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Research Article

Integral HOMFLY-PT and sl(n)-Link Homology

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Using the diagrammatic calculus for Soergel bimodules, developed by Elias and Khovanov, as well as Rasmussen's spectral sequence, we construct an integral version of HOMFLY-PT and sl(n)-link homology.

1. Introduction

During the past half-decade, categorification and, in particular, that of topological invariants has flourished into a subject of its own right. It has been a study finding connections and ramifications over a vast spectrum of mathematics, including areas such as low-dimensional topology, representation theory, and algebraic geometry. Following the original work of Khovanov on the categorification of the Jones polynomial in [1], came a slew of link homology theories lifting other quantum invariants. With a construction that utilized matrix factorizations, a tool previously developed in an algebra-geometric context, Khovanov and Rozansky produced the sl(n) and HOMFLY-PT link homology theories. Albeit computationally intensive, it was clear from the onset that thick interlacing structure was hidden within. The most insightful and influential work in uncovering these innerconnections was that of Rasmussen in [2], where he constructed a spectral sequence from the HOMFLY-PT to the sl(n)-link homology. This was a major step in deconstructing the web of how these theories come together, yet many structural questions remained and still remain unanswered, waiting for a new approach. Close to the time of the original work, Khovanov produced an equivalent categorification of the HOMFLY-PT polynomial in [3], but this time using Hochschild homology of Soergel bimodules and Rouquier complexes of [4]. The latter proved to be more computation-friendly and was used by Webster to calculate many examples in [5].

In the meantime, a new flavor of categorification came into light. With the work of Khovanov and Lauda on the categorification of quantum groups in [6], a diagrammatic calculus originating in the study of 2 categories arrived into the foreground. This graphical approach proved quite fruitful and was soon used by Elias and Khovanov to rewrite the work of Soergel in [7], and en suite by Elias and the author to repackage Rouquier's complexes and to prove that they are functorial over braid cobordisms [8] (not just projectively functorial as was known before). We note the closely related independent construction of Chuang and Rouquier in [9, 10]. An immediate advantage to the diagrammatic construction was a comparative ease of calculation.

As there has yet to be seen an integral version of either HOMFLY-PT or $\mathrm{sl}(n)$-link homology, with the original Khovanov homology being defined over $\mathbb{Z}$ and torsion playing an interesting role, a natural question arose as to whether this graphical calculus could be used to define these. The definition of such integral theories is precisely the purpose of this paper. The one immediate disadvantage to the graphical approach is that at the present moment there does not exist a diagrammatic calculus for the Hochschild homology of Soergel bimodules. Hence, to define integral HOMFLY-PT homology, our paper takes a rather roundabout way, jumping between matrix factorizations and diagrammatic Rouquier complexes whenever one is deemed more advantageous than the other. For the $\mathrm{sl}(n)$ version of the story, we add the Rasmussen spectral sequence into the mix and essentially repeat his construction in our context.

When choosing what to define in full and what to leave out, we assume the reader's familiarity with [8]. The organization of the paper is the following: in Section 2, we give a brief account of the necessary tools (matrix factorizations, Soergel bimodules, Hochschild homology, Rouquier complexes, and corresponding diagrammatics)—the emphasis here is brevity and we refer the reader to more original sources for particulars and details; in Sections 3 and 4, we describe the integral HOMFLY-PT complex and prove the Reidemeister moves, utilizing all of the background in Section 2; Section 5 is devoted to the Rasmussen spectral sequence and integral $\mathrm{sl}(n)$-link homology. We conclude it with some remarks and questions.

Throughout the paper, we will refer to a positive crossing as the one labelled D_+ and negative crossing as the one labelled D_- in Figure 1. For resolutions of a crossing, we will refer to D_o and D_s of Figure 1 as the "oriented" and "singular" resolutions, respectively. We will use the following conventions for the HOMLFY-PT polynomial:

$$aP(D_-) - a^{-1}P(D_+) = \left(q - q^{-1}\right)P(D_o), \tag{1.1}$$

with P of the unknot being 1. Substituting $a = q^n$ we arrive at the quantum $\mathrm{sl}(n)$-link polynomial.

2. The Toolkit

We will require some knowledge of matrix factorizations, Soergel bimodules, and Rouquier complexes, as well as the corresponding diagrammatic calculus of Elias and Khovanov [7]. In this section the reader will find a brief survery of the necessary tools, and for more details we refer him to the following papers: matrix factorizations [2, 11], Soergel bimodules and Rouquier complexes and diagrammatics [3, 4, 7, 8], and Hochschild homology [3, 12].

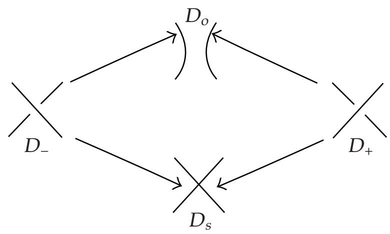


Figure 1: Crossings and resolutions (note: these are braid diagrams).

2.1. Matrix Factorizations

Definition 2.1. Let R be a Noetherian commutative ring, $w \in R$, and C^*, $* \in \mathbb{Z}$, a free graded R-module. A *$\mathbb{Z}$-graded matrix factorization with potential $w \in R$* consists of C^* and a pair of differentials $d_\pm : C^* \to C^{*\pm 1}$, such that $(d_+ + d_-)^2 = wId_{C^*}$.

A morphism of two matrix factorizations C^* and D^* is a homomorphism of graded R-modules $f : C^* \to D^*$ that commutes with both d_+ and d_-. The tensor product $C^* \otimes D^*$ is taken as the regular tensor product of complexes, and is itself a matrix factorization with diffentials d_+ and d_-. A useful and easy exercise is the following.

Lemma 2.2. *Given two matrix factorizations C^* and D^* with potenials w_c and w_d, respectively, the tensor product $C^* \otimes D^*$ is a matrix factorization with potential $w_c + w_d$.*

Remark 2.3. Following Rasmussen [2], we work with $\mathbb{Z}$-graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded, matrix factorizations as in [11]. The $\mathbb{Z}$-grading implies that $(d_+ + d_-)^2 = wId_{C^*}$ is equivalent to

$$\begin{aligned} d_+^2 = d_-^2 &= 0, \\ d_+d_- + d_-d_+ &= wId_{C^*}. \end{aligned} \tag{2.1}$$

In the case that $w = 0$, we acquire a new $\mathbb{Z}/2\mathbb{Z}$-graded chain complex structure with differential $d_+ + d_-$.

Suppressing the underlying ring R and potential w, we will denote the category of graded matrix factorizations by mf.

We also need the notion of complexes of matrix factorizations. If we visualize a collection of matrix factorizations as sitting horizontally in the plane at each integer level, with differentials d_+ and d_- running right and left, respectively, we can think of morphisms $\{d_v\}$ between these as running in the vertical direction. All together, we have that

$$d_\pm : C^{i,j} \longrightarrow C^{i\pm 1,j}, \qquad d_v : C^{i,j} \longrightarrow C^{i,j+1}, \tag{2.2}$$

where we think of i as the *horizontal* grading and j as the *vertical* grading, and will denote these as gr_h and gr_v, respectively.

In addition we will be taking tensor products of complexes of matrix factorizations (in the obvious way) and, just to add to the confusion, we will also have homotopies of these complexes as well homotopies of matrix factorizations themselves. These notions will land us in different categories for which we now give some notation.

(i) hmf will denote the *homotopy category of matrix factorizations.*

(ii) $\mathcal{KOM}(mf)$ will denote the *category of complexes of matrix factorizations.*

(iii) $\mathcal{KOM}_h(mf)$ will denote *homotopy category of complexes of matrix factorizations.*

(iv) $\mathcal{KOM}_h(hmf)$ will denote the obvious conglomerate.

2.2. Diagrammatics of Soergel Bimodules

The category of Soergel bimodules $\mathcal{SC}_1$ is a monoidal category generated by objects B_i, where $i \in I$ is a finite indexing set, which satisfy

$$\begin{aligned} &B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}, \\ &B_i \otimes B_j \cong B_j \otimes B_i, \quad \text{for distant } i, j, \\ &B_i \otimes B_j \otimes B_i \oplus B_j \cong B_j \otimes B_i \otimes B_j \oplus B_i, \quad \text{for adjacent } i, j. \end{aligned} \tag{2.3}$$

These objects B_i are graded and the notation $\{j\}$ refers a grading shift of $+j$. Technically speaking this should be called the category of Bott-Samuelson bimodules and the "real" category of Soergel bimodules is obtained as described at the end of this section. A key feature of this category is that the Grothendieck group of $\mathcal{SC}(I)$ is isomorphic to the Hecke algebra $\mathcal{H}$ of type A_∞ over the ring $\mathbb{Z}[t, t^{-1}]$. We refer the reader to [7, 8] for defenitions and relvant details.

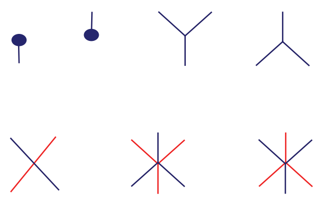
More concretely, the Soergel bimodule $B_i = R\bigotimes_{R^i} R\{-1\}$, where $R = \mathbb{Z}[x_1 - x_2, \dots, x_{n-1} - x_n]$ with $\deg x_i = 2$, $\{m\}$ denotes the grading shift by m, and R^i is the subring of invariants corresponding to the permutation $(i, i+1)$ under the natural action of S_n on the variables. There is some flexibility as to the exact description of R, but we work with the most convenient for the constructions below (note that our grading shift of -1 in the definition of B_i is absent from the contruction of [3]). We have that $B_\emptyset = R$ itself, and $B_{\underline{i}} = B_{i_1} \bigotimes_R B_{i_2} \bigotimes_R \cdots \bigotimes_R B_{i_d}$, where $\underline{i}$ is denotes the sequence $\{i_1, i_2, \dots, i_d\}$, that is,

$$\begin{aligned} B_{\underline{i}} &= \left(R \bigotimes_{R^{i_1}} R\{-1\} \right) \otimes \left(R \bigotimes_{R^{i_2}} R\{-1\} \right) \otimes \cdots \otimes \left(R \bigotimes_{R^{i_d}} R\{-1\} \right) \\ &= R \bigotimes_{R^{i_1}} R \otimes R \bigotimes_{R^{i_2}} R \otimes \cdots \otimes R \bigotimes_{R^{i_d}} R\{-d\} \\ &= R \bigotimes_{R^{i_1}} \otimes R \bigotimes_{R^{i_2}} R \bigotimes_{R^{i_3}} \cdots R \bigotimes_{R^{i_d}} R\{-d\}. \end{aligned} \tag{2.4}$$

One useful feature of this categorification is that it is easy to calculate the dimension of Hom spaces in each degree. Let $\mathrm{HOM}(M,N) \overset{\mathrm{def}}{=} \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}(M,N\{m\})$ be the graded vector space (actually an R-bimodule) generated by homogeneous morphisms of all degrees. Then $\mathrm{HOM}(B_{\underline{i}},B_{\underline{j}})$ is a free left R-module, and its graded rank over R is given by a natural bilinear form $(b_{\underline{i}},b_{\underline{j}})$ defined on the Hecke algebra $\mathcal{H}$. For more information on this categorification and related topics we refer the reader to [7, 13].

The graphical counterpart, which we will also refer to as $\mathcal{SC}_1$ was given a diagrammatic presentation by generators and relations, allowing morphisms to be viewed as isotopy classes of certain graphs.

An object in $\mathcal{SC}_1$ is given by a sequence of indices $\underline{i}$, which is visualized as d points on the real line $\mathbb{R}$, labelled or "colored" by the indices in order from left to right. Sometimes these objects are also called $B_{\underline{i}}$. Morphisms are given by pictures embedded in the strip $\mathbb{R}\times[0,1]$ (modulo certain relations), constructed by gluing the following generators horizontally and vertically:



For instance, if "blue" corresponds to the index i and "red" to j, then the lower right generator is a morphism from jij to iji. The generating pictures above may exist in various colors, although there are some restrictions based on adjacency conditions.

We can view a morphism as an embedding of a planar graph, satisfying the following properties:

(1) edges of the graph are colored by indices from 1 to n;

(2) edges may run into the boundary $\mathbb{R}\times\{0,1\}$, yielding two sequences of colored points on $\mathbb{R}$, the top boundary $\underline{i}$, and the bottom boundary $\underline{j}$. In this case, the graph is viewed as a morphism from $\underline{j}$ to $\underline{i}$;

(3) only four types of vertices exist in this graph: univalent vertices or "dots," trivalent vertices with all three adjoining edges of the same color, 4-valent vertices whose adjoining edges alternate in colors between distant i and j, and 6-valent vertices whose adjoining edges alternate between adjacent i and j.

The degree of a graph is +1 for each dot and −1 for each trivalent vertex. 4-valent and 6-valent vertices are of degree 0. The term *graph* henceforth refers to such a graph embedding.

By convention, we color the edges with different colors, but do not specify which colors match up with which $i \in I$. This is legitimate, as only the various adjacency relations between colors are relevant for any relations or calculations. We will specify adjacency for all pictures, although one can generally deduce it from the fact that 6-valent vertices only join adjacent colors, and 4-valent vertices join only distant colors.

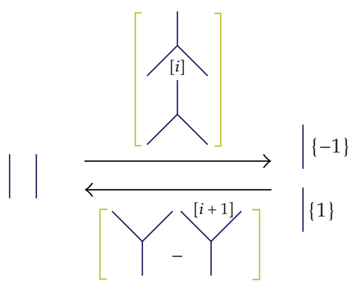
In addition to the bimodules $B_{\underline{i}}$ above, we will require the use of the bimodule $R\otimes_{R^{i,i+1}}R\{-3\}$, where $R^{i,i+1}$ is the ring of invariants under the transpositions $(i,i+1)$ and

$(i+1,\ i+2)$, and will use a black squiggly line, as in (2.7) below, to represent it. This bimodule comes into play in the isomorphisms:

$$B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus \left(R \bigotimes_{R^{i,i+1}} R\{-3\} \right),$$
$$B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus \left(R \bigotimes_{R^{i,i+1}} R\{-3\} \right), \tag{2.5}$$

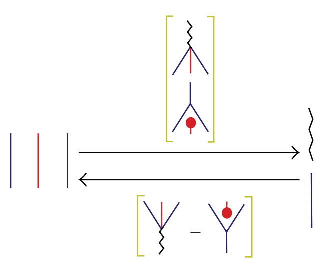
which we will use in the proof of Reidemeister move III. As usual in a diagrammatic category, composition of morphisms is given by vertical concatenation, and the monoidal structure is given by horizontal concatenation.

We then allow $\mathbb{Z}$-linear sums of graphs, and apply relations to obtain our category $\mathcal{SC}_1$. The relations come in three flavors: one color, two distant colors, two adjacent and one distant, and three mutually distant colors. We do not list all of them here, just the consequences necessary for the calculations at hand, and refer the reader to [7, 11] for a complete picture. Our graphs are invariant under isotopy and in addition, we have the following isomorphisms or "decompositions":



(2.6)

The vertical juxtapositions of diagrams corresponds to direct sums of morphisms and $[i]$ corresponds to the morphism induced by multiplication by the polynomial x_i. Note that this relation is precisely that of 1 described diagrammatically.



(2.7)

Here, we have the graphical counterpart of 4 and 5.

Remark 2.4. Primarily we will work in another category denoted $\mathcal{SC}_2$, the category formally containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$, but whose morphisms are forced to be degree 0. In addition, we let $\mathcal{SC}$ be the Karoubi envelope, or idempotent completion, of the category $\mathcal{SC}_2$. Recall that the Karoubi envelope of a category $\mathcal{C}$ has as objects pairs (B, e) where B is an object in $\mathcal{C}$ and e an idempotent endomorphism of B. This object acts as though it were the "image" of this projection e, and in an additive category behaves like a direct summand. For more information on Karoubi envelopes, see Wikipedia. It is really here that the object $R\bigotimes_{R^{i,i+1}} R\{-3\}$ of 4 and 5 resides. In practice all our calculations will be done in $\mathcal{SC}_2$, but since $\mathcal{SC}_2$ includes fully faithfully into $\mathcal{SC}$ they will be valid there as well.

The important fact here is that there is a functor from $\mathcal{SC}$ to the category of R-bimodules, sending a line colored i to B_i and each generator to an appropriate bimodule map. The functor gives an equivalence of categories between this diagrammatic category and the subcategory $\mathcal{SC}_1$ of R-bimodules mentioned in the previous section, so the use of the same name is legitimate.

Our diagrammatic category has many wonderful properties, such as the self-adjointness of B_i, which permits us to "twist" morphisms around and view any morphism as one from or to the empty diagram. This allows for a very hands-on, explicit, understanding of hompaces between objects in $\mathcal{SC}_1$, which was key in proving functoriality in [8].

2.3. Hochschild (Co)Homology

Let A be a $\Bbbk$-algebra and M an A-bimodule, or equivalently a left $A \otimes A^{op}$-module or a right $A^{op} \otimes A$-module. The definitions of the *Hochschild (co)homology* groups $HH_*(A, M)$ ($HH^*(A, M)$) are the following:

$$HH_*(A, M) := \operatorname{Tor}_*^{A\otimes A^{op}}(M, A), \qquad HH^*(A, M) := \operatorname{Ext}^*_{A\otimes A^{op}}(A, M). \tag{2.8}$$

To compute them we take a projective resolution of the A-bimodule A, with the natural left and right action, by projective A-bimodules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \tag{2.9}$$

and tensor this with M over $A \otimes A^{op}$ to get

$$\cdots \longrightarrow P_2 \bigotimes_{A\otimes A^{op}} M \longrightarrow P_1 \bigotimes_{A\otimes A^{op}} M \longrightarrow P_0 \bigotimes_{A\otimes A^{op}} M \longrightarrow 0. \tag{2.10}$$

The homology of this complex is isomorphic to $HH_*(A, M)$.

Example 2.5. For any bimodule M, we have

$$HH_0(A, M) \cong \frac{M}{[A, M]}, \qquad HH^0(A, M) \cong M^A, \tag{2.11}$$

where $[A, M]$ is the subspace of M generated by all elements of the form $am - ma$, $a \in A$ and $m \in M$, and $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. We leave this as an exercise or refer the reader to [12].

For the polynomial algebra $A = \Bbbk[x_1, \ldots, x_n]$, with $\Bbbk$ commutative, we can use the Koszul resolution of A by free $A \otimes A$-modules, which is the tensor product of the following complexes:

$$0 \longrightarrow A \otimes A \xrightarrow{x_i \otimes 1 - 1 \otimes x_i} A \otimes A \longrightarrow 0, \tag{2.12}$$

for $1 \geq i \geq n$. This resolution has length n, and its total space is naturally isomorphic to the exterior algebra on n generators tensored with $A \otimes A$. Hence, we have that the complex which computes Hochschild homology of a bimodule M over A is made up of 2^n copies of M, with the differentials coming from multiplication by $x_i \otimes 1 - 1 \otimes x_i$. In other words

$$0 \longrightarrow C_n(M) \longrightarrow \cdots \longrightarrow C_1(M) \longrightarrow C_0(M) \longrightarrow 0, \tag{2.13}$$

with

$$C_j(M) = \bigoplus_{I \subset \{1, \ldots, n\}, |I| = j} M \bigotimes_{\mathbb{Z}} \mathbb{Z}[I], \tag{2.14}$$

where $\mathbb{Z}[I]$ is the rank 1 free abelian group generated by the symbol $[I]$ (i.e., it is there to keep track where exactly we are in the complex). Here, the differential takes the form

$$d(m \otimes [I]) = \sum_{i \in I} \pm (x_i m - m x_i) \otimes [I \setminus \{i\}], \tag{2.15}$$

and the sign is taken as negative if I contains an odd number of elements less than i.

Remark 2.6. For the polynomial algebra, the Hochschild homology and cohomology are isomorphic,

$$HH_i(A, M) \cong HH^{n-i}(A, M), \tag{2.16}$$

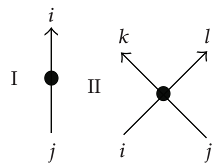
for any bimodule M. This comes from self-duality of the Koszul resolution for such algebras. Hence, we will be free to use either homology or cohomology groups in the constructions below.

For us, taking Hochschild homology will come into play when looking at closed braid diagrams. To a given resolution of a braid diagram we will assign a Soergel bimodule; “closing off” this diagram will correspond to taking Hochschild homology of the associated bimodule. More details are below in Section 3.2.

3. The Integral HOMFLY-PT Complex

3.1. The Matrix Factorization Construction

As stated above we will work with $\mathbb{Z}$-graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded, matrix factorizations and follow closely the conventions laid out in [2]. We begin by first assigning the appropriate complex to a single crossing and then extend this to general braids.



(a) (b)

Figure 2

Gradings

Our complex will be triply graded, coming from the internal or "quantum" grading of the underlying ring, the homological grading of the matrix factorizations, and finally an overall homological grading of the entire complex. It will be convenient to visualize our complexes in the plane with the latter two homological gradings lying in the horizontal and vertical directions, respectively. We will denoted these gradings by $(i, j, k) = (q, 2gr_h, 2gr_v)$ and their shifts by curly brackets, that is, $\{a, b, c\}$ will indicate a shift in the quantum grading by a, in the horizontal grading by b, and in the vertical grading by c. Note that following the conventions in [2], we have doubled the latter two gradings, as illustrated in Figure 2.

Definition 3.1 (edge ring). Given a diagram D with vertices labelled by $x_1, \ldots, x_n$, define the *edge ring* of D as $R(D) := \mathbb{Z}[x_1, \ldots, x_n]/\langle \mathrm{rel}(v_i) \rangle$, where i runs over all internal vertices, or marks, with the defining relations being $x_i - x_j$ for type I and $x_k + x_l - x_i - x_j$ for type II vertices (see Figure 2).

Consider the two types of crossings D_+ and D_-, as in Figure 1, with outgoing edges labeled by k, l, and incoming edges labelled by i, j. Let

$$R_c := \frac{\mathbb{Z}[x_i, x_j, x_k, x_l]}{(x_k + x_l - x_i - x_j)} \cong \mathbb{Z}[x_i, x_j, x_k] \tag{3.1}$$

be the underlying ring associated to a crossing. The complex for the positive crossing D_+ is given by the following diagram:

$$\begin{tikzcd} R_c\{0,-2,0\} \arrow[r, "(x_k - x_i)"] & R_c\{0,0,0\} \\ R_c\{2,-2,-2\} \arrow[u, "(x_j - x_k)"] \arrow[r, "-(x_k - x_i)(x_k - x_j)"] & R_c\{0,0,-2\} \arrow[u, "1"'] \end{tikzcd} \tag{3.2}$$

The complex for the negative crossing D_+ is given by the following diagram:

$$\begin{tikzcd} R_c\{0,-2,2\} \arrow[r, "-(x_k - x_i)(x_k - x_j)"] & R_c\{-2,0,2\} \\ R_c\{0,-2,0\} \arrow[u, "1"] \arrow[r, "(x_k - x_i)"] & R_c\{0,0,0\} \arrow[u, "(x_j - x_k)"] \end{tikzcd} \tag{3.3}$$

Remark 3.2. The horizontal and vertical differentials d_+ and d_v are homogeneous of degrees $(2,2,0)$ and $(0,0,2)$, respectively. For those more familiar with [11] and hoping to reconcile the differences, note that in R_c multiplication by $x_kx_l - x_ix_j = -(x_k - x_i)(x_k - x_j)$, so up to some grading shifts we are working with the same underlying complex as in the original construction, but over $\mathbb{Z}$, not $\mathbb{Q}$.

To write down the complex for a general braid we tensor the above for every crossing, keeping track of markings, replace the underlying ring with a copy of the edge ring $R(D)$ and replace d_v with $(-1)^i d_v$ to make it anticommute with d_h (here i is the degree if d_v). More precisely, given a diagram D of a braid let

$$C(D) := \bigotimes_{\text{crossings}} \left(C(D_c) \bigotimes_{R_c} R(D) \right). \tag{3.4}$$

Definition 3.3 (HOMFLY-PT homology). Given a braid diagram D of a link L we define its HOMFLY-PT homology to be the group

$$H(L) := H(H(C(D), d_+), d_v^*)\{-w + b, w + b - 1, w - b + 1\}, \tag{3.5}$$

where w and b are the writhe and the number of strands of D, respectively.

Remark 3.4. In [2], this is what Rasmussen calls the "middle HOMFLY homology." The relation between this link homology theory and the HOMFLY-PT polynomial is that for any link $L \subset S^3$

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim H^{i,j,k}(L) = \frac{-P(L)}{q - q^{-1}}. \tag{3.6}$$

The Reduced Complex

There is a natural subcomplex $\overline{C}(D) \subset C(D)$ defined as follows: let $\overline{R}(D) \subset R(D)$ to be the subring generated by $x_i - x_j$ where i, j run over all edges of D and let $\overline{C}(D)$ be the subcomplex gotten by replacing in $C(D)$ each copy of $R(D)$ by one of $\overline{R}(D)$. A quick glance at the complexes $C(D_+)$ and $C(D_-)$ will reassure the reader that this is indeed a subcomplex, as the coefficients of both d_v and d_+ lie in $\overline{R}(D)$. We will refer to $\overline{C}(D)$ as the *reduced complex* for D.

(i) If i is an edge of D we can also define the complex $\overline{C}(D, i) := C(D)/(x_i)$. It is not hard to see that $\overline{C}(D, i) \cong \overline{C}(D)$ and is, hence, independent of the choice of edge i. See [2, Section 2.8] for a discussion as well as [11].

Below we will work primarily with the reduced complex $\overline{C}(D)$, and will stick with the grading conventions of [2], which are different than that of [11].

Definition 3.5 (reduced homology). Given a braid diagram D of a link L we define its reduced HOMFLY-PT homology to be the group

$$\overline{H}(L) := H\left(H\left(\overline{C}(D), d_+\right), d_v^*\right)\{-w+b-1, w+b-1, w-b+1\}, \tag{3.7}$$

where w and b are the writhe and the number of strands of D, respectively.

Remark 3.6. For any link $L \subset S^3$, we have

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \overline{H}^{i,j,k}(L) = P(L). \tag{3.8}$$

We can look at the complex $C(D)$ in two essential ways: either as the tensor product, over appropriate rings, of $C(D_+)$ and $C(D_-)$ for every crossing in our diagram D (as described above), or as a tensor product of corresponding complexes over all resolutions of the diagram. Although this is really just a matter of point of view, the latter approach is what we find in the original construction of Khovanov and Rozansky, as well as in the Soergel bimodule construction to be described below. To clarify this approach, consider the oriented D_o and singular D_s resolution of a crossing as in Figure 1. Assign to D_o the complex

$$0 \longrightarrow R_c \xrightarrow{(x_k - x_i)} R_c \longrightarrow 0 \tag{3.9}$$

and to D_s the complex

$$0 \longrightarrow R_c \xrightarrow{-(x_k - x_i)(x_k - x_j)} R_c \longrightarrow 0. \tag{3.10}$$

Then, we have

$$\begin{aligned} C(D_+) &: 0 \longrightarrow C(D_s) \longrightarrow C(D_o) \longrightarrow 0, \\ C(D_-) &: 0 \longrightarrow C(D_o) \longrightarrow C(D_s) \longrightarrow 0, \end{aligned} \tag{3.11}$$

where the maps are given by d_v as defined above. (For simplicity we leave out the internal grading shifts.) Let a resolution of a link diagram D be a resolution of each crossing in either of the two ways above, and let the complex assigned to each resolution be the tensor product of the corresponding complexes for each resolved crossing. Then, modulo grading shifts, our total complex can be viewed as

$$C(D) = \bigoplus_{\text{resolutions}} C(D_{\text{res}}), \tag{3.12}$$

where D_{res} is the diagram of a given resolution. This closely mimics the "state-sum model" for the Jones polynomial, due to Kauffman [14], or the MOY calculus of [15] for other quantum polynomials.

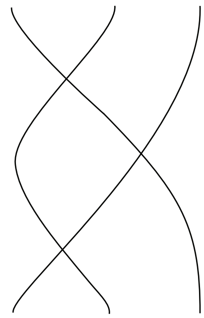


Figure 3: A braid diagram.

3.2. The Soergel Bimodule Construction

We now turn to the Soergel bimodule construction for the HOMFLY-PT homology found in [3]. Recall from Section 2.2 that the Soergel bimodule $B_i = R\bigotimes_{R^i} R\{-1\}$ where $R = \mathbb{Z}[x_1 - x_2, \ldots, x_{n-1} - x_n]$ is the ring generated by consecutive differences in variables $x_1, \ldots, x_n$ (n is the number of strands in the braid diagram), and $R^i \subset R$ is the subring of S_2-invariants corresponding to the permutation action $x_i \leftrightarrow x_{i+1}$. Furthermore define the map $B_i \to R$ by $1 \otimes 1 \mapsto 1$, and the map $R \to B_i$ by $1 \mapsto (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})$. We resolve a crossing in position $[i, i{+}1]$ in the either of two ways, as in Figure 1, assigning R to the oriented resolution and B_i to the singular resolution. For a positive crossing, we have the following complex:

$$C(D_+) : 0 \longrightarrow R\{2\} \longrightarrow B_i\{1\} \longrightarrow 0, \tag{3.13}$$

and for a negative crossing the complex

$$C(D_-) : 0 \longrightarrow B_i\{-1\} \longrightarrow R\{-2\} \longrightarrow 0. \tag{3.14}$$

We place B_i in homological grading 0 and increase/decrease by 1, that is, in the complex for D_+, $R\{2\}$ is in homological grading -1. Note, this grading convention differs from [3], and is the convention used in [8]. The complexes above are known as Rouquier complexes, due to Rouquier who studied braid group actions with relation to the category of Soergel bimodules; for more information we refer the reader to [3, 4, 8].

Given a braid diagram D we tensor the above complexes for each crossing, arriving at a total complex of length k, where k is the number of crossings of D, or equivalently the length of the corresponding braid word (Figure 3). Each entry in the complex can be thought of as a resolution of the diagram consisting of the tensor product of the appropriate Soergel bimodules. For example, to the graph in Section 3.2, we assign the bimodule $B_1 \otimes B_2 \otimes B_1$. That is, modulo grading shifts, we can view our total complex as

$$C(D) = \bigoplus_{\text{resolutions}} C(D_{\text{res}}). \tag{3.15}$$

To proceed, we take Hochschild homology $HH(C(D_{\text{res}}))$ for each resolution of D and arrive at the complex

$$HH(C(D)) = \bigoplus_{\text{resolutions}} HH(C(D_{\text{res}})), \tag{3.16}$$

with the induced differentials. Finally, taking homology of $HH(C(D))$ with respect to these differentials gives us our link homology.

Definition 3.7 (reduced homology). Given a braid diagram D of a link L we define its reduced HOMFLY-PT homology to be the group

$$H(HH(C(D))). \tag{3.17}$$

Of course, now that, we have defined reduced HOMFLY-PT homology in two different ways, it would be nice to reconcile the fact that they are indeed the same.

Claim 1. Up to grading shifts the two definitions of reduced HOMFLY-PT homology agree, that is, $H(H(\overline{C}(D), d_+), d_v^*) \cong H(HH(C(D)))$ for a diagram D of a link L.

Proof. The proof in [3] works without any changes for matrix factorizations and Soergel bimodules over $\mathbb{Z}$. We sketch it here for completeness and the fact that we will be referring to some of its details a bit later. Letus first look at the matrix factorization $C(D_s)$ (unreduced version) associated to a singular resolution D_s. Now $C(D_s)$ can be thought of as a Koszul complex of the sequence $(x_k + x_l - x_i - x_j,\ x_k x_l - x_i x_j)$ in the polynomial ring $\mathbb{Z}[x_i, x_j, x_k, x_l]$ (donot forget that in R_c multiplication by $x_k x_l - x_i x_j = -(x_k - x_i)(x_k - x_j)$). This sequence is regular so the complex has cohomology only in the right-most degree. The cohomology is the quotient ring

$$\frac{\mathbb{Z}[x_i, x_j, x_k, x_l]}{(x_i + x_j - x_k - x_l, x_k x_l - x_i x_j)}. \tag{3.18}$$

This is naturally isomorphic to the Soergel bimodule B_i' (notice that this is the "unreduced" Soergel bimodule) over the polynomial ring $R' = \mathbb{Z}[x_i, x_j]$. The left and right action of R' on B_i' corresponds to multiplication by x_i, x_j and x_k, x_l, respectively. Quotienting out by $x_k + x_l - x_i - x_j$ and $x_k x_l - x_i x_j$ agrees with the definition of B_i' as the tensor product $R' \bigotimes_{R_i'} R'$ over the subalgebra R_i' of symmetric polynomials in x_i, x_j.

Now letus consider a general resolution D_{res}. The matrix factorization for D_{res} is, once again, just a Koszul complex corresponding to a sequence of two types of elements. The first ones are as above, that is, they are of the form $x_k + x_l - x_i - x_j$ and $x_k x_l - x_i x_j$ and come from the singular resolutions D_s, and the remaining are of the form $x_i - x_j$ that come from "closing off" our braid diagram D, which in turn means equating the corresponding marks at the top and bottom the diagram. Now it is pretty easy to see that the polynomials of the first type, coming from the D_ss form a regular sequence and we can quotient out by them immediately, just like above. The quotient ring we get is naturally isomorphic to the Soergel bimodule $B'(D_{\text{res}})$ associated to the resolution D_{res}. At this point all, we have left is to deal with the remaining elements of the form $x_i - x_j$ coming from closing off D; to be more concrete, the Koszul complex we started with for D_{res} is quasi-isomorphic to the Koszul complex of the ring $B'(D_{\mathrm{res}})$ corresponding to these remaining elements. This in turn precisely computes the Hochschild homology of $B'(D_{\mathrm{res}})$.

Finally if we downsize from B_i' to B_i and from $C(D_{\mathrm{res}})$ to $\overline{C}(D_{\mathrm{res}})$ we get the required isomorphism. For more details we refer the reader to [3]. □

Gradings et al.

We come to the usual rigmarole of grading conventions, which seems to be evepresent in link homology. Perhaps when using the Rouquier complexes above we could have picked conventions that more closely matched those of Section 3.1. However, we chose not to for a couple of reasons: first there would inevitably be some grading conversion to be done either way due to the inherent difference in the nature of the constructions, and second we use Rouquier complexes to aid us in just a few results (namely the proof of Reidemeister moves II and III), and leave them shortly after attaining these; hence, it is convenient for us, as well as for the reader familiar with the Soergel bimodule construction of [3] and the diagrammatic construction of [7], to adhere to the conventions of the former and the subsequent results in [8]. For completeness, we descibe the conversion rules. Recall that in the matrix factorization construction of Section 3.1 we denoted the gradings as $(i, j, k) = (q, 2gr_h, 2gr_v)$.

(i) To get the cohomological grading in the Soergel construction take $(j - k)/2$ from Section 3.1.

(ii) The Hochschild grading here matches the "horizontal" or j grading of Section 3.1.

(iii) To get the "quantum" grading i of Section 3.1 of an element x, take Hochschild grading of x minus $\deg(x)$, that is, $\deg(x) = j(x) - i(x)$.

3.2.1. Diagrammatic Rouquier Complexes

We now restate the last section in the diagrammatic language of [8] as outlined above in Section 2.2. The main advantage of doing this is the inherent ability of the graphical calculus developed by Elias and Khovanov in [7] to hide and, hence simplify, the complexity of the calculations at hand. Recall that we work in the integral version of Soergel category $\mathcal{SC}_2$ as defined in Section 2.3 of [8], which allows for constructions over $\mathbb{Z}$ without adjoining inverses (see Section 5.2 in [8] for a discussion of these facts). Recall, that an object of $\mathcal{SC}_2$ is given by a sequence of indices $\underline{i}$, visualized as d points on the real line and morhisms are given by pictures or graphs embedded in the strip $\mathbb{R} \times [0, 1]$. We think of the indices as "colors," and depict them accordingly. The Soergel bimodule B_i is represented by a vertical line of "color" i (i.e., by the identity morphism from B_i to itself) and the maps we find in the Rouquier complexes above, Section 3.2, are given by those referred to as "start dot" and "end-dot." More precisely, the complexes $C(D_-)$ and $C(D_+)$ become as illustrated in Figure 4.

For completeness and ease we remind the reader of the diagrammatic calculus rubric used to contruct Rouquier complexes for a given braid diagram.

3.2.2. Conventions

We use a colored circle to indicate the empty graph, but maintain the color for reasons of sanity. It is immediately clear that in the complex associated to a tensor product of d Rouquier

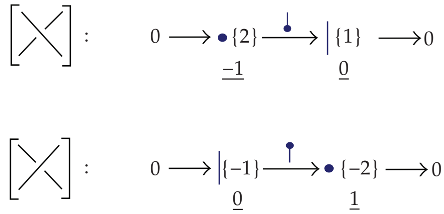


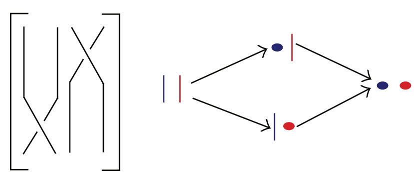
Figure 4: Diagrammatic Rouquier complex for right and left crossings.

complexes, each summand will be a sequence of k lines where $0 \leq k \leq d$ (interspersed with colored circles, but these represent the empty graph so could be ignored). Each differential from one summand to another will be a "dot" map, with an appropriate sign.

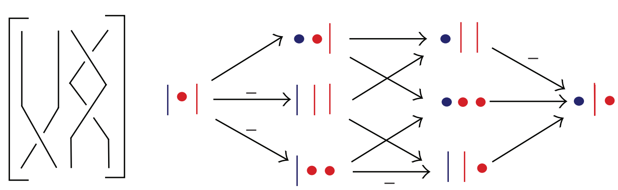
(1) The dot would be a map of degree 1 if B_i had not been shifted accordingly. In $\mathcal{SC}_2$, all maps must be homogeneous, so we could have deduced the degree shift in B_i from the degree of the differential. Because of this, it is not useful to keep track of various degree shifts of objects in a complex. Hence at times we will draw all the objects without degree shifts, and all differentials will therefore be maps of graded degree 1 (as well as homological degree 1). It follows from this that homotopies will have degree -1, in order to be degree 0 when the shifts are put back in. One could put in the degree shifts later, noting that $B_\emptyset$ always occurs as a summand in a tensor product exactly once, with degree shift 0.

(2) We will use blue for the index associated to the leftmost crossing in the braid, then red and dotted orange for other crossings, from left to right. The adjacency of these various colors is determined from the braid.

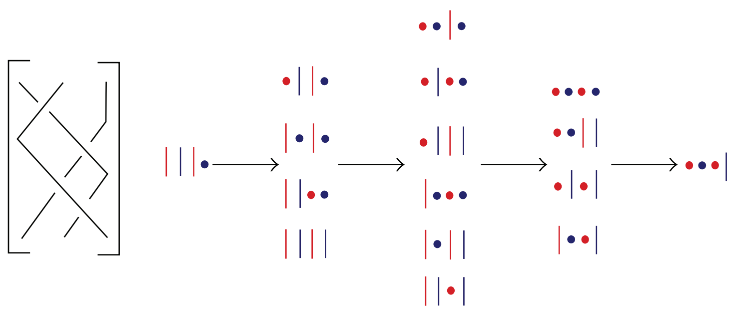
(3) We read tensor products in a braid diagram from bottom to top. That is, in the following diagram, we take the complex for the blue crossing, and tensor by the complex for the red crossing. Then we translate this into pictures by saying that tensors go from left to right. In other words, in the complex associated to this braid, blue always appears to the left of red.



(4) One can deduce the sign of a differential between two summands using the Liebnitz rule, $d(ab) = d(a)b + (-1)^{|a|}ad(b)$. In particular, since a line always occurs in the basic complex in homological dimension ± 1, the sign on a particular differential is exactly given by the parity of lines appearing to the left of the map. For example,



(5) When putting an order on the summands in the tensored complex, we use the following standardized order. Draw the picture for the object of smallest homological degree, which we draw with lines and circles. In the next homological degree, the first summand has the first color switched (from line to circle, or circle to line), the second has the second color switched, and so forth. In the next homological degree, two colors will be switched, and we use the lexicographic order: 1st and 2nd, then 1st and 3rd, then 1st and 4th,... then 2nd and 3rd, and so forth. This pattern continues.



4. Checking the Reidemeister Moves

We will use the matrix factorization construction of Section 3.1 to check Reidemeister move I, as it is not very difficult to verify even over $\mathbb{Z}$ that this goes through, and the diagrammatic calculus of Section 3.2.1 for the remaining moves. There are two main reasons for the interplay: first, checking Reidemeister II and III over $\mathbb{Z}$ using the matix factorization approach is rather computationally intensive (it was already quite so over $\mathbb{Q}$ in [11] with all the algebraic advantages of working over a field at hand); second, at this moment there does not exist a full diagrammatic description of Hochschild homology of Soergel bimodules, which prevents us from using a pictorial calculus to compute link homology from closed braid diagrams. Of course, for Reidemeister II and III we could have used the computations of [8], where we prove the stronger result that Rouquier complexes are functorial over braid cobordisms. Instead, the proofs we exhibit below use essentially the same strategy as the original paper [11], but are so much simpler and more concise that they underline well the usefulness of the diagrammatic calculus for computations.

A small lemma from linear algebra, which Bar-Natan refers to as "Gaussian Elimination for Complexes" in [16], will be very helpful to us.

Lemma 4.1. *If $\phi : B \to D$ is an isomorphism (in some additive category $\mathcal{C}$), then the four-term complex segment below*

$$\cdots [A] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}B\\ C\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix}D\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F] \cdots \tag{4.1}$$

is isomorphic to the (direct sum) complex segment

$$\cdots [A] \xrightarrow{\begin{pmatrix} 0 \\ \beta \end{pmatrix}} \begin{bmatrix} B \\ C \end{bmatrix} \xrightarrow{\begin{pmatrix} \phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{bmatrix} D \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix} 0 & \nu \end{pmatrix}} [F] \cdots . \tag{4.2}$$

Both of these complexes are homotopy equivalent to the (simpler) complex segment

$$\cdots [A] \xrightarrow{(\beta)} [C] \xrightarrow{(\epsilon - \gamma\phi^{-1}\delta)} [E] \xrightarrow{(\nu)} [F] \cdots . \tag{4.3}$$

Here the capital letters are arbitrary columns of objects in $\mathcal{C}$ *and all Greek letters are arbitrary matrices representing morphisms (all the matrices are block matrices);* $\phi : B \to D$ *is an isomorphism, that is, it is invertible.*

4.1. Reidemeister I

Proof. The complex $C(D_{I_a})$ for the left-hand side braid in Reidemester Ia, see Figure 5, has the form

$$\begin{tikzcd} \mathbb{Z}[x_1, x_2]\{0,-2,0\} \arrow[r, "0"] & \mathbb{Z}[x_1, x_2]\{0,0,0\} \\ \mathbb{Z}[x_1, x_2]\{2,-2,-2\} \arrow[r, "0"] \arrow[u, "(x_2 - x_1)"] & \mathbb{Z}[x_1, x_2]\{0,0,-2\} \arrow[u, "1"'] \end{tikzcd} \tag{4.4}$$

Up to homotopy, the right-hand side of the complex dissapears and only the top left corner survives after quotienting out by the relation $x_2 - x_1$. Note that the overall degree shifts of the total complex make sure that the left-over entry sits in the correct trigrading.

Similarly, the complex $C(D_{I_b})$ for the left-hand side braid in Reidemester Ib, has the form

$$\begin{tikzcd} \mathbb{Z}[x_1, x_2]\{0,-2,2\} \arrow[r, "0"] & \mathbb{Z}[x_1, x_2]\{-2,0,2\} \\ \mathbb{Z}[x_1, x_2]\{0,-2,0\} \arrow[r, "0"] \arrow[u, "1"] & \mathbb{Z}[x_1, x_2]\{0,0,0\} \arrow[u, "(x_2 - x_1)"'] \end{tikzcd} \tag{4.5}$$

The left-hand side is annihilated and the upper-right corner remains modulo the relation $x_2 - x_1$. □

4.2. Reidemeister II

Proof. Consider the braid diagrams for Reidemeister type IIa in Figure 5. Recall the decomposition $B_i \otimes B_i \cong B_i\{-1\} \oplus B_i\{1\}$ in $\mathcal{SC}_2$ and its pictorial counterpart 6. The complex we are interested in is, as illustrated in Figure 6.

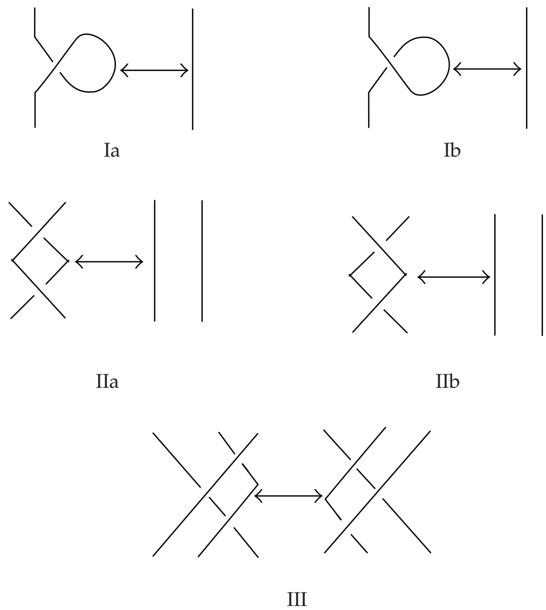


Figure 5: The Reidemeister moves.

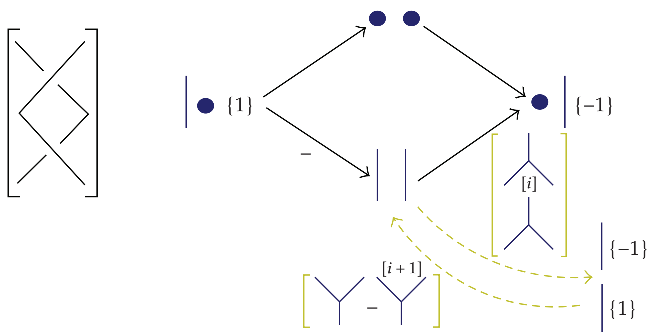


Figure 6: Reidemeister IIa complex with decomposition 6.

Inserting the decomposed $B_i \otimes B_i$ and the corresponding maps, we find two isomorphisms staring at us; we pick the left most one and mark it for removal, (see Figure 7).

After changing basis and removing the acyclic complex, as in Lemma 4.1, we arrive at the complex below with two more entries marked for removal, (see Figure 8.)

With the marked acyclic subcomplex removed, we arrive at our desired result, the complex assigned to the no crossing braid of two strands as in Figure 5. The computation for Reidemeister IIb is virtually identical. □

4.3. Reidemeister III

Proof. Luckily, we only have to check one version of Reidemeister move III, but as the reader will see below even that is pretty easy and not much harder than that of Reidemeister II above. We follow closely the structure of the proof in [11], utilizing the bimodule $R\bigotimes_{R^{i,i+1}}R\{-3\}$ and

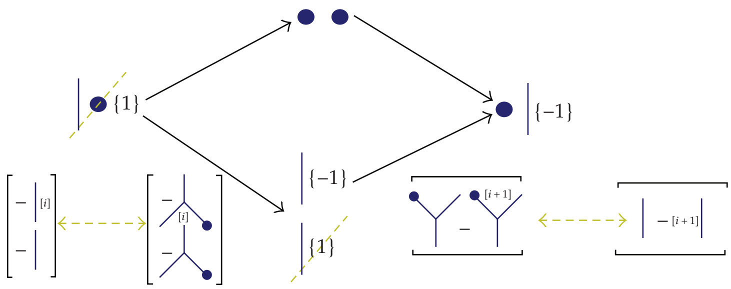


Figure 7: Reidemeister IIa complex, removing one of the acyclic subcomplexes.

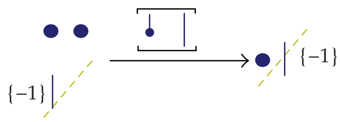


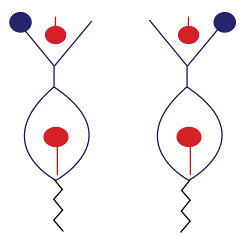
Figure 8: Reidemeister IIa complex, removing a second acyclic subcomplex.

decomposition 7 to reduce the complex for one of the RIII braids to that which is invariant under the move or, equivalently in our case, invariant under color flip. We start with the braid on the left-hand side of III in Figure 5; the corresponding complex, with decomposition 6 and 7 given by dashed/yellow arrows, is, (as illustrated in Figure 9).

We insert the decomposed bimodules and the appropriate maps; then we change bases as in Lemma 4.1 (the higher matrix of the two is before base-change, and the lower is after), (see Figure 10.)

We strike out the acyclic subcomplex and mark another one for removal; yet again we change bases (the lower matrix is the one after base change), (see Figure 11.)

Now we are almost done; if we can prove that the maps



are invariant under color change, we would arrive at a complex that is invariant under Reidemeister move III. To do this we must stop for a second, go back to the source and examine the original, algebraic, definitions of the morphisms in [7]; upon doing so we are relieved to see that the maps we are interested in are actually equal to zero (they are defined by sending $1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 \mapsto 0$). In all, we have arrived, (see Figure 12).

Repeating the calculation for the braid on the right-hand side of RIII, Figure 5, amounts to the above calculation with the colors switched—a quick glance will convince the reader that the end result is the same complex rotated about the x-axis. $\square$

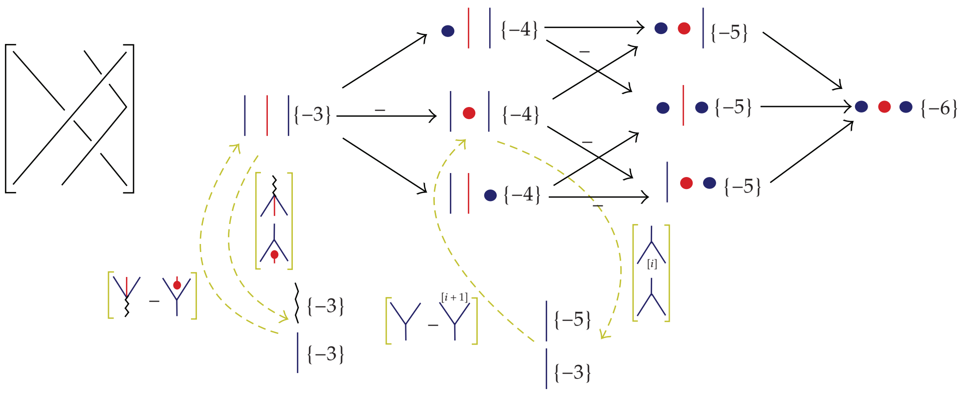


Figure 9: Reidemeister III complex with decompositions 6 and 7.

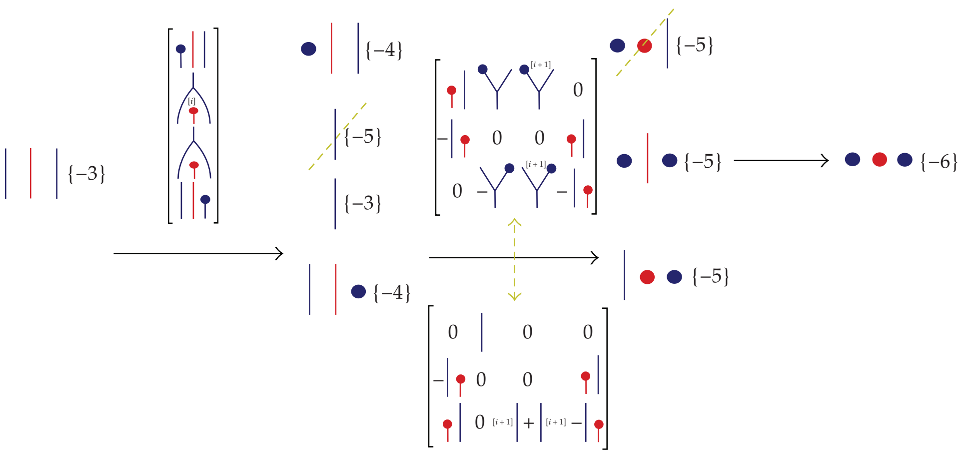


Figure 10: Reidemeister III complex, with an acyclic subcomplex marked for removal.

4.4. Observations

Having seen this interplay between the different constructions, perhaps it is a good moment to highlight exactly what categories we do need to work in so as to arrive at a genuine link invariant, or a braid invariant at that. Let us start with the latter: we can take the category of complexes of Soergel bimodules $\mathcal{KOM}(\mathcal{SC})$ (either the diagrammatic or "original" version) and construct Rouquier complexes; if we mod out by homotopies and work in $\mathcal{KOM}_h(\mathcal{SC})$, we arrive at something that is not only an invariant of braids but of braid cobordisms as well (over $\mathbb{Z}$ or $\mathbb{Q}$ if we wish). Now if we repeat the construction in the category of complexes of graded matrix factorizations $\mathcal{KOM}(mf)$, we have some choices of homotopies to quotient out by. First, we can quotient out by the homotopies in the category of graded matrix factorizations and work in $\mathcal{KOM}(hmf)$ and second, we can quotient in the category of the complexes and work in $\mathcal{KOM}_h(mf)$, or we can do both and work in $\mathcal{KOM}_h(hmf)$. It is immediate that working in $\mathcal{KOM}_h(mf)$ is necessary, but one could hope that it is also sufficient. A close look at the argument of Claim 1, where the two constructions are proven equivalent, shows that if we start with the Koszul complex associated to the resolution of

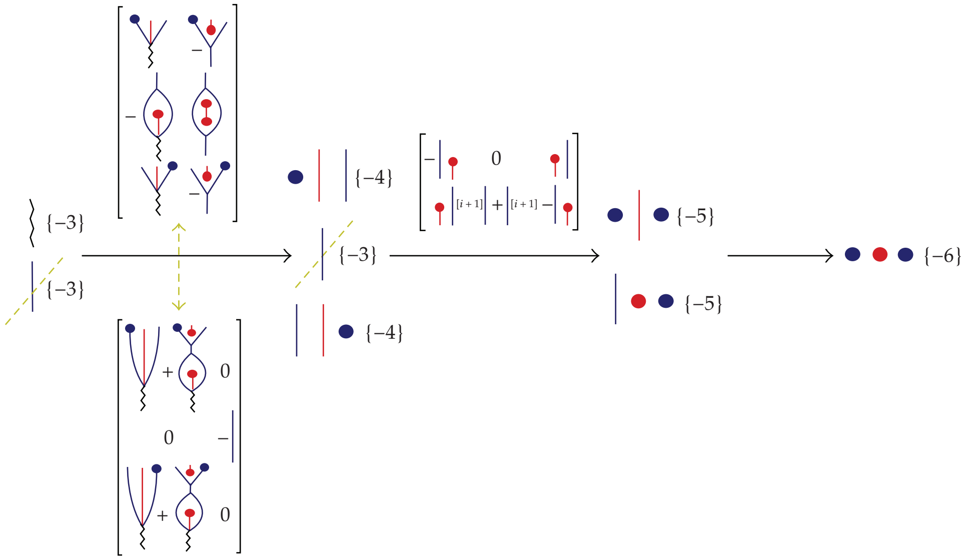


Figure 11: Reidemeister III complex, with another acyclic subcomplex marked for removal.

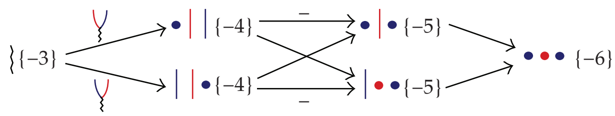


Figure 12: Reidemeister III complex—the end result, after removal of all acyclic subcomplexes.

a braid D_{res} the polynomial relations coming from the singular vertices in D_{res} form a regular sequence and, hence, the homology of this complex is the quotient of the edge ring $R(D_{\mathrm{res}})$ by these relations and is supported in the right-most degree. It is this quotient that is isomorphic to the corresponding Soergel bimodule, that is, the Koszul complex is quasi-isomorphic, as a bimodule, to $B'(D_{\mathrm{res}})$. Hence, we really do need to work in $\mathcal{KOM}_h(hmf)$, to have a braid invariant or an invariant of braid cobordisms, or a link invariant.

Anyone, who has suffered throught the proofs of, say, Reidemeister III in [11] would probably find the above a relief. Of course, much of the ease in computation using this pictorial language is founded upon the intimate understanding and knowledge of hom spaces between objects in $\mathcal{SC}$, which is something that is only available to us due to the labors of Elias and Khovanov in [7]. With that said, it would not be suprising if this diagrammatic calculus would aid other calculations of link homology in the future.

All in all, we have arrived at an integral version of HOMFLY-PT link homology; combining with the results of [8], we have the following.

Theorem 4.2. *Given a link $L \subset S^3$, the groups $H(L)$ and $\overline{H}(L)$ are invariants of L and when tensored with $\mathbb{Q}$ are isomorphic to the unreduced and reduced versions, respectively, of the Khovanov-Rozansky HOMFLY-PT link homology. Moreover, these integral homology theories give rise to functors from the category of braid cobordisms to the category of complexes of graded R-bimodules.*

5. Rasmussen's Spectral Sequence and Integral sl(n)-Link Homology

It is time for us to look more closely at Rasmussen's spectral sequence from HOMFLY-PT to sl(n)-link homology. For this we need an extra "horizontal" differential d_- in our complex, and here is the first time we encounter matrix factorizations with a nonzero potential; as before, to a link diagram D we will associate the tensor product of complexes of matrix factorizations with potential for each crossing. These will be complexes over the ring

$$R_c = \frac{\mathbb{Z}[x_i, x_j, x_k, x_l]}{(x_k + x_l - x_i - x_j)} \cong \mathbb{Z}[x_i, x_j, x_k], \tag{5.1}$$

with total potential

$$\begin{aligned} W_p[x_i, x_j, x_k, x_l] &= p(x_k) + p(x_l) - p(x_i) - p(x_j), \\ &= p(x_k) + p(x_i + x_j - x_k) - p(x_i) - p(x_j), \end{aligned} \tag{5.2}$$

where the $p(x) \in \mathbb{Z}[x]$. We do not specify the potential $p(x)$ at the moment as the spectral sequence works for any choice; later on when looking at sl(n)-link homology we will set $p(x) = x^{n+1}$.

To define d_-, let $p_i = W_p/(x_k - x_i)$ and $p_{ij} = -W_p/(x_k - x_i)(x_k - x_j)$ (recall that in R_c, $(x_k - x_i)(x_k - x_j) = x_i x_j - x_k x_l$, and note that substituting either $x_k = x_i$ or $x_k = x_j$ into W_p makes it vanish, so p_{ij} is indeed a polynomial in R_c).

To the positive crossing D_+, we assign the following complex:

$$\begin{tikzcd} R_c\{0,-2,0\} \arrow[r, shift left, "(x_k - x_i)"] & R_c\{0,0,0\} \arrow[l, shift left, "p_i"] \\ R_c\{2,-2,-2\} \arrow[u, "(x_j - x_k)"] \arrow[r, shift left, "-(x_k - x_i)(x_k - x_j)"] & R_c\{0,0,-2\} \arrow[u, "1"'] \arrow[l, shift left, "p_{ij}"] \end{tikzcd} \tag{5.3}$$

To the negative crossing D_-, we assign the following complex:

$$\begin{tikzcd} R_c\{0,-2,2\} \arrow[r, shift left, "-(x_k - x_i)(x_k - x_j)"] & R_c\{-2,0,2\} \arrow[l, shift left, "p_{ij}"] \\ R_c\{0,-2,0\} \arrow[u, "1"] \arrow[r, shift left, "(x_k - x_i)"] & R_c\{0,0,0\} \arrow[u, "(x_j - x_k)"'] \arrow[l, shift left, "p_i"] \end{tikzcd} \tag{5.4}$$

The total complex for a link L with diagram D will be defined analagously to the one above, that is,

$$C_p(D) := \bigotimes_{\text{crossings}} \left(C(D_c) \bigotimes_{R_c} R(D) \right), \tag{5.5}$$

as will be the reduced $\overline{H}_p(L, i)$ and unreduced $H_p(L)$ versions of link homology.

The main result of [2] is the following.

Theorem 5.1 (Rasmussen [2]). *Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p(x) \in \mathbb{Q}[x]$, there is a spectral sequence $E_k(p)$ with $E_1(p) \cong \overline{H}(L)$ and $E_\infty(p) \cong \overline{H}_p(L,i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L,i).*

In particular setting $p(x) = x^{n+1}$ one would arrive at a spectral sequence from the HOMFLY-PT to the $\mathrm{sl}(n)$-link homology. Rasmussen's result pertains to rational link homology with matrix factorizations defined over the ring $\mathbb{Q}[x_1,\ldots,x_n]$ and potentials polynomials in $\mathbb{Q}[x]$. We will essentially repeat his construction in our setting and, for the benefit of those familiar with the results of [2], will stay as close as possible to the notation and conventions therein. This will be a rather condensed version of the story and we refer the reader to the original paper for more details.

We will work primarily with reduced link homology (although all the results follow through for both versions) and with closed link diagrams, where all three differentials d_v, d_+, and d_- anticommute. We have some choices as to the order of running the differentials, so let us define

$$\overline{H}^+(D,i) = H\left(\overline{C}(D,i), d_+\right). \tag{5.6}$$

Here, $\overline{H}^+(D,i)$ inherits a pair of anticommuting differentials d_-^* and d_v^*, where d_-^* lowers gr_h by 1 while preserving gr_v and d_v^* raises gr_v by 1 while preserving gr_h. Hence, $(\overline{H}_p^+(D,i), d_v^*, d_-^*)$ defines a double complex with total differential $d_{v-} := d_v^* + d_-^*$.

Definition 5.2. Let $E_k(p)$ be the spectral sequence induced by the horizontal filtration on the complex $(\overline{H}_p^+(D,i), d_{v-})$.

After shifting the triple grading of $E_k(p)$ by $\{-w+b-1, w+b-1, w-b+1\}$ it is immediate that the first page of the spectral sequence is isomorphic to $\overline{H}(L,i)$ (the part of the differential $d_v^* + d_-^*$ which preserves horizontal grading on $E_0(p) = \overline{H}^+(D,i)\{-w+b-1, w+b-1, w-b+1\}$ is precisely d_v^*, that is, $d_0(p) = d_v^*$ and

$$E_1(p) = H\left(\overline{H}^+(D,i), d_v^*\right)\{-w+b-1, w+b-1, w-b+1\} \cong \overline{H}(L,i), \tag{5.7}$$

where D is a diagram for L. It also follows that $d_k(p)$ is homogenous of degree $-k$ with respect to gr_h and degree $1-k$ with respect to gr_v, and in the case that $p(x) = x^{n+1}$ it is also homogeneous of degree $2nk$ with respect to the q-grading.

Claim 2. Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p(x) \in \mathbb{Z}[x]$, the spectral sequence $E_k(p)$ has $E_1(p) \cong \overline{H}(L,i)$ and $E_\infty(p) \cong \overline{H}_p(L,i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L,i).

Proof. We argue as in [2, Section 5.4]. Suppose that, we have two closed diagrams D_j and D_j' that are related by the jth Reidemeister move, and suppose that there is a morphism

$$\sigma_j : \overline{H}_p^+(D_j,i) \longrightarrow \overline{H}_p^+\left(D_j', i\right) \tag{5.8}$$

in the category $\mathcal{KOM}(mf)$ that extends to a homotopy equivalence in the category of modules over the edge ring R. Then σ_j induces a morphism of spectral sequences $(\sigma_j)_k : E_k(D_j, i, p) \to E_k(D'_j, i, p)$ which is an isomorphism for $k > 0$. See [2] for more details and discussion. Hence, in practice, we have to exhibit morphisms and prove invariance for the first page of the spectral sequence, that is, for the HOMLFY-PT homology, which is basically already done. However, we ought to be a bit careful, of course, as here we are working with $\overline{H}_p^+(D, i)$ and not with the complex $\overline{C}(D, i)$ defined in Section 4.

Reidemeister I is done, as in this case $d_+ = 0$ and, hence, the complex $\overline{H}_p^+(D, i) = \overline{C}_p(D, i)$ and the same argument as the one in Section 4.1 works here.

For Reidemesiter II and III, we have to observe that for a closed diagram, we have morphisms $\sigma_j : \overline{C}_p(D_j, i) \to \overline{C}_p(D'_j, i)$ for $j = \mathrm{II}, \mathrm{III}$, which are homotopy equivalences of complexes (these can be extrapolated from Section 4 above, or from [8], where all chain maps are exhibited concretely). Therefore we get induced maps $(\sigma_j)_k$ on the spectral sequence with the property that $(\sigma_j)_1 = \sigma_{j*}$ is an isomorphism.

To get the last part of the claim, that is, that the reduced homology depends only on the link component and not on the edge therein we refer the reader to [2], as the arguments from there are valid verbatum. □

Setting $p(x) = x^{n+1}$, we get that the differentials $d_k(p)$ preserve $q + 2ngr_h$ and, hence, the graded Euler characteristic of $H(\overline{H}_p^+(D, i), d_{v-})$ with respect to this quantity is the same as that of $E_1(x^{n+1})$. Tensoring with $\mathbb{Q}$, to get rid of torsion elements, and computing we see that the Euler characteristic of the $E_\infty(x^{n+1})$ is the quantum $\mathrm{sl}(n)$-link polynomial $P_L(q^n, q)$ of L. See [2, Section 5.1] for details. We have arrived at the following.

Theorem 5.3. *The $E_\infty(x^{n+1})$ of the spectral sequence defined in 11 is an invariant of L and categorifies the quantum* $\mathrm{sl}(n)$*-link polynomial $P_L(q^n, q)$.*

Remark 5.4. Well, we have a categorification over $\mathbb{Z}$ of the quantum $\mathrm{sl}(n)$-link polynomial, but what homology theory exactly are we dealing with? Is it isomorphic to $H(H(H(\overline{C}_{x^{n+1}}(D, i), d_+), d_-^*), d_v^*)$ or to $H(H(\overline{C}_{x^{n+1}}(D, i), d_+ + d_-), d_v^*)$ and are these two isomorphic here? The answer is not immediate. In [2], Rasmussen bases the corresponding results on a lemma that utilizes the Kunneth formula, which is much more manageable in this context when looked at over $\mathbb{Q}$. Of course, for certain classes of knots things are easier. For example, if we take the class of knots that are *KR-thin*, then the spectral sequence converges at the E_1 term, as this statement only depends on the degrees of the differentials, and we have that $E_\infty(x^{n+1}) \cong H(H(\overline{C}_{x^{n+1}}(D, i), d_+), d_v^*)$. However, that is a bit of a "red herring" as stated.

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