J_{k,ml}$. More precisely, let $\Phi(\tau, z) = \sum c(n,r)q^n\xi^r \in J_{k,m}$. Then $(\Phi|_{k,m}V_l)(\tau, z) = \sum c^*(n,r)q^n\xi^r$ with

$$c^*(n,r) = \sum_{a|(n,r,l)} a^{k-1} c\left(\frac{nl}{a^2}, \frac{r}{a}\right), \quad \text{for } l \in \mathbb{N}, \tag{4.1}$$

and for $l = 0$, we have $c^*(0,0) = c(0,0)(-2k/B_{2k})$ and for $l = 0$ and $n > 0$ we have $c^*(n,r) = c(0,0)\sigma_{k-1}(n)$. This includes the theory of Eisenstein series in a nice way [9].

Definition 4.1. The lifting $\mathcal{V}$ is given by the linear map

$$\mathcal{V} : \begin{cases} J_{k,1} \longrightarrow M_k^{(2)}, \\ \Phi \longmapsto \displaystyle\sum_{l=0}^{\infty} (\Phi|_{k,1}V_l)\tilde{q}^l. \end{cases} \tag{4.2}$$

The image of this lifting is the Maass Spezialschar M_k^{Maass} of weight k. The subspace of cusp forms we denote with S_k^{Maass}.

Remark 4.2.

(i) The lifting is invariant by the Klingen parabolic of $Sp_2(\mathbb{Z})$. Since the Fourier coefficients satisfy $A(n,r,m) = A(m,r,n)$, the map $\mathcal{V}$ is well defined.

(ii) If we restrict the Saito-Kurokawa lifting to Jacobi cusp forms, we get Siegel cusp forms.

(iii) Let $\Phi \in J_{k,m}$ and $l, \mu \in \mathbb{N}_0$. Then we have

$$\mathfrak{D}_{k,2\underline{\mu}}^{J,ml}(\Phi|_{k,m}V_l) = \left(\mathfrak{D}_{k,2\underline{\mu}}^{J,m}\Phi\right)\Big|_k T_l. \tag{4.3}$$

Here T_l is the Hecke operator on the space of elliptic modular forms.

(iv) Let $F \in M_k^{\text{Maass}}$ be the lift of $\Phi \in J_{k,1}$. Then F is a Hecke eigenform if and only if Φ is a Hecke-Jacobi eigenform.

From this consideration we conclude [9] the following.

Proposition 4.3. *Let $F \in M_k^{(2)}$ be a Siegel modular form. Then the following properties are equivalent.*

Arithmetic

Let $A(n,r,m)$ denote the Fourier coefficients of F then

$$A(n,r,m) = \sum_{d|(n,r,m)} d^{k-1} A\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right). \tag{4.4}$$

Lifting

Let Φ_1^F be the first Fourier-Jacobi coefficient of F. Then all other Fourier-Jacobi coefficients satisfy the identity

$$\Phi_m^F = \Phi_1^F|_{k,1}V_m. \tag{4.5}$$

Let $F \in S_k^{(2)}$ be a Hecke eigenform. Then F is a Saito-Kurokawa lift if and only if the spinor L-function $Z(F,s)$ of degree 4 has a pole (see Evdokimov [15]). Similar results are also obtained by Oda [16].

4.2. The Diagonal of $\mathbb{W}_k$ and the Proof of Theorem 1.3

Let (f_j) be the normalized Hecke eigenbasis of M_k. With this notation, we introduce the diagonal space

$$M_k^D = \left\{ \sum_j \alpha_j f_j \otimes f_j \in M_k^{\mathrm{Sym}} \right\}, \tag{4.6}$$

and the corresponding cuspidal subspace S_k^D. Now we are ready to distinguish the Maass Spezialschar in the vector spaces $\mathbb{W}_k$ and $\mathbb{W}_k^{\mathrm{cusp}}$. This leads to Theorem 1.3 stated in the introduction. Before we give a proof we note the following.

Remark 4.4. (i) Theorem 1.3 describes a link between Siegel modular forms and elliptic Hecke eigenforms.

(ii) Let $F \in M_{20}^{(2)}$ and let (f_j) be a Hecke eigenbasis of S_{24}. Then $F \in M_k^{\mathrm{Maass}}$ if and only if

$$\mathfrak{D}_{20,\underline{4}}F = \alpha_0 E_{24} \otimes E_{24} + \alpha f_1 \otimes f_1 + \gamma f_2 \otimes f_2; \tag{4.7}$$

here $\alpha_0, \alpha, \gamma \in \mathbb{C}$.

Proof. We first show that if F is in the Maass Spezialschar then $\mathfrak{D}_{k,\underline{2\nu}}F$ is an element of the diagonal space. Let $\nu \in \mathbb{N}_0$ and let Φ_1^F be the first Fourier-Jacobi coefficient of F. Then we have

$$\left(\mathfrak{D}_{k,\underline{2\nu}}(\mathcal{V}\Phi)\right)(\tau,\widetilde{\tau}) = \sum_{l=0}^{\infty} \left(\mathfrak{D}_{k,\underline{2\nu}}^{J,l}(\Phi|_{k,1}V_l)\right)(\tau)\widetilde{q}^l. \tag{4.8}$$

Here we applied the Fourier-Jacobi expansion of the differential operator $\mathfrak{D}_{k,\underline{2\nu}}$ acting on Siegel modular forms. Then we used the formula (4.3) to interchange the operators $\mathfrak{D}_{k,\underline{2\nu}}^{J,l}$ and V_l to get

$$\left(\mathfrak{D}_{k,\underline{2\nu}}F\right)(\tau,\widetilde{\tau}) = \sum_{l=0}^{\infty} \left(\mathfrak{D}_{k,\underline{2\nu}}^{J,1}\Phi\right)\Big|_k T_l \widetilde{q}^l. \tag{4.9}$$

Now let $(f_j^{k+2\nu})_{j=1}^{\widehat{d}_{k+2\nu}}$ be a normalized Hecke eigenbasis of $S_{k+2\nu}$. Let $1 \leq j_1, j_2 \leq \widehat{d}_{k+2\nu}$. Then we have

$$\left\langle (\mathfrak{D}_{k,\underline{2\nu}} F)(*, \widetilde{\tau}), f_{j_1}^{k+2\nu} \right\rangle = \left\langle k\left(\mathfrak{D}_{k,\underline{2\nu}}^{J,1} \Phi\right), f_{j_1}^{k+2\nu} \right\rangle f_{j_1}^{k+2\nu}, \tag{4.10}$$

which leads to the desired result

$$\left\langle (\mathfrak{D}_{k,\underline{2\nu}} F), f_{j_1}^{k+2\nu} \otimes f_{j_2}^{k+2\nu} \right\rangle = 0, \quad \text{for } j_1 \neq j_2. \tag{4.11}$$

It remains to look at the Eisenstein part if $\nu = 0$. Since the space of Eisenstein series has the basis E_k and is orthogonal to the functions given in (4.10), we have proven that the Spezialschar property of F implies that $\mathbb{D}_k F \in \mathbb{W}_k^{\mathbb{D}}$.

Now let us assume that $F \notin M_k^{\text{Maass}}$. Then we show that $\mathbb{D}_k F \notin \mathbb{W}_k^{\mathbb{D}}$. Since the map

$$\left(\mathfrak{D}_{k,\underline{0}} \oplus \mathfrak{D}_{k,\underline{2}}\right) : M_k^{\text{Maass}} \longrightarrow M_k^D \oplus S_{k+2}^D \tag{4.12}$$

is an isomorphism, we can assume that $(\mathfrak{D}_{k,\underline{0}} \oplus \mathfrak{D}_{k,\underline{2}})(F)$ projected on $M_k^D \oplus S_{k+2}^D$ is identically zero. Altering F by an element of the Maass Spezialschar does not change the property we have to prove. If $\mathfrak{D}_{k,\underline{0}} F \notin M_k^D$ or $\mathfrak{D}_{k,\underline{2}} F \notin S_{k+2}^D$, we are done; otherwise we can assume that

$$\left(\mathfrak{D}_{k,\underline{0}} \oplus \mathfrak{D}_{k,\underline{2}}\right)(F) \equiv 0. \tag{4.13}$$

Then we have the order $F = 2\nu_0 \geqslant 4$ and $k \geqslant 20$, since $F \notin M_k^{\text{Maass}}$. Let

$$F\begin{pmatrix} \tau & z \\ z & \widetilde{\tau} \end{pmatrix} = \sum_{\nu=\nu_0}^{\infty} \chi_{2\nu_0}^F(\tau, \widetilde{\tau}) z^{2\nu} \tag{4.14}$$

be the Taylor expansion of F with $\chi_{2\nu_0}^F(\tau, \widetilde{\tau}) \in S_{k+2\nu_0}$ not identically zero. Let $\Phi_{10} \in S_{10}^{(2)}$ be the Siegel cusp form of weight 10 and degree 2. It has the properties that $\chi_0^{\Phi_{10}} \equiv 0$ and $\chi_2^{\Phi_{10}}(\tau, \widetilde{\tau}) = c\Delta(\tau)\Delta(\widetilde{\tau})$ with $c \neq 0$. Since order $F = 2\nu_0$, we also have

$$\Phi_{10}^{\nu_0} \| F. \tag{4.15}$$

This means that there exists a $G \in S_{k-10\nu_0}$ such that χ_0^G is nontrivial and

$$F = (\Phi_{10})^{\nu_0} G. \tag{4.16}$$

Hence we have for the first nontrivial Taylor coefficient of F, the formula

$$\begin{aligned} \chi_{2\nu_0}^F(\tau, \widetilde{\tau}) &= \left(\chi_2^{\Phi_{10}}(\tau, \widetilde{\tau})\right)^{\nu_0} \chi_0^G(\tau, \widetilde{\tau}), \\ &= c^{\nu_0} \Delta(\tau)^{\nu_0} \Delta(\widetilde{\tau})^{\nu_0} \chi_0^G(\tau, \widetilde{\tau}). \end{aligned} \tag{4.17}$$

And the coefficient $a_1(\widetilde{\tau})$ of q is identically zero. Now let us assume for a moment that $\chi_{2\nu_0}^F \in S_{k+2\nu_0}^D$. Then we have

$$\chi_{2\nu_0}^F(\tau, \widetilde{\tau}) = \sum_{l=1}^{\widehat{d}_{k+2\nu}} \alpha_l f_l^{k+2\nu_0}(\tau) f_l^{k+2\nu_0}(\widetilde{\tau}), \tag{4.18}$$

and the coefficient of q is given by $\sum_{l=1}^{\widehat{d}_{k+2\nu_0}} \alpha_l f_l^{k+2\nu_0}(\widetilde{\tau})$. Since $(f_l^{k+2\nu_0})_{l=1}^{\widehat{d}_{k+2\nu_0}}$ is a basis we have $\alpha_1 = \cdots = \alpha_{\widehat{d}_{k+2\nu_0}} = 0$. But since we assumed that order $F = 2\nu_0$, we have a reductio ad absurdum. Hence we have shown that $\chi_{2\nu_0}^F \notin S_{k+2\nu_0}$ which proves our theorem. □

Corollary 4.5. *Klingen Eisenstein series are not in the Maass Spezialschar.*

Remark 4.6. Let k be a natural even number. Let F be a Siegel modular form of degree two and weight k. Then we have

$$F \in M_k^{\text{Maass}} \Longleftrightarrow \mathfrak{D}_{k,\underline{2\nu}} F \in M_{k+2\nu}^D, \quad \forall \nu \in \mathbb{N}_0. \tag{4.19}$$

4.3. The Spezialschar

Let $G^+Sp_n(\mathbb{Q})$ be the rational symplectic group with positive similitude μ. In the sense of Shimura, we attach to Hecke pairs the corresponding Hecke algebras

$$\begin{aligned} \mathscr{H}^n &= \left(\Gamma_n, G^+Sp_n(\mathbb{Q})\right), \\ \mathscr{H}_0^n &= \left(\Gamma_n, Sp_n(\mathbb{Q})\right). \end{aligned} \tag{4.20}$$

We also would like to mention that in the setting of elliptic modular forms the classical Hecke operator $T(p)$ can be normalized such that it is an element of the full Hecke algebra $\mathscr{H}^1$, but not of the even one $\mathscr{H}_0^1$. Let g $\in G^+Sp_n(\mathbb{Q})$ with similitude $\mu(g)$. Then we put

$$\widetilde{g} = \mu(g)^{-1/2}\ g \tag{4.21}$$

to obtain an element of $Sp_n(\mathbb{R})$. We further extend this to $\mathscr{H}^n$.

Definition 4.7. Let $T \in \mathscr{H}^n$. Then we define

$$\bowtie_T = \left(\widetilde{T} \times 1_{2n}\right) - \left(1_{2n} \times \widetilde{T}\right). \tag{4.22}$$

Here $\times$ is the standard embedding of (Sp_n, Sp_n) into Sp_{2n}.

Now we study the action $|_k \bowtie_T$ on the space of modular forms of degree $2n$ for all $T \in \mathscr{H}^n$ or $T \in \mathscr{H}_0^n$. The first thing we would like to mention is that for $F \in M_k^{(2n)}$ the function $F|_k \bowtie_T$ is in general not an element of $M_k^{(2n)}$ anymore. Anyway at the moment we are much more interested in the properties of the kernel of a certain map related to this action. In particular, in the case $n = 1$ we get a new description of the Maass Spezialschar.

Definition 4.8. Let n and k be natural numbers. Let $M_k^{(2n)}$ be the space of Siegel modular forms of degree $2n$ and weight k. Then we introduce the Spezialschar corresponding to the Hecke algebras $\mathscr{H}^n$ and $\mathscr{H}_0^n$.

$$\begin{aligned} \operatorname{Spez}\left(M_k^{(2n)}\right) &= \left\{F \in M_k^{(2n)} \mid F|_k \bowtie_T = 0, \quad \forall T \in \mathscr{H}^n\right\}, \\ \operatorname{Spez}_0\left(M_k^{(2n)}\right) &= \left\{F \in M_k^{(2n)} \mid F|_k \bowtie_T = 0, \quad \forall T \in \mathscr{H}_0^n\right\}. \end{aligned} \tag{4.23}$$

Moreover, $\operatorname{Spez}(S_k^{(2n)})$ and $\operatorname{Spez}_0(S_k^{(2n)})$ are the cuspidal part of the corresponding Spezialschar.

It is obvious that these subspaces of $M_k^{(2n)}$ are candidates for finding spaces of modular forms with distinguished Fourier coefficients. Further it turns that these spaces are related to the Maass Spezialschar and the Ikeda lift [17]. This leads to Theorem 1.1 stated in the introduction.

4.3.1. *Proof of Theorem 1.1*

Proof. Let $F \in M_k^{(2)}$. Then we have $F \in M_k^{\mathrm{Maass}}$ if and only if $\mathfrak{D}_{k,\underline{2\nu}} F \in M_{k+2\nu}^D$ for all $\nu \in \mathbb{N}_0$. This follows from Remark 4.6. On the other side, the property $\mathfrak{D}_{k,\underline{2\nu}} F \in M_{k+2\nu}^D$ is equivalent to the identity

$$\left(\mathfrak{D}_{k,\underline{2\nu}} F\right)\big|_{k+2\nu} \bowtie_T = 0, \qquad \forall T \in \mathscr{H}. \tag{4.24}$$

This follows from the fact that the Hecke operators are self-adjoint and that the space of elliptic modular forms has multiplicity one. To make the operator well defined, we used the embedding $\mathbb{H} \times \mathbb{H}$ into the diagonal of $\mathbb{H}_2$. We can now interchange the differential operators $\mathfrak{D}_{k,\underline{2\nu}}$ and the Petersson slash operator $|_*$. This leads to

$$\mathfrak{D}_{k,\underline{2\nu}} F \in M_{k+2\nu}^D \Longleftrightarrow \mathfrak{D}_{k,\underline{2\nu}}\left(F|_k \bowtie_T\right) = 0. \tag{4.25}$$

So finally it remains to show that if $\mathfrak{D}_{k,\underline{2\nu}}(F|_k \bowtie_T) = 0$ for all $\nu \in \mathbb{N}_0$ then it follows $F|_k \bowtie_T = 0$. By looking at the Taylor expansion of the function $F|_k \bowtie_T \begin{pmatrix} \tau & z \\ z & \tilde{\tau} \end{pmatrix}$ with respect to z around 0, we get with the same argument as given in the proof of Theorem 3.4 the desired result. □

5. Maass Relations Revised

We introduced two Hecke algebras $\mathscr{H}$ and $\mathscr{H}_0$ related to elliptic modular forms. For the corresponding Spezialschar $\operatorname{Spez}(M_k^{(2)})$ and $\operatorname{Spez}_0(M_k^{(2)})$, we obtain the following theorem.

Theorem 5.1. *Let k be an even natural number. Then the even Spezialschar $\mathrm{Spez}_0(M_k^{(2)})$ related to the Hecke algebra $\mathscr{H}_0$ which is locally generated by $T(p^2)$ is equal to the Spezialschar $\mathrm{Spez}(M_k^{(2)})$ related to the Hecke algebra $\mathscr{H}$ which is locally generated by $T(p)$.*

$$\mathrm{Spez}_0\left(M_k^{(2)}\right) = \mathrm{Spez}\left(M_k^{(2)}\right). \tag{5.1}$$

Proof. Let $F \in M_k^{(2)}$. We proceed as follows. In the proof of Theorem 1.1, it has been shown that

$$F \in \mathrm{Spez}\left(M_k^{(2)}\right) \Longleftrightarrow \left(\mathfrak{D}_{k,\underline{2\nu}}F\right)\big|_{k+2\nu}\bowtie_T = 0, \quad \forall T \in \mathscr{H},\ \nu \in \mathbb{N}_0. \tag{5.2}$$

Now we show that

$$\left(\mathfrak{D}_{k,\underline{2\nu}}F\right)\big|_{k+2\nu}\bowtie_{T(p)} = 0 \Longleftrightarrow \left(\mathfrak{D}_{k,\underline{2\nu}}F\right)\big|_{k+2\nu}\bowtie_{T(p^2)} = 0, \tag{5.3}$$

for all $\nu \in \mathbb{N}_0$ and prime numbers p. This would finish the proof since

$$F \in \mathrm{Spez}_0\left(M_k^{(2)}\right) \Longleftrightarrow \left(\mathfrak{D}_{k,\underline{2\nu}}F\right)\big|_{k+2\nu}\bowtie_T = 0, \quad \forall T \in \mathscr{H}_0,\ \nu \in \mathbb{N}_0. \tag{5.4}$$

(this can also be obtained by following the procedure of the proof of Theorem 1.1).

To verify (5.3), we show that to be an element of the kernel of the operator $|\bowtie_{T(p^2)}$ implies already to be an element of the kernel of $|\bowtie_{T(p)}$.

To see this we give a more general proof. Let $\phi \in M_k^{\mathrm{Sym}}$ and let $\phi|_k\bowtie_{T(p^2)} = 0$. Let (f_j) be a normalized Hecke eigenbasis of M_k. Then we have

$$\phi = \sum_{i,j}\alpha_{i,j} f_i \otimes f_j. \tag{5.5}$$

Let us assume that there exists an $\alpha_{i_0,j_0} \neq 0$ with $i_0 \neq j_0$. Let us denote $\lambda_l(p^2)$ to be the eigenvalue of f_l with respect to the Hecke operator $T(p^2)$. Then we have

$$0 = \phi|_k\bowtie_{T(p^2)} = \sum_{i,j}\alpha_{i,j}\left(\lambda_i\left(p^2\right) - \lambda_j\left(p^2\right)\right) f_i \otimes f_j. \tag{5.6}$$

From this follows that $\lambda_{i_0}(p^2) = \lambda_{j_0}(p^2)$ for all prime numbers p. It is easy to see at this point that f_{i_0} and f_{j_0} have to be cusp forms. In the setting of cusp forms, we can apply a result on multiplicity one for SL_2 of D. Ramakrishnan [18, Section 4.1] and other people obtain $f_{i_0} = f_{j_0}$. Since this is a contradiction we have $\phi \in M_k^D$. In other words, we have $\phi|_k\bowtie_{T(p)} = 0$. □

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