Research Article

Some Results on Warped Product Submanifolds of a Sasakian Manifold

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Received 5 October 2009; Revised 18 December 2009; Accepted 13 February 2010

Academic Editor: Frédéric Robert

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We study warped product Pseudo-slant submanifolds of Sasakian manifolds. We prove a theorem for the existence of warped product submanifolds of a Sasakian manifold in terms of the canonical structure F .

1. Introduction

The notion of slant submanifold of almost contact metric manifold was introduced by Lotta [1]. Latter, Cabrerizo et al. investigated slant and semislant submanifolds of a Sasakian manifold and obtained many interesting results [2, 3].

The notion of warped product manifolds was introduced by Bishop and O'Neill in [4]. Latter on, many research articles appeared exploring the existence or nonexistence of warped product submanifolds in different spaces (cf. [5–7]). The study of warped product semislant submanifolds of Kaehler manifolds was introduced by Sahin [8]. Recently, Hasegawa and Mihai proved that warped product of the type $N_{\perp} \times_{\lambda} N_T$ in Sasakian manifolds is trivial where N_T and $N_{\perp}$ are ϕ -invariant and anti-invariant submanifolds of a Sasakian manifold, respectively [9].

In this paper we study warped product submanifolds of a Sasakian manifold. We will see in this paper that for a warped product of the type $M = N_1 \times_{\lambda} N_2$, if N_1 is any Riemannian submanifold tangent to the structure vector field ξ of a Sasakian manifold $\bar{M}$ then N_2 is an anti-invariant submanifold and if ξ is tangent to N_2 then there is no warped product. Also, we will show that the warped product of the type $M = N_{\perp} \times_{\lambda} N_{\theta}$ of a Sasakian manifold $\bar{M}$ is trivial and that the warped product of the type $N_T \times_{\lambda} N_{\perp}$ exists and obtains a result in terms of canonical structure.

2. Preliminaries

Let $\overline{M}$ be a $(2m + 1)$ -dimensional manifold with almost contact structure (ϕ, ξ, η) defined by a $(1, 1)$ tensor field ϕ , a vector field ξ , and the dual 1-form η of ξ , satisfying the following properties [10]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \quad (2.1)$$

There always exists a Riemannian metric g on an almost contact manifold $\overline{M}$ satisfying the following compatibility condition:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.2)$$

An almost contact metric manifold $\overline{M}$ is called *Sasakian* if

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (2.3)$$

for all X, Y in $T\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of g on $\overline{M}$. From (2.3), it follows that

$$\overline{\nabla}_X \xi = -\phi X. \quad (2.4)$$

Let M be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M , respectively, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.5)$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.6)$$

for each $X, Y \in TM$ and $N \in T^\perp M$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \quad (2.7)$$

where g denotes the Riemannian metric on $\overline{M}$ as well as the one induced on M .

For any $X \in TM$, we write

$$\phi X = PX + FX, \quad (2.8)$$

where PX is the tangential component and FX is the normal component of ϕX .

Similarly, for any $N \in T^\perp M$, we write

$$\phi N = tN + fN, \quad (2.9)$$

where tN is the tangential component and fN is the normal component of ϕN . We shall always consider ξ to be tangent to M . The submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in T^\perp M$, for any $X \in TM$.

For each nonzero vector X tangent to M at x , such that X is not proportional to ξ , we denote by $\theta(X)$ the angle between ϕX and PX .

M is said to be *slant* [3] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $x \in M$. The angle θ is called *slant angle* or *Wirtinger angle*. Obviously, if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. If the slant angle of M is different from 0 and $\pi/2$ then it is called *proper slant*.

A characterization of slant submanifolds is given by the following.

Theorem 2.1 (see [3]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\delta \in [0, 1]$ such that*

$$P^2 = \delta(-I + \eta \otimes \xi). \quad (2.10)$$

Furthermore, in such case, if θ is slant angle, then $\delta = \cos^2 \theta$.

Following relations are straightforward consequences of (2.10)

$$g(PX, PY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (2.11)$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)] \quad (2.12)$$

for any X, Y tangent to M .

3. Warped and Doubly Warped Product Manifolds

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and λ a positive differentiable function on N_1 . The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_\lambda N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + \lambda^2 g_2. \quad (3.1)$$

A warped product manifold $N_1 \times_\lambda N_2$ is said to be *trivial* if the warping function λ is constant. We recall the following general formula on a warped product [4]:

$$\nabla_X V = \nabla_V X = (X \ln \lambda) V, \quad (3.2)$$

where X is tangent to N_1 and V is tangent to N_2 .

Let $M = N_1 \times_{f_1} N_2$ be a warped product manifold then N_1 is totally geodesic and N_2 is totally umbilical submanifold of M , respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by Ünal [11]. A *doubly warped product manifold* of N_1 and N_2 , denoted as $_{f_2}N_1 \times_{f_1} N_2$ is the manifold $N_1 \times N_2$ endowed with a metric g defined as

$$g = f_2^2 g_1 + f_1^2 g_2 \quad (3.3)$$

where f_1 and f_2 are positive differentiable functions on N_1 and N_2 , respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1)Z + (Z \ln f_2)X \quad (3.4)$$

for each X in TN_1 and Z in TN_2 [7].

If neither f_1 nor f_2 is constant we have a nontrivial doubly warped product $M =_{f_2} N_1 \times_{f_1} N_2$. Obviously in this case both N_1 and N_2 are totally umbilical submanifolds of M .

Now, we consider a doubly warped product of two Riemannian manifolds N_1 and N_2 embedded into a Sasakian manifold $\overline{M}$ such that the structure vector field ξ is tangent to the submanifold $M =_{f_2} N_1 \times_{f_1} N_2$. Consider ξ is tangent to N_1 , then for any $V \in TN_2$ we have

$$\nabla_V \xi = (\xi \ln f_1)V + (V \ln f_2)\xi. \quad (3.5)$$

Thus from (2.4), (2.5), (2.8), and (3.5), we get

$$\overline{\nabla}_V \xi = (\xi \ln f_1)V + (V \ln f_2)\xi + h(V, \xi) = -PV - FV. \quad (3.6)$$

On comparing tangential and normal parts and using the fact that ξ , V , and PV are mutually orthogonal vector fields, (3.6) implies that

$$V \ln f_2 = 0, \quad \xi \ln f_1 = 0, \quad h(V, \xi) = -FV, \quad PV = 0. \quad (3.7)$$

This shows that f_2 is constant and N_2 is an anti-invariant submanifold of $\overline{M}$, if the structure vector field ξ is tangent to N_1 .

Similarly, if ξ is tangent to N_2 and for any $U \in TN_1$ we have

$$\overline{\nabla}_U \xi = (\xi \ln f_2)U + (U \ln f_1)\xi + h(U, \xi) = -PU - FU, \quad (3.8)$$

which gives

$$U \ln f_1 = 0, \quad \xi \ln f_2 = 0, \quad PU = 0, \quad h(U, \xi) = -FU. \quad (3.9)$$

That is, f_1 is constant and N_1 is an anti-invariant submanifold of $\overline{M}$.

Note 1. From the above conclusion we see that for warped product submanifolds $M = N_1 \times_\lambda N_2$ of a Sasakian manifold $\overline{M}$, if the structure vector field ξ is tangent to the first factor N_1 then second factor N_2 is an anti-invariant submanifold. On the other hand the warped product $M = N_1 \times_\lambda N_2$ is trivial if the structure vector field ξ is tangent to N_2 .

To study the warped product submanifolds $N_1 \times_\lambda N_2$ with structure vector field ξ tangent to N_1 , we have obtained the following lemma.

Lemma 3.1 (see [12]). *Let $M = N_1 \times_\lambda N_2$ be a proper warped product submanifold of a Sasakian manifold $\overline{M}$, with $\xi \in TN_1$, where N_1 and N_2 are any Riemannian submanifolds of $\overline{M}$. Then*

- (i) $\xi \ln \lambda = 0$,
- (ii) $A_{FZ}X = -th(X, Z)$,
- (iii) $g(h(X, Z), FY) = g(h(X, Y), FZ)$,
- (iv) $g(h(X, Z), FW) = g(h(X, W), FZ)$

for any $X, Y \in TN_1$ and $Z, W \in TN_2$.

4. Warped Product Pseudoslant Submanifolds

The study of semislant submanifolds of almost contact metric manifolds was introduced by Cabrerizo et.al. [2]. A semislant submanifold M of an almost contact metric manifold $\overline{M}$ is a submanifold which admits two orthogonal complementary distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ such that $\mathfrak{D}$ is invariant under ϕ and $\mathfrak{D}^\theta$ is slant with slant angle $\theta \neq 0$, that is, $\phi\mathfrak{D} = \mathfrak{D}$ and ϕZ makes a constant angle θ with TM for each $Z \in \mathfrak{D}^\theta$. In particular, if $\theta = \pi/2$, then a semislant submanifold reduces to a contact CR-submanifold. For a semislant submanifold M of an almost contact metric manifold, we have

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \{\xi\}. \quad (4.1)$$

Similarly we say that M is an *pseudo-slant submanifold* of $\overline{M}$ if $\mathfrak{D}$ is an anti-invariant distribution of M , that is, $\phi\mathfrak{D} \subseteq T^\perp M$ and $\mathfrak{D}^\theta$ is slant with slant angle $\theta \neq 0$. The normal bundle $T^\perp M$ of an pseudo-slant submanifold is decomposed as

$$T^\perp M = FTM \oplus \mu, \quad (4.2)$$

where μ is an invariant subbundle of $T^\perp M$.

From the above note, we see that for warped product submanifolds $N_1 \times_\lambda N_2$ of a Sasakian manifold $\overline{M}$, one of the factors is an anti-invariant submanifold of $\overline{M}$. Thus, if the manifolds N_θ and $N_\perp$ are slant and anti-invariant submanifolds of Sasakian manifold $\overline{M}$, then their possible warped product pseudo-slant submanifolds may be given by one of the following forms:

- (a) $N_\perp \times_\lambda N_\theta$,
- (b) $N_\theta \times_\lambda N_\perp$.

The above two types of warped product pseudo-slant submanifolds are trivial if the structure vector field ξ is tangent to N_θ and $N_\perp$, respectively. Here, we are concerned with the other two cases for the above two types of warped product pseudo-slant submanifolds $N_\perp \times_\lambda N_\theta$ and $N_\theta \times_\lambda N_\perp$ when ξ is in $TN_\perp$ and in TN_θ , respectively.

For the warped product of the type (a), we have

Theorem 4.1. *There do not exist the warped product Pseudo-slant submanifolds $M = N_\perp \times_\lambda N_\theta$ where $N_\perp$ is an anti-invariant and N_θ is a proper slant submanifold of a Sasakian manifold $\overline{M}$ such that ξ is tangent to $N_\perp$.*

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$, we have

$$(\overline{\nabla}_X \phi)Z = \overline{\nabla}_X \phi Z - \phi \overline{\nabla}_X Z. \quad (4.3)$$

Using (2.3), (2.5), (2.6), and the fact that ξ is tangent to $N_\perp$, we obtain

$$-\eta(Z)X = -A_{FZ}X + \nabla_X^\perp FZ - P\nabla_X Z - F\nabla_X Z - th(X, Z) - fh(X, Z). \quad (4.4)$$

Comparing tangential and normal parts, we get

$$\eta(Z)X = A_{FZ}X + P\nabla_X Z + th(X, Z) \quad (4.5)$$

Equation (4.5) takes the form on using (3.2) as

$$\eta(Z)X = A_{FZ}X + (Z \ln \lambda)PX + th(X, Z). \quad (4.6)$$

Taking product with PX , the left hand side of the above equation is zero using the fact that X and PX are mutually orthogonal vector fields. Then

$$0 = g(A_{FZ}X, PX) + (Z \ln \lambda)g(PX, PX) + g(th(X, Z), PX). \quad (4.7)$$

Using (2.7), (2.11) and the fact that ξ is tangent to $N_\perp$, we get

$$(Z \ln \lambda)\cos^2\theta\|X\|^2 = g(h(X, Z), FPX) - g(h(X, PX), FZ). \quad (4.8)$$

As $\theta \neq \pi/2$, then interchanging X by PX and taking account of (2.10), we obtain

$$(Z \ln \lambda)\cos^4\theta\|X\|^2 = -\cos^2\theta g(h(PX, Z), FX) + \cos^2\theta g(h(X, PX), FZ) \quad (4.9)$$

or

$$(Z \ln \lambda)\cos^2\theta\|X\|^2 = g(h(X, PX), FZ) - g(h(PX, Z), FX). \quad (4.10)$$

Adding equations (4.8) and (4.10), we get

$$2(Z \ln \lambda) \cos^2 \theta \|X\|^2 = g(h(X, Z), FPX) - g(h(PX, Z), FX). \quad (4.11)$$

The right hand side of the above equation is zero by Lemma 3.1(iv); then

$$(Z \ln \lambda) \cos^2 \theta \|X\|^2 = 0. \quad (4.12)$$

Since N_θ is proper slant and X is nonnull, then

$$Z \ln \lambda = 0. \quad (4.13)$$

In particular, for $Z = \xi \in TN_\perp$, Lemma 3.1 (i) implies that $\xi \ln \lambda = 0$. This means that λ is constant on $N_\perp$. Hence the theorem is proved. $\square$

Now, the other case is dealt with in the following theorem.

Theorem 4.2. *Let $M = N_T \times_\lambda N_\perp$ be a warped product submanifold of a Sasakian manifold $\overline{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$. Then $(\overline{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in TN_T$ and $Z \in TN_\perp$.*

Proof. As $M = N_T \times_\lambda N_\perp$ is a warped product submanifold with ξ tangent to N_T , then by (2.3),

$$(\overline{\nabla}_X \phi)Z = 0, \quad (4.14)$$

for any $X \in TN_T$ and $Z \in TN_\perp$. Using this fact in the formula

$$(\overline{\nabla}_U \phi)V = \overline{\nabla}_U \phi V - \phi \overline{\nabla}_U V \quad (4.15)$$

for each $U, V \in T\overline{M}$, thus, we obtain

$$\overline{\nabla}_X \phi Z = \phi \overline{\nabla}_X Z. \quad (4.16)$$

Then from (2.5) and (2.6), we get

$$-A_{FZ}X + \nabla_X^\perp FZ = \phi(\nabla_X Z + h(X, Z)). \quad (4.17)$$

Which on using (2.8) and (2.9) yields

$$-A_{FZ}X + \nabla_X^\perp FZ = P\nabla_X Z + F\nabla_X Z + th(X, Z) + fh(X, Z). \quad (4.18)$$

From the normal components of the above equation, formula (3.2) gives

$$\nabla_X^\perp FZ = (X \ln \lambda)FZ + fh(X, Z). \quad (4.19)$$

Taking the product in (4.19) with FW_1 for any $W_1 \in TN_\perp$, we get

$$g\left(\nabla_X^\perp FZ, FW_1\right) = (X \ln \lambda)g(FZ, FW_1) + g(fh(X, Z), FW_1) \quad (4.20)$$

or

$$g\left(\nabla_X^\perp FZ, FW_1\right) = (X \ln \lambda)g(\phi Z, \phi W_1) + g(\phi h(X, Z), \phi W_1). \quad (4.21)$$

Then from (2.2), we have

$$g\left(\nabla_X^\perp FZ, FW_1\right) = (X \ln \lambda)g(Z, W_1). \quad (4.22)$$

On the other hand, we have

$$\left(\overline{\nabla}_X F\right)Z = \nabla_X^\perp FZ - F\nabla_X Z. \quad (4.23)$$

Taking the product in (4.23) with FW_1 for any $W_1 \in TN_\perp$ and using (4.22), (2.2), (3.2), and the fact that ξ is tangential to N_T , we obtain that

$$g\left(\left(\overline{\nabla}_X F\right)Z, FW_1\right) = 0, \quad (4.24)$$

for any $X \in TN_T$ and $Z, W_1 \in TN_\perp$.

Now, if $W_2 \in TN_T$ then using the formula (4.23), we get

$$g\left(\left(\overline{\nabla}_X F\right)Z, \phi W_2\right) = g\left(\nabla_X^\perp FZ, \phi W_2\right) - g(F\nabla_X Z, \phi W_2). \quad (4.25)$$

As N_T is an invariant submanifold, then $\phi W_2 \in TN_T$ for any $W_2 \in TN_T$, thus using the fact that the product of tangential component with normal is zero, we obtain that

$$g\left(\left(\overline{\nabla}_X F\right)Z, \phi W_2\right) = 0, \quad (4.26)$$

for any $X, W_2 \in TN_T$ and $Z \in TN_\perp$. Thus from (4.24) and (4.26), it follows that $\left(\overline{\nabla}_X F\right)Z \in \mu$. Thus the proof is complete. $\square$

Acknowledgment

The authors are thankful to the referee for his valuable suggestion and comments which have improved this paper.

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